CHAPTER I

INTRODUCTION

1. The present thesis is devoted to the study of certain problems relating to absolute convergence of Fourier series. A Fourier series, as is well known, is a trigonometric series

$$(1) \qquad \frac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx),$$

whose coefficients a_n, b_n are related to a 2π - periodic and Lebesgue-integrable function f by means of the socalled Euler-Fourier formulae:

$$(2) \qquad \begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt, \quad n = 0, 1, 2, \cdots \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt, \quad n = 1, 2, \cdots \end{aligned}$$

Considerable amount of work has been done with regard to the summability problems of Fourier series and, in fact, one can hardly find problems of intrinsic merit in that field now, but much remains to be done with regard to the question of almost everywhere convergence and almost everywhere divergence of Fourier series. Also, one of the outstanding problems, which await their solutions in the theory of Fourier series, is to ascertain the structural properties of functions which have absolutely convergent Fourier series. We have obtained results pertaining to both of these questions.

This chapter is of introductory character and seeks to give a brief survey of problems dealt within the thesis.

2. The problem of the absolute convergence of Fourier series appears to have been considered for the first time by S. Bernstein[1)] in the year 1914. It will be convenient to introduce some definitions and notations before we mention Bernstein's theorem and its generalisations by others. Let

$$\omega(\delta) = \omega(\delta; f) = \sup |f(x_1) - f(x_2)|,$$

for $x_1, x_2 \in [0, 2\pi]$, $|x_2 - x_1| \leq \delta$. The function $\omega(\delta)$ is called the modulus of continuity of f. We say that f satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, in $(0, 2\pi)$, or in symbols $f \in Lip\,\alpha\,(0, 2\pi)$, if there exists a constant C, independent of δ, such that

$$\omega(\delta) \leq \delta^{\alpha}.$$

Bernstein proved the following theorem:

(1.1) _If_ $f \in Lip\,\alpha\,(0, 2\pi)$, $\alpha > \frac{1}{2}$, _then the series_

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1) Bernstein [1]

(3) $$\sum_{n=1}^{\infty} \left(|a_n| + |b_n| \right) .$$

is convergent. For $\alpha = \frac{1}{2}$, (3) need not converge.[1)]

Obviously the convergence of (3) implies the absolute convergence of the Fourier series (1); although (1) may converge absolutely without the series (3) being convergent.[2)] As a matter of fact, in our study of the problem of absolute convergence of a Fourier series we shall mostly investigate conditions under which (3) converges and deduce from it the absolute convergence of (1).

Now as the Fourier series of a function $f \in Lip\, \frac{1}{2}$ need not be absolutely convergent it is natural to investigate additional conditions satisfied by f which would ensure the absolute convergence of its Fourier series. L. Neder[3)] has proved the following result in this connection:

(1.2) If the Lipschitz condition in Bernstein's theroem is strengthened to the condition:

$$\omega(\delta) \leq \frac{c\,\delta^{1/2}}{l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)} ,$$

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1) For some generalizations of this theorem, see Bernstein [2], [3], as also Salem [22] .

2) Zygmund [40], p. 232.

3) Neder [20] .

<u>where</u> $\varepsilon > 0$ <u>and</u>

$$l_1(\delta) = \log(e + \delta^{-1})$$
$$l_2(\delta) = \log\log(e^e + \delta^{-1}), \text{ etc.,}$$

(C <u>is a constant depending on</u> f , <u>but not on</u> δ) <u>then the Fourier series of</u> f <u>converges absolutely</u>.

The problem of the absolute convergence of a Fourier series may be formulated in the following more general setting:

Given a Fourier series (1), we want to know as to what values of the exponent β will make the series

$$(4) \qquad \sum_{n=1}^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta} \right)$$

converge*. In this connection, the following result was obtained by O. Szàsz[1)] in the year 1922:

(1.3) <u>If</u> $f \in Lip\,\alpha\ (0, 2\pi)$, $0 < \alpha \leq 1$, <u>then the series</u> (4) <u>converges for all</u> $\beta > 2/(2\alpha+1)$; <u>but not necessarily for</u> $\beta = 2/(2\alpha+1)$.

