CHAPTER II

CERTAIN THEOREMS ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES.

1. It is known that if f is of bounded variation, belongs to $Lip\,\alpha(0, 2\pi)$, $\alpha > 0$, and has Fourier coefficients a_n, b_n, then

$$\sum_{n=1}^{\infty}\left(|a_n|^{\beta} + |b_n|^{\beta}\right) \tag{1}$$

is convergent for $\beta > 2/(\alpha+2)$; but not necessarily for $\beta = 2/(\alpha+2)$.[1)]

It is of interest to ask as to what condition, satisfied by f, will make (1) convergent for $\beta = 2/(\alpha+2)$. In this connection, we prove the following

THEOREM 1. <u>If</u> f <u>is of bounded variation and satisfies the condition:</u>

$$\omega(\delta) = \omega(\delta; f) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta)\, l_2(\delta) \cdots l_k^{1+\varepsilon}(\delta)\right]^{\alpha+2}}, \tag{2}$$

<u>where</u> $\varepsilon > 0$, $\alpha \geq 0$ <u>and</u>

$$l_1(\delta) = \log(e + \delta^{-1})$$
$$l_2(\delta) = \log\log(e^e + \delta^{-1}), \text{ etc.},$$

<u>then</u> (1) <u>is convergent for</u> $\beta = 2/(\alpha+2)$.

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1) Zygmund [39], Waraszkiewicz [29].

Similar theorems have been proved by L. Neder[1] and A.C.Zaanen[2] in connection with the classical theorems of S.Bernstein[3] and its generalization due to O. Szàsz.[4]

If we take $\alpha = 0$ and $k=1$, then this theorem reduces to the more general form of Zygmund's theorem (1.5).[5]

<u>PROOF</u>: We shall prove the theorem for $k=2$. First we see that

$$f(x+h) - f(x-h) \sim 2\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \sin nh .$$

Therefore

$$(3) \qquad \frac{1}{\pi}\int_0^{2\pi} \{f(x+h) - f(x-h)\}^2 dx = 4\sum_{n=1}^{\infty} \rho_n^2 \sin nh ,$$

where $\rho_n^2 = a_n^2 + b_n^2$.

Now

$$\frac{1}{\pi}\int_0^{2\pi} \{f(x+h) - f(x-h)\}^2 dx \leq \frac{1}{\pi}\,\omega(2h)\int_0^{2\pi} |f(x+h) - f(x-h)|\,dx ,$$

Putting $h = \frac{\pi}{2N}$, where N is a positive integer, we have

$$(4) \qquad \begin{aligned} &\frac{1}{\pi}\int_0^{2\pi} \left\{f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right)\right\}^2 dx \\ &\leq \frac{1}{\pi}\,\omega\left(\frac{\pi}{N}\right)\int_0^{2\pi} \left|f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right)\right| dx ; \end{aligned}$$

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1) Neder [20]
2) Zaanen [36]
3) Bernstein [1], [2], [3]
4) Szàsz [23]
5) Zygmund [37]

and

$$\int_0^{2\pi} \left| f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right)\right| dx$$

$$= \int_0^{2\pi} \left| f\left(x+\frac{\pi}{N}\right) - f(x)\right| dx$$

$$\leq \sum_{\nu=0}^{2N-1} \int_{\nu\pi/N}^{(\nu+1)\pi/N} \left| f\left(x+\frac{\pi}{N}\right) - f(x)\right| dx$$

$$\leq \int_0^{\pi/N} \sum_{\nu=0}^{2N-1} \left| f\left(x+\frac{\nu+1}{N}\pi\right) - f\left(x+\frac{\nu}{N}\pi\right)\right| dx .$$

Since

$$\sum_{\nu=0}^{2N-1} \left| f\left(x+\frac{\nu+1}{N}\pi\right) - f\left(x+\frac{\nu}{N}\pi\right)\right| \leq V,$$

where V is the total variation of f in $(0,2\pi)$, we have

$$(5) \qquad \int_0^{2\pi} \left| f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right)\right| dx \leq \frac{\pi V}{N} .$$

