CHAPTER IV

ABSOLUTE CONVERGENCE OF FOURIER SERIES OF THE CONVOLUTION OF TWO FUNCTIONS.

Let the functions f, g and h be each L-integrable in $(0, 2\pi)$ and periodic outside with period 2π. We recall that h is called the Fourier convolution (faltung or composition) of f and g if

$$(1) \qquad h(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)g(t)dt.$$ [1]

Moreover, a function h is said to be a Young's continuous function if there exist two functions f and g, each of the Lebesgue class L_2, such that h is the Fourier convolution of these functions[2]. Although there is no obvious criterion to verify directly whether a given function is a Young's continuous function, it is known that a function is a Young's continuous function if and only if its Fourier series is absolutely convergent.[3] Min-Teh Cheng[4] has obtained Young's continuous functions as convolution of two functions one of which satisfies a stronger and the other a weaker condition than that of being a function of L_2. In fact, he has proved the following

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1) Zygmund [40], Vol. I, p. 36 2) Young [35]
3) Hardy and Littlewood [12], Chen [7]
4) Cheng [8]

THEOREM-A. *If* $f \in Lip(\alpha, p)$ *and* $g \in Lip(\alpha', q)$, *where* $1 < p \leq 2$, $q > 1$ *and* $\alpha' = \frac{1}{2p} < \alpha$, *then the function* h *given by* (1) *is a Young's continuous function.*

In this chapter we shall establish two theorems regarding the construction of Young's continuous functions in a way similar to that of Min-Teh Cheng. The first one is the following

THEOREM 11. *If* $f \in Lip(\alpha, p)$ *and* $g \in Lip(\alpha', q)$, *where* $1 < p \leq 2$, $\alpha + \alpha' > \frac{1}{p}$ *and* q *is given by* $p^{-1} + q^{-1} = 1$, *then the function* h *given by* (1) *is a Young's continuous function.*

The case $p = 2$ in this theorem is trivial.

A word of explanation is in order here to show the connection between our theorem and that of Min-Teh Cheng. In Theorem-A, for a given p, α' is constant and $\alpha > \frac{1}{2p}$. In particular, the least value of α' is $\frac{1}{4}$ and the value of α is always greater than $\frac{1}{4}$. In Theorem 11, each of α and α' may assume any value between 0 and 1, provided that they remain connected by the inequality: $\alpha + \alpha' > \frac{1}{p}$ (which is required of them in Theorem-A as well) and q is the conjugate index of p.

In order to prove our theorem, we shall need the following

LEMMA 1. Let $f \in Lip(\alpha, p)$, $1 < p \leq 2$, $0 < \alpha \leq 1$ and q be given by $p^{-1} + q^{-1} = 1$. If

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

then

$$\sum_{-\infty}^{\infty} |c_n|^{\beta} |n|^{T} < \infty, \tag{2}$$

for all β such that $\beta\{p(\alpha+1)-1\} > p(T+1)$; but not necessarily when $\beta\{p(\alpha+1)-1\} = p(T+1)$.

PROOF: Since

$$f(x+h) - f(x-h) \sim \sum_{-\infty}^{\infty} 2i\, c_n e^{inx} \sin nh,$$

by applying Hausdorff-Young inequality, we get

$$\left(\sum_{-\infty}^{\infty} |2 c_n \sin nh|^{q}\right)^{1/q} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x+h) - f(x-h)|^{p} dx\right)^{1/p},$$

for $f \in L_p$, $1 < p \leq 2$.[1)]

Now, taking $h = \frac{\pi}{2N}$, where N is a positive integer,

$$\sum_{1}^{N} \left|c_n \sin \frac{n\pi}{2N}\right|^{q} \leq \sum_{1}^{\infty} \left|c_n \sin \frac{n\pi}{2N}\right|^{q}$$

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1) Hardy and Littlewood [13]

$$\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right) \right|^p dx \right)^{q/p}$$

$$= O\left(N^{-q/p} \right).$$

Putting $N = 2^\nu$, and considering only the terms with indices exceeding $\frac{1}{2}N$, we obtain

$$\sum_{2^{\nu-1}+1}^{2^\nu} \left| c_n \sin \frac{n\pi}{2^{\nu+1}} \right|^q = O\left(2^{-\nu\alpha q} \right).$$