A.C.Zaanen[2)], however, has proved the following theorem:

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1) Szàsz [23] 2) Zaanen [36]

* The problem is further generalised when we establish a relation between α, β, and T which makes $\sum_{n=2}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}) \log^T n < \infty$, see, for example, Cheng [8] and Yadav [30].

(1.4) If for certain $\varepsilon > 0$,

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)]^{\frac{2\alpha+1}{2}}},$$

then (4) converges for $\beta = 2/(2\alpha+1)$.

We observe that Bernstein's theorem (1.1) reduces to a particular case of Szàsz's result, for if we take $\alpha > \frac{1}{2}$, then $2/(2\alpha+1) < 1$, so that β could be taken greater or equal to 1.

In the year 1928, A. Zygmund[1] proved the following result which is closely related to Bernstein's theorem:

(1.5) If f is of bounded variation and belongs to $Lip\,\alpha\,(0, 2\pi)$, $\alpha > 0$, then the Fourier series of f converges absolutely.

In fact, the theorem holds even if the condition ' f belongs to $Lip\,\alpha\,(0, 2\pi)$, $\alpha > 0$ ' is replaced by the following less stringent condition:

(5) $$\omega(\delta) = O\left(\log \frac{1}{\delta}\right)^{-2-\eta}, \quad \eta > 0.$$

The fact that the Fourier series of f need not be absolutely convergent when $\eta = 0$ in condition (5) has been proved recently by Ching-tsün Loo.[2]

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1) Zygmund [37] 2) Loo [18]

It is important to note that the Fourier series of a function of bounded variation or even of an absolutely continuous function may not converge absolutely. This is seen from the classical example of the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}, \tag{6}$$

which is the Fourier series of an absolutely continuous function, and which does not converge absolutely.

The following generalisation of Zygmund's theorem (1.5) was obtained by Z. Waraszkiewicz[1] and A. Zygmund[2]:

(1.6) <u>If f is of bounded variation and belongs to</u> $Lip\ \alpha(0, 2\pi)$, $\alpha > 0$ <u>then the series</u> (4) <u>converges for all</u> $\beta > 2/(\alpha+2)$; <u>but not necessarily</u> for $\beta = 2/(\alpha+2)$.

We have studied in Chapter II of this thesis the question of convergence of (4) for $\beta = 2/(\alpha+2)$ and have proved the following theorem:[3]

(1.7) <u>If</u> f <u>is of bounded variation and satisfies the condition</u>:

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\epsilon}(\delta)]^{\alpha+2}},$$

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1) Waraszkiewicz [29] 2) Zygmund [39]
3) Yadav and Goyal [34]

then (4) converges for $\beta = 2/(\alpha+2)$.

3. There is yet another aspect of the problem of absolute convergence of Fourier series of functions belonging to $Lip\,\alpha$ which is forced to our attention when we compare the theorems (1.1) and (1.5) of Bernstein and Zygmund respectively. The series $\sum(|a_n|+|b_n|)$ may diverge if $f \in Lip\,\alpha$, $\alpha \leq \frac{1}{2}$, but according to Zygmund's theorem the series converges if $f \in Lip\,\alpha$, $\alpha > 0$, and also f is of bounded variation. It is of interest to enquire as to how rapid is the divergence in the first case and how fast is the convergence in the second case. The 'rapidity' in either case could be measured, in a sense, by considering certain convergence factors or divergence factors as the case may be and this has been done by various authors. G.H. Hardy[1], for example, proved the following theorems:

(1.8) *If* $f \in Lip\,\alpha(0,2\pi)$, $0 < \alpha \leq 1$, *then*

$$\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}}\left(|a_n|+|b_n|\right) < \infty, \tag{7}$$

for $\beta < \alpha$.

(1.9) *If* f *is of bounded variation and belongs to* $Lip\,\alpha(0,2\pi)$, $0 < \alpha \leq 1$, *then*

1) Hardy [10]

$$(8) \qquad \sum_{n=1}^{\infty} n^{\beta/2} (|a_n| + |b_n|) < \infty,$$

<u>for</u> $\beta < \alpha$.