Therefore it follows from (3), (4) and (5) that

$$\sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{n\pi}{2N} \leq \omega\left(\frac{\pi}{N}\right)\frac{V}{N} .$$

This implies that

$$\sum_{n=1}^{N} \rho_n^2 \sin^2 \frac{n\pi}{2N} \leq \omega\left(\frac{\pi}{N}\right)\frac{V}{N} .$$

Putting $N=2^\nu$, where ν is an integer greater or equal to ν_0, where $\nu_0 + (\log 2)^{-2}+3$ and taking into account only the terms with indices n exceeding $\frac{1}{2}N$, we get from the last inequality

$$(6) \quad \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \sin^2 \frac{n\pi}{2^{\nu+1}} \leq \omega\left(\frac{\pi}{2^\nu}\right) V. 2^{-\nu}.$$

Since $\sin \frac{n\pi}{2^{\nu+1}} > \frac{1}{\sqrt{2}}$ for $2^{\nu-1}+1 \leq n \leq 2^\nu$, we obtain

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \leq \omega\left(\frac{\pi}{2^\nu}\right). V. 2^{-\nu}.$$

Hence by (2)

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \leq \frac{V. 2^{-\nu+1} c\,(\pi 2^{-\nu})^\alpha}{\left[l_1\left(\frac{\pi}{2^\nu}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^\nu}\right)\right]^{\alpha+2}}$$

$$\leq \frac{c. 2^{-\nu(1+\alpha)}}{\left[l_1\left(\frac{\pi}{2^\nu}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^\nu}\right)\right]^{\alpha+2}}.$$

Now

$$\begin{aligned} l_1\left(\frac{\pi}{2^\nu}\right) &= \log\left(e + \frac{2^\nu}{\pi}\right) \\ &> \log 2^{\nu-2} \\ &= (\nu-2)\log 2; \\ l_2\left(\frac{\pi}{2^\nu}\right) &= \log\log\left(e^e + \frac{2^\nu}{\pi}\right) \\ &> \log\log 2^{\nu-2} \\ &> \frac{1}{2}\log(\nu-2). \end{aligned}$$

Therefore

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \leq \frac{c \,.\, 2^{-\nu(1+\alpha)}}{\left[(\nu-2)\log^{1+\varepsilon}(\nu-2)\right]^{\alpha+2}} \,.$$

Applying Hölder's inequality, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^\beta \leq \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2\Big)^{\beta/2} \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\Big)^{1-\beta/2}$$

$$= \frac{c \,.\, 2^{-\nu(1+\alpha)\beta/2}}{\left[(\nu-2)\log^{1+\varepsilon}(\nu-2)\right]^{(\alpha+2)\beta/2}} \cdot 2^{\nu(1-\beta/2)}$$

$$= \frac{c}{(\nu-2)\log^{1+\varepsilon}(\nu-1)} \,,$$

for $\beta = 2/(\alpha+2)$.

Hence

$$\sum \rho_n^\beta = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^\beta$$

$$\leq \sum_{\nu=\nu_0}^{\infty} \frac{c}{(\nu-2)\log^{1+\varepsilon}(\nu-2)}$$

$$< \infty \,;$$

since $|a_n|^\beta$, as also $|b_n|^\beta$, does not exceed ρ_n^β, it follows that

$$\sum_{n=1}^{\infty} \left(|a_n|^\beta + |b_n|^\beta \right) < \infty .$$

This completes the proof of the theorem.