The following extension to theorem (1.8) has been given by A.C.Zaanen[1]:

(1.10) <u>If for certain</u> $\varepsilon > 0$,

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{l_1(\delta)\, l_2(\delta) \dots l_k^{1+\varepsilon}(\delta)},$$

<u>then</u> (7) <u>holds for</u> $\beta = \alpha$.

We shall also prove, in Chapter II, the following theorem which is analog~~i~~ous addition to the theorem (1.9)[2]:

(1.11) <u>If f is of bounded variation and satisfies the condition</u>:

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{[l_1(\delta)\, l_2(\delta) \dots l_k^{1+\varepsilon}(\delta)]^2},$$

<u>then</u> (8) <u>holds for</u> $\beta = \alpha$.

4. In the year 1928, Hardy and Littlewood[3] introduced the notion of a class of functions called by them $Lip(\alpha, p), \alpha > 0, 1 \leq p$, which is more general than the class $Lip\,\alpha$. A function f is said to belong to the class $Lip(\alpha, p), 0 < \alpha \leq 1, 1 \leq p$, if

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1) Zaanen [36] 2) Yadav and Goyal, loc. cit.
3) Hardy and Littlewood [13]

$$\left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx \right)^{1/p} = O(|h|^\alpha).$$

It is not difficult to see that the conclusions of Bernstein's theorem (1.1) and Szàsz theorem (1.3) remain valid if the hypothesis that $f \varepsilon \mathrm{Lip}\,\alpha$ is replaced by the less stringent hypothesis that $f \varepsilon \mathrm{Lip}(\alpha, 2)$. Even more general results could be proved under the new hypothesis and, indeed, O. Szàsz[1)] proved the following generalisation of (1.3):

(1.12) Let $0 < \alpha \leq 1$, $1 < p \leq 2$. If $f \varepsilon \mathrm{Lip}(\alpha, p)$ then the series (4) converges for all $\beta > p/\{p(\alpha+1)-1\}$; but not necessarily for $\beta = p/\{p(\alpha+1)-1\}$.

Pursuing the study of such problems still further we have proved in Chapter II of the present thesis the following theorem:

(1.13) Let $0 < \alpha \leq 1$, $1 < p \leq 2$ and $h > 0$. If

$$\left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx \right)^{1/p} \leq \frac{c h^\alpha}{[l_1(h) l_2(h) \ldots l_k^{1+\varepsilon}(h)]^\gamma},$$

where $\gamma = \{p(\alpha+1)-1\}/p$, then the series (4) is convergent for $\beta = p/\{p(\alpha+1)-1\}$.

This theorem contains as a special case the theorem (1.19) due to Min-Teh Cheng.

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1) Szàsz [24]

We remark that more general forms of theorems (1.13) - (1.16) (corresponding to a theorem of Szàsz[1]) can be proved where the logarithms in hypotheses are replaced by a more general function.

It is also known that for a function $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$, $1 < p \leq 2$,

$$(8) \qquad \sum_{n=1}^{\infty} n^{\beta}\left(|a_n| + |b_n|\right) < \infty$$

for every $\beta < \alpha - \frac{1}{p}$[2].

In Chapter II, we prove the following extension of the above result:

(1.14) _Let_ $0 < \alpha \leq 1$, $1 < p \leq 2$ _and_ $h > 0$. _If for certain_ $\varepsilon > 0$,

$$\left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx\right)^{1/p} \leq \frac{c\, h^{\alpha}}{l_1(h).\, l_2(h) \ldots l_k^{1+\varepsilon}(h)},$$

then (8) _holds for_ $\beta = \alpha - 1/p$.

5. Some of the ideas that we have discussed above could be fruitfully applied to the theory of ~~F~~ourier transforms also. Taking clue from the theorems of Bernstein (Theorem 1.1) and Szàsz (Theorm 1.3), E.C.Titchmarsh[3] proved the following theorem:

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1) Szàsz [25]
2) Zygmund [40]
3) Titchmarsh [26]

(1.15) Let $0<\alpha\leq 1$, $1<p\leq 2$ and $h>0$. If $f \in L_p(-\infty,\infty)$ and if

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^p dx = O(h^{\alpha p}), \text{ as } h\to 0,$$

then the Fourier transform g of f belongs to L_β, for

$$p/(\alpha p+p-1) < \beta < p/(p-1).$$

The range of β cannot be extended in this theorem. We have, however, proved the following theorem in Chapter II:

(1.16) Let $0<\alpha\leq 1$, $1<p\leq 2$ and $h>0$. If $f \in L_p(-\infty,\infty)$, and if for certain $\varepsilon>0$

$$\left(\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^p dx\right)^{1/p} = O\left(\frac{h^\alpha}{[l_1(h)l_2(h)\dots l_k^{1+\varepsilon}(h)]^\lambda}\right),$$

where $l_1(h)=\log(h^{-1})$, $l_n(h)=\log l_{n-1}(h)$ and $\lambda=(\alpha p+p-1)/p$, then g belongs to L_β, for $\beta=p/(\alpha p+p-1)$.

If we take $\alpha<1$ and $p=2$, theorem (1.15) can be put in a completely satisfactory form:[1]

(1.17) If $f\in L_2(-\infty,\infty)$ and $0<\alpha<1$, then a necessary and sufficient condition for

$$\left(\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right)\{g(x)\}^2 dx = O(\eta^{-2\alpha}), \text{ as } \eta\to\infty,$$

1) Titchmarsh [27]

is that

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^2 dx = O\left(|h|^{2\alpha}\right), \quad h \to 0.$$

We have also proved, in Chapter II, the following analogue of the above theorem for Fourier series:

(1.18) Let $f \in L_2(0, 2\pi)$ and let

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx}. \quad \text{(Complex form)*}$$

A necessary and sufficient condition that

$$\sum_{-\infty}^{-N} |c_n|^2 + \sum_{N}^{\infty} |c_n|^2 = O\left(N^{-2\alpha}\right), \quad as\ N \to \infty,$$

is that

$$\int_{0}^{2\pi} |f(x+h)-f(x-h)|^2 dx = O\left(|h|^{2\alpha}\right), \quad as\ h \to 0,$$

$0 < \alpha < 1$.

6. If we take $\alpha p > 1$ in Szàsz theorem (1.12), then $p/(\alpha p + p - 1) < 1$, and hence (1.12) implies the absolute convergence of the Fourier series of f in this

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* Every (real) Fourier series (1) can be put in this (complex) form if we define

$a_{-n} = a_n$, $b_{-n} = -b_n$; $c_n = \frac{1}{2}(a_n - i b_n)$, $n = 0, 1, 2, \cdots$; $c_{-n} = \bar{c}_n$,
and express cosines and sines in terms of exponential functions.

case. However, Min-Teh Cheng[1] proved, in 1942, the following theorem:

(1.19) <u>If</u> $1 < p \leq 2$, $\varepsilon > 0$, $h > 0$ <u>and</u>

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h \left(\log h^{-1}\right)^{-p-\varepsilon}\right),$$

<u>then the Fourier series of</u> f <u>converges absolutely. This is no longer true for</u> $\varepsilon = 0$.

As a matter of fact, this theorem has been proved by Min-Teh Cheng in the following more general form:

(1.20) <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$ <u>and</u>

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h \left(\log h^{-1}\right)^{-1-\alpha p}\right),$$

<u>then</u>

$$\sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right) \log^{T} n < \infty, \tag{9}$$

<u>for</u> $T < \alpha + p^{-1} - 1$. <u>Moreover</u> (9) <u>may not hold for</u> $T = \alpha + p^{-1} - 1$.

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1) Cheng [8]

Min-Teh Cheng has proved this theorem by first establishing an inequality for $Lip(\alpha, p)$ classes corresponding to the Hausdorff-Young inequality for L_p classes. We have given, in Chapter III, an alternative proof of (1.20) based only on the Hausdorff-Young inequality and without appealing to the inequality for $Lip(\alpha, p)$ classes obtained by Min-Teh Cheng. Thus the proof which we have given is shorter and more direct. We have also generalized this theorem to the following theorem:[1]

(1.21) <u>If</u> $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$ <u>and</u>

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h^\delta (\log h^{-1})^{-1-\alpha p}\right),$$

<u>where</u> $\delta = 1 + p(1-\beta)/\beta$, <u>then</u>

$$(10) \qquad \sum_{n=2}^{\infty} \left(|a_n|^\beta + |b_n|^\beta\right) \log^T n < \infty,$$

<u>for all</u> $\beta > p(T+1)/(1+\alpha p)$. <u>For</u> $\beta = p(T+1)/(1+\alpha p)$, (10) <u>does not hold.</u>

We also observe that Min-Teh Cheng's theorem (1.19) is contained in our more general result (1.13).