2. G.H. Hardy[1] has proved the following

THEOREM-A: If $f \in \mathrm{lip}\, \alpha$, $0 < \alpha \leq 1$, then

$$\sum_{n=1}^{\infty} n^{\beta - 1/2} \left(|a_n| + |b_n| \right) < \infty , \tag{7}$$

for $\beta < \alpha$. If f is, in addition, of bounded variation, then

$$\sum_{n=1}^{\infty} n^{\beta/2} \left(|a_n| + |b_n| \right) < \infty ; \tag{8}$$

for $\beta < \alpha$. For $\beta = \alpha$, (7) and (8) need not hold.

A.C.Zaanen[2] has made the following addition to Theorem-A:

THEOREM-B. If for certain $\varepsilon > 0$

$$\omega(\delta) \leq \frac{c\,\delta^\alpha}{l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)} ,$$

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1) Hardy [10]

2) Zaanen, loc. cit.

then (7) holds for $\beta = \alpha$.

In this connection, we shall prove the following

THEOREM 2. If f is of bounded variation and satisfies the condition:

$$(9) \qquad \omega(\delta) \leq \frac{c\,\delta^{\alpha}}{[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)]},$$

then

$$\sum_{n=1}^{\infty} n^{\alpha/2}\left(|a_n| + |b_n|\right) < \infty.$$

PROOF: We shall prove this theorem also for $k = 2$. From (6), we have

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^{2} \leq c\,\omega\left(\frac{\pi}{2^{\nu}}\right) . 2^{-\nu};$$

and hence under the condition (9)

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^{2} \leq \frac{c \,.\, 2^{-\nu} \,.\, 2^{-\nu\alpha}}{\left[l_1\left(\frac{\pi}{2^{\nu}}\right) l_2^{1+\varepsilon}\left(\frac{\pi}{2^{\nu}}\right)\right]^2}$$

$$\leq \frac{c \,.\, 2^{-\nu(1+\alpha)}}{\left[(\nu-2) \log^{1+\varepsilon}(\nu-2)\right]^2}.$$

Therefore, by Schwarz's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leq \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \right)^{1/2} \cdot 2^{\nu/2}$$

$$\leq \frac{c \cdot 2^{-\nu\alpha/2}}{(\nu-2) \log^{1+\varepsilon}(\nu-2)} ;$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha/2} \rho_n \leq \frac{c}{(\nu-2) \log^{1+\varepsilon}(\nu-2)} .$$

Now

$$\sum n^{\alpha/2} \rho_n = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha/2} \rho_n$$

$$\leq \sum_{\nu=\nu_0}^{\infty} \frac{c}{(\nu-2) \log^{1+\varepsilon}(\nu-2)}$$

$$< \infty .$$

This proves that

$$\sum_{n=1}^{\infty} n^{\alpha/2} \left(|a_n| + |b_n| \right) < \infty .$$

3. It can be easily seen that the hypotheses of Bernstein's theorem (1.1) and its generalization (1.3) by Szàsz are unnecessarily stringent. As a matter of fact,

they remain true and the proofs unchanged, if the condition $f \in Lip\,\alpha$ is replaced by the weaker condition $f \in Lip(\alpha, 2)$. O.Szàsz[1] has actually proved the following generalization of (1.3):

THEOREM-C. If $f \in Lip(\alpha, p)$, where $0 < \alpha \leq 1$, $1 < p \leq 2$, then (1) is convergent for all $\beta > p/\{p(\alpha+1)-1\}$; but not necessarily for $\beta = p/\{p(\alpha+1)-1\}$.

Similarly it is known that, if $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$, $1 < p \leq 2$, then

$$(10) \qquad \sum_{n=1}^{\infty} n^{\beta} (|a_n| + |b_n|) < \infty,$$

for all $\beta < \alpha - \frac{1}{p}$.[2]

We shall make the following extensions to these theorems:

THEOREM 3. Let $0 < \alpha \leq 1$, $1 < p \leq 2$, $\varepsilon > 0$ and $h > 0$. If

$$(11) \qquad \left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx \right)^{1/p} \leq \frac{c\, h^{\alpha}}{[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)]^{\gamma}},$$

where $\gamma = \{p(\alpha+1)-1\}/p$, then (1) converges for $\beta = p/\{p(\alpha+1)-1\}$.

THEOREM 4. If, with the same α, p, ε and h as in Theorem 3,

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1) Szàsz [24] 2) Zygmund [40]

$$(12) \quad \left(\int_0^{2\pi} |f(x+h)-f(x)|^p dx \right)^{1/p} \le \frac{c\, h^{\alpha}}{l_1(h)\, l_2(h) \cdots l_k^{1+\epsilon}(h)},$$

<u>then</u> (10) <u>holds for</u> $\beta = \alpha - \frac{1}{p}$.

We also observe that our Theorem 3 includes, as a special case, the theorem (1.19) due to Min-Teh Cheng.

<u>PROOF OF THEOREM 3:</u> We shall prove the theorem for $k=2$. Since

$$f(x+h)-f(x-h) \sim 2 \sum_{n=1}^{\infty} (-a_n \sin nx + b_n \cos nx) \sin nh,$$

by Hausdorff-Young inequality, we have

$$\left(\sum_{n=1}^{\infty} |2 \rho_n \sin nh|^{p'} \right)^{1/p'} \le \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x+h)-f(x-h)|^p dx \right)^{1/p},$$

where $\rho_n^{p'} = |a_n|^{p'} + |b_n|^{p'}$ and p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Putting $h = \frac{\pi}{2N}$, where N is a positive integer, we obtain by the condition (11),

$$\left(\sum_{n=1}^{N} |\rho_n \sin \tfrac{n\pi}{2N}|^{p'} \right)^{1/p'} \le \left(\sum_{n=1}^{\infty} |\rho_n \sin \tfrac{n\pi}{2N}|^{p'} \right)^{1/p'}$$

$$\le \frac{1}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x+\tfrac{\pi}{2N}) - f(x-\tfrac{\pi}{2N})|^p dx \right)^{1/p}$$

$$\le \frac{c\, N^{-\alpha}}{[l_1(\frac{\pi}{N})\, l_2^{1+\epsilon}(\frac{\pi}{N})]^{\gamma}}.$$

Taking $N = 2^{\nu}$, where ν is an integer greater or equal to $\nu_0 \geq (\log 2)^{-2} + 3$, and taking into account only the terms with indices n exceeding $\frac{1}{2}N$, we get from the last inequality

$$\left(\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \left| \rho_n \sin \frac{n\pi}{2^{\nu+1}} \right|^{p'} \right)^{1/p'} \leq \frac{c \cdot 2^{-\nu\alpha}}{\left[l_1\left(\frac{\pi}{2^{\nu}}\right) l_2^{1+\epsilon}\left(\frac{\pi}{2^{\nu}}\right) \right]^{\gamma}} .$$

From this, it follows, as in the proof of Theorem 1, that

$$(13) \qquad \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^{p'} \leq \frac{c \cdot 2^{-\nu\alpha p'}}{\left[(\nu-2) \log^{1+\epsilon}(\nu-2) \right]^{\gamma p'}} .$$

Now, by Hölder's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^{\beta} \leq \left(\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^{p'} \right)^{\beta/p'} \left(\sum_{n=2^{\nu-1}+1}^{2^{\nu}} 1 \right)^{1-\beta/p'}$$

$$\leq \frac{c \cdot 2^{-\nu\alpha\beta}}{\left[(\nu-2) \log^{1+\epsilon}(\nu-2) \right]^{\gamma\beta}} \cdot 2^{\nu(1-\beta/p')}$$

$$= \frac{c \cdot 2^{-\nu\alpha p/\{p(\alpha+1)-1\}}}{(\nu-2) \log^{1+\epsilon}(\nu-2)} \cdot 2^{\nu\alpha p/\{p(\alpha+1)-1\}}$$

$$\left[\text{Putting } \beta = \frac{1}{\gamma} = p/\{p(1+\alpha)-1\} \right]$$

$$= \frac{c}{(\nu-2) \log^{1+\epsilon}(\nu-2)} .$$

Therefore

$$\sum \rho_n^{\beta} = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta}$$
$$\leq \sum_{\nu=\nu_0}^{\infty} \frac{c}{(\nu-2)\log^{1+\epsilon}(\nu-2)}$$
$$< \infty .$$

This completes the proof of the theorem.