In the theorem (1.20), it is natural to ask: What condition must f satisfy in order that (9) may hold for

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1) Yadav [30]

$T = \alpha + p^{-1} - 1$? In this connection, we have established the following result:

(1.22) Let $0 < \alpha \leq 1$, $1 < p \leq 2$ and $h > 0$. If

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h\,(\log h^{-1})^{-(1+\alpha p)}(\log\log h^{-1})^{-(1+\varepsilon)p}\right),$$

then (9) holds for $T = \alpha + p^{-1} - 1$.

Similar addition can be made to the theorem (1.21).

We may mention here that the method of proofs which we have developped in order to prove theorems (1.20)-(1.22) can be used to prove these results in their more general forms as indicated in Chapter III.

7. Next we give an application of the theorem (1.21) to construct Young's continuous functions. Let the functions f, g and h be each L-integrable in $(0, 2\pi)$ and periodic outside with period 2π. We say that h is the Fourier faltung (composition or convolution) of f and g if

$$h(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\,g(t)\,dt.$$

Moreover, a function h is said to be a Young's continuous function if there exist two functions f and g, each of the Lebesgue class L_2, such that h is the Fourier faltung of these functions.[1] Although there is no obvious criterion to verify directly whether a given function h is a Young's continuous function, it is known that a necessary and sufficient condition for a function to be a Young's continuous function is that its Fourier series be absolutely convergent.[2] Min-Teh Cheng[3] has obtained Young's continuous functions as a convolution of f and g, where f satisfies stronger conditions and g weaker ones. Indeed, he proves the theorem:

(1.23) <u>If</u> $f \varepsilon Lip(\alpha, p)$, $\alpha p > \frac{1}{2}$, $0 < \alpha \leq 1$, $1 < p \leq 2$ <u>and if</u> $g \varepsilon Lip(\frac{1}{2p}, q)$, <u>for</u> $q > 1$, <u>then the function</u>

$$h(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)g(t)\,dt \tag{11}$$

<u>is a Young's continuous function.</u>

We have also obtained Young's continuous functions with the help of our theorem (1.21). We establish in Chapter IV the following result:

(1.24) <u>Let</u> $0 < \alpha \leq 1$, $1 < p \leq 2$ <u>and</u> $t > 0$. <u>If</u>

$$\int_0^{2\pi} |f(x+t) - f(x)|^p\,dx = O\left(t^{2-p}(\log t^{-1})^{-1-\alpha p}\right),$$

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1) Young [35]
2) Hardy and Littlewood [12] and Chen [7]
3) Cheng, loc. cit.

<u>as</u> $t \to 0$, <u>and</u> $g \in Lip(\frac{1}{q}, q)$, <u>where</u> $p(1-\alpha) < 1$ <u>and</u> q <u>is given by</u> $p^{-1} + q^{-1} = 1$, <u>then the function</u> h <u>in</u> (11) <u>is a Young's continuous function.</u>[1]

We have also proved the following theorem:

(1.25) <u>If</u> $f \in Lip(\alpha, p)$ <u>and</u> $g \in lip(\alpha', q)$, <u>where</u> $1 < p \leq 2, \alpha + \alpha' > \frac{1}{p}$ <u>and</u> q <u>is given by</u> $p^{-1} + q^{-1} = 1$, <u>then the function</u> h <u>given by</u> (11) <u>is a Young's continuous function</u>.

8. We have further considered absolute convergence of the Fourier series of an absolutely continuous function. We have already seen [series (6)] that the Fourier series of an absolutely continuous function is not necessarily absolutely convergent. Therefore, in addition to being absolutely continuous, a function must satisfy some other conditions in order that its Fourier series may converge absolutely. Various such additional conditions have been obtained by L.Tonelli[2], A.Zygmund[3] and G.H.Hardy and J.E.Littlewood[4] mostly in terms of the integrability of the derevative of an absolutely continuous function. We shall also prove, in Chapter V, the following theorem which is different in character from those of the above authors:

(1.26) <u>If</u> F <u>is a</u> 2π-<u>periodic absolutely continuous function with its derivative</u> $F'(x) = f(x)$ <u>such that</u>

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1) Yadav [33] 2) Tonelli [28]
4) Hardy Littlewood [14] , see also Fejér [9]
3) Zygmund [38]

(12) $$\omega(\delta) \leq c\,\delta^{\alpha}, \quad \alpha > -1/2,$$

where ω is the modulus of continuity of f, then the Fourier series of F converges absolutely.

We have also proved the following routine generalization of the above theorem:

(1.27) If $0 < \beta \leq 1$ and f satisfies the condition (12), then

$$\sum_{\nu=1}^{\infty} \left(|A_\nu|^{\beta} + |B_\nu|^{\beta} \right) < \infty,$$

for $\beta > \dfrac{2}{2\alpha+3}$.

Lastly we establish in Chapter V one more theorem on the absolute convergence of Fourier series of functions of bounded variation. Apart from an additional condition on function, the hypothesis of the theorem involves certain restrictions on its Fourier coefficients also. Our theorem is in a sequel to certain theorems of R.Mohanty[1] and S. Izumi.[2] It runs as follows:[3]

(1.28) Let

$$f \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

If f is such that

(i) f is of bounded variation in $(0, \pi)$

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1) Mohanty [19]
2) S.Izumi [15]
3) Yadav [32]

(ii) $\int_0^\delta |d\phi(x,h)| = O(|h|^\alpha), \quad \alpha > 0$, where

$\phi(x,h) = f(x+h) - f(x-h)$ and $\delta > 0$; and

(iii) the sequence $\{n^\gamma \Delta(n k_n)\}$ is of bounded variation for some $\gamma > 0$, then

$$\sum_{n=1}^{\infty} |k_n| < \infty .$$

We have also the cosine series analogue.

9. Some of the techniques of proofs of the theorems on absolute convergence of Fourier series can be applied to obtain results in other directions also. This we have done in Chapter VI where we prove a theorem on the almost everywhere convergence of Fourier series. According to A. Beurling[1] a function f is said to be a contraction of a function g if

$$|f(x) - f(y)| \leq |g(x) - g(y)|,$$

for all x, y in $[0, 2\pi]$. Employing the notion of contraction, Beurling proved, in the year 1949, the following theorem on the absolute convergence of Forier series:

(1.29) If f and g are continuous even functions of period 2π, with Fourier cosine coefficients c_n and g_n, if f is a contraction of g, and if $|g_n| \leq \gamma_n$, where $\gamma_n \downarrow 0$ and $\sum \gamma_n < \infty$, then $\sum |c_n| < \infty$.

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1) Beurling [4]

Recently R.P.Boas[1] has given an elementary proof of the following generalization of this theorem:

(1.30) <u>Theorem (1.29) remains true when the hypothesis that $\gamma_n \downarrow 0$ is replaced by</u>

$$\sum_{n=1}^{\infty} n^{-3/2} \left\{ \sum_{k=1}^{n} k^2 \gamma_k^2 \right\}^{1/2} + \sum_{n=1}^{\infty} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{1/2} < \infty .$$

We have used the method of proof of the theorem (1.26) to prove a similar theorem on the almost everywhere convergence of Fourier series. Thus we prove in Chapter VI the following theorem:[2]

(1.31) <u>If f and g are even functions of class L_2, each of period 2π, with Fourier cosine coefficients c_n and g_n, if f is a contraction of g, and if $|g_n| \leq \gamma_n$, where</u>

$$\sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2} \left\{ \sum_{k=1}^{\nu} k^2 \gamma_k^2 \right\}^{1/2} + \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu} \left\{ \sum_{k=\nu+1}^{\infty} \gamma_k^2 \right\}^{1/2} < \infty ,$$

<u>then the Fourier series of f converges almost everywhere.</u>

Part of the research work incorporated in this thesis has been published by the author in the form of two research papers in mathematical journals. Reprints of these papers constitute an appendix to the thesis.

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1) Boas [5]   2) Yadav [31]