PROOF OF THEOREM 4: Since the condition (12) is nothing but the condition (11) with $\gamma = 1$, we can, by putting in the inequality (13) (or directly as above), obtain

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta'} \leq \frac{c \cdot 2^{-\nu\alpha\beta'}}{[(\nu-2)\log^{1+\epsilon}(\nu-2)]^{\beta'}} .$$

Therefore, by Hölder's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leq \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta'}\Big)^{1/\beta'} \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\Big)^{1-1/\beta'}$$
$$\leq \frac{c \cdot 2^{-\nu\alpha}}{(\nu-2)\log^{1+\epsilon}(\nu-2)} \cdot 2^{(\nu-1)(1-1/\beta')} ;$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\beta}\rho_n \leq 2^{\nu\beta} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n$$

$$\leq \frac{c \cdot 2^{-\nu(\alpha-\beta)}}{(\nu-2)\log^{1+\epsilon}(\nu-2)} \cdot 2^{\nu/p}$$

$$= \frac{c}{(\nu-2)\log^{1+\epsilon}(\nu-2)},$$

for $\beta = \alpha - 1/p$.

Therefore

$$\sum n^{\beta} \rho_n = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} n^{\beta} \rho_n$$

$$\leq \sum_{\nu=\nu_0}^{\infty} \frac{c}{(\nu-2)\log^{1+\epsilon}(\nu-2)}$$

$$< \infty.$$

This proves the theorem.

4. In the rest of this Chapter we shall establish two theorems which are related to two theorems of E.C.Titchmarsh[1]. Taking clue from the theorem of S.Bernstein on the absolute convergence of Fourier series and its generalization by O.Szàsz, Titchmarsh[2] proved the following theorem on Fourier transforms:

THEOREM-D. Let $f \in L_p(-\infty,\infty)$, $1 < p \leq 2$, and let

1) Titchmarsh [27]

2) Titchmarsh [26]

$$\left(\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^p dx \right)^{1/p} = O\left(|h|^{\alpha} \right),\ 0 < \alpha \leq 1,$$

as $h \to 0$. Then the Fourier Transform g of f belongs to L_β, for

$$p/(p+\alpha p-1) < \beta \leq p/(p-1).$$

The range of β cannot be extended in this theorem. We shall, however, prove the following

THEOREM 5. Let $0<\alpha\leq 1$, $1<p\leq 2$, $\varepsilon>0$ and $h>0$. If

$$\left(\int_{-\infty}^{\infty} |f(x+h) - f(x-h)|^p dx \right)^{1/p} = O\left(\frac{h^{\alpha}}{[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)]^{\lambda}} \right), \tag{14}$$

then $g \in L_\beta$ for $\beta = p/(p+\alpha p-1)$.

This theorem corresponds to our Theorem 3 on the absolute convergence of Fourier series.

If we take $\alpha<1$ and $p=2$, Theorem-D can be put in a completely satisfactory form:[1]

THEOREM-E. If $f \in L_2(-\infty,\infty)$ and $0<\alpha<1$, then a necessary and sufficient condition for

$$\left(\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty} \right) \{g(x)\}^2 dx = O(\eta^{-2\alpha}),\ \text{as } \eta \to \infty,$$

1) Titchmarsh [27]

is that

$$\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^2 dx = O\left(|h|^{2\alpha}\right), \text{ as } h \to 0.$$

We shall also prove here the following analogue for Fourier series:

THEOREM 6. *Let* $f \in L_2(0, 2\pi)$ *and be periodic outside with period* 2π, *and let*

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx}.$$

A necessary and sufficient condition that

$$(15) \quad \sum_{-\infty}^{-N} |c_n|^2 + \sum_{N}^{\infty} |c_n|^2 = O\left(N^{-2\alpha}\right), \text{ as } N \to \infty,$$

is that

$$(16) \quad \int_0^{2\pi} |f(x+h)-f(x-h)|^2 dx = O\left(|h|^{2\alpha}\right), \text{ as } h \to 0;$$

$0 < \alpha < 1.$

PROOF OF THEOREM 5: We shall prove the theorem for $k=2$. We first observe that the Fourier transform of $f(x+h)-f(x-h)$, for a fixed h, is $-2i \sin xh \cdot g(x)$. Therefore

$$\left(\int_{-\infty}^{\infty} |2\sin xh \cdot g(x)|^{p'} dx\right)^{1/p'} \leq K\left\{\int_{-\infty}^{\infty} |f(x+h)-f(x-h)|^{p} dx\right\}^{1/p}$$

where p' is given by $\frac{1}{p}+\frac{1}{p'}=1$, and K is a constant depending on p .[1])

It follows from the condition (14) that

$$\int_{-\infty}^{\infty} |2 \sin xh . g(x)|^{p'} dx = O\left(h^{\alpha p'}(l_1(h) . l_2^{1+\varepsilon}(h))^{-\lambda p'}\right).$$

Since $|\sin xh| > A x h$, for $x < \frac{1}{h}$, we get

$$\int_{-\infty}^{\infty} |2 \sin xh . g(x)|^{p'} dx > A \int_0^{1/h} x^{p'} h^{p'} |g(x)|^{p'} dx ;$$

and hence

$$\int_0^{1/h} x^{p'} |g(x)|^{p'} dx = O\left(h^{(\alpha-1)p'}(l_1(h) l_2^{1+\varepsilon}(h))^{-\lambda p'}\right).$$

Now put

$$\phi(\xi) = \int_2^{\xi} |x g(x)|^{\beta} dx ,$$

then, since $\beta \leq \frac{p}{p-1} = p'$, we obtain by Hölder's inequality

$$\phi(\xi) \leq \left(\int_2^{\xi} |x g(x)|^{p'} dx\right)^{\beta/p'} \left(\int_2^{\xi} dx\right)^{1-\beta/p'}$$

$$= O\left(\xi^{-(\alpha-1)\beta+1-\beta/p'} (\log \xi)^{-\lambda\beta} (\log \log \xi)^{-(1+\varepsilon)\lambda\beta}\right)$$

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1) Titchmarsh, loc. cit.

$$= O\left(\xi^{1-\alpha\beta+\beta/p}\,(\log\xi)^{-1}(\log\log)^{-(1+\varepsilon)}\right).$$

Hence

$$\int_2^{\xi}|g(x)|^{\beta}dx = \int_2^{\xi} x^{-\beta}\phi'(x)\,dx$$

$$= \xi^{-\beta}\phi(\xi) + \beta\int_2^{\xi} x^{-\beta-1}\phi(x)\,dx$$

$$= O\left(\xi^{1-\beta-\alpha\beta+\beta/p}\,(\log\xi)^{-1}(\log\log\xi)^{-(1+\varepsilon)}\right) +$$

$$+\left(\int_2^{\xi} x^{-\beta-\alpha\beta+\beta/p}(\log x)^{-1}(\log\log x)^{-(1+\varepsilon)}dx\right)$$

$$= O\left(\int_2^{\xi}\frac{dx}{x\log x(\log\log x)^{1+\varepsilon}}\right.$$

$$= O(1), \quad \text{as } \xi \to \infty,$$

for $\beta = p/(p+\alpha p-1)$.

Similarly we can show that

$$\int_{-\xi}^{-2}|g(x)|^{\beta}dx = O(1), \quad \text{as } \xi \to \infty.$$

This completes the proof of the theorem.

<u>PROOF OF THEOREM 6:</u> The sufficiency of the condition (16) is easy to prove and we supply a proof only for the sake of completeness. Since

$$f(x+h) - f(x-h) \sim \sum_{-\infty}^{\infty} 2i \sin nh\, e^{inx},$$

it follows from Parseval's theorem that

$$(17) \quad \sum_{-\infty}^{\infty} 4 \sin^2 nh \, |c_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x-h)|^2 dx.$$

Now taking $h = \frac{\pi}{4N}$, we get from condition (16)

$$\begin{aligned}
\sum_{N}^{2N} |c_n|^2 &\leq 2 \sum_{N}^{2N} \sin^2 \frac{n\pi}{4N} |c_n|^2 \\
&\leq \frac{1}{2} \sum_{-\infty}^{\infty} 4 \sin^2 \frac{n\pi}{4N} |c_n|^2 \\
&= \frac{1}{4\pi} \int_0^{2\pi} \left| f\left(x+\frac{\pi}{4N}\right) - f\left(x-\frac{\pi}{4N}\right) \right|^2 dx \\
&= O(N^{-2\alpha}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{N}^{\infty} |c_n|^2 &= \sum_{N}^{2N} + \sum_{2N}^{4N} + \cdots \\
&= O\{ N^{-2\alpha} + (2N)^{-2\alpha} + \cdots \} \\
&= O(N^{-2\alpha});
\end{aligned}$$

and (2) follows from

$$\sum_{-\infty}^{-N} |c_n|^2 = \sum_{N}^{\infty} |c_n|^2 = O(N^{-2\alpha}).$$

To prove that the condition is necessary, we put

$$\phi(n) = \sum_{k=n}^{\infty} |c_n|^2.$$

Then

$$\sum_{k=0}^{n} |c_k|^2 = \sum_{k=0}^{n} \{\phi(k) - \phi(k+1)\}$$

$$= \phi(0) - \phi(n+1).$$

Also, from (5)

$$\phi(n) = O(n^{-2\alpha})$$

Therefore, by Abel's transformation, we obtain

$$\sum_{0}^{N} n^2 |c_n|^2 = \sum_{0}^{N} n^2 \{\phi(n) - \phi(n+1)\}$$

$$= \sum_{0}^{N-1} \{n^2 - (n+1)^2\}\{\phi(0) - \phi(n+1)\}$$

$$+ N^2\{\phi(0) - \phi(N+1)\}$$

$$= -\phi(0)\sum_{0}^{N-1}(2n+1) + \sum_{0}^{N-1}(2n+1)\phi(n+1)$$

$$+ N^2\phi(0) - N^2\phi(N+1)$$

$$\leq O\left(\sum_{0}^{N-1} (2n+1)\, \phi(n+1)\right)$$
$$= O\left(\sum_{0}^{N-1} n^{1-2\alpha}\right)$$
$$= O\left(N^{2-2\alpha}\right);$$

and hence from (7)

$$\int_0^{2\pi} |f(x+h)-f(x-h)|^2 dx = 8\pi \sum_{-\infty}^{\infty} \sin^2 nh\, |c_n|^2$$
$$= 8\pi \left(\sum_{-[1/h]}^{[1/h]} + \sum_{-\infty}^{-[1/h]-1} + \sum_{[1/h]+1}^{\infty}\right)$$
$$= O\left(h^2 \sum_{-[1/h]}^{[1/h]} n^2 |c_n|^2\right) +$$
$$+ \left(\sum_{-\infty}^{-[1/h]-1} |c_n|^2 + \sum_{[1/h]+1}^{\infty} |c_n|^2\right)$$
$$= O\left(|h|^{2\alpha}\right).$$

This proves Theorem 6 completely.