Since $\sin \frac{n\pi}{2^{\nu+1}} > \frac{1}{\sqrt{2}}$, for $2^{\nu-1} < n \leq 2^{\nu+1}$, it follows that

$$\sum_{2^{\nu-1}+1}^{2^\nu} |c_n|^q = O\left(2^{-\nu\alpha q} \right).$$

Therefore, by Hölder's inequality,

$$\sum_{2^{\nu-1}+1}^{2^\nu} |c_n|^\beta \leq \left(\sum_{2^{\nu-1}+1}^{2^\nu} |c_n|^q \right)^{\beta/q} \left(\sum_{2^{\nu-1}+1}^{2^\nu} 1 \right)^{1-\beta/q}$$

$$= O\left(2^{-\nu\alpha\beta} \cdot 2^{\nu(1-\beta/q)} \right);$$

and hence

$$\sum_{2^{\nu-1}+1}^{2^\nu} |c_n|^\beta |n|^T \leq 2^{\nu T} \cdot \sum_{2^{\nu-1}+1}^{2^\nu} |c_n|^\beta$$

$$= O\left(2^{\nu(T-\alpha\beta+1-\beta/q)} \right).$$

Thus

$$\sum_{-\infty}^{\infty} |c_n|^{\beta} |n|^{T} = 2\sum_{1}^{\infty} |c_n|^{\beta} n^{T}$$

$$= 2|c_1|^{\beta} + 2\sum_{\nu=1}^{\infty} \sum_{2^{\nu-1}+1}^{2^{\nu}} |c_n|^{\beta} n^{T}$$

$$= O\left(\sum_{1}^{\infty} 2^{\nu(T-\alpha\beta+1-\beta/q)}\right)$$

$$= O(1),$$

since $\beta\{p(\alpha+1)-1\} > p(T+1)$.

This proves the first part of the lemma.

We observe that Lemma 1 is a generalization of the following lemma due to Cheng:

<u>Let</u> $f \in Lip(\alpha, p)$, $1 < p \leq 2$, $0 < \alpha \leq 1$ <u>and</u> q <u>be given by</u> $p^{-1} + q^{-1} = 1$, <u>then</u>

$$\sum_{-\infty}^{\infty} |c_n n^{\gamma}|^{q} < \infty,$$

<u>for</u> $\gamma < \alpha$; <u>but not necessarily for</u> $\gamma = \alpha$.

We shall also need the following lemma which is obtained by Hardy and Littlewood.[1]

1) Hardy and Littlewood, loc. cit.

LEMMA 2. If $f \varepsilon Lip(\alpha, p)$, $p \geq 1$ and $0 < \alpha \leq 1$, then $c_n = o(|n|^{-\alpha})$.

PROOF OF THEOREM 11: In the first place we see, that $f \varepsilon L_p$ and $g \varepsilon L_q$, and hence the existence of the integral

$$\int_0^{2\pi} f(x+t)\, g(t)\, dt$$

follows by Hölder's inequality. Secondly, it will suffice to prove that the Fourier series of h is absolutely convergent.

Let

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad g \sim \sum_{-\infty}^{\infty} d_n e^{inx}.$$

Then we know that

$$h \sim \sum_{-\infty}^{\infty} c_n d_n e^{inx}.$$

Now

$$(3) \quad \sum_{-N}^{N} |c_n d_n| \leq \Big(\sum_{-N}^{N} |c_n n^{\delta/p}|^p\Big)^{1/p} \Big(\sum_{-N}^{N} |d_n n^{-\delta/q}|^q\Big)^{1/q},$$

where δ is chosen such that $p(\alpha+1-\delta) > 2$ and $(\delta+\alpha')q > 1$. This choice of δ is possible because of the hypothesis $\alpha+\alpha' > \frac{1}{p}$.* The first sum on the right

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* To see this, we note that δ must simultaneously satisfy the conditions: (i) $\delta < 1+\alpha-2/p$ and (ii) $\delta > 1/q - \alpha'$. That is, δ must be such that $1/q - \alpha' < \delta < 1+\alpha-2/p$. Therefore such a will exist if $1/q - \alpha' < 1+\alpha-2/p$, i.e. if $\alpha+\alpha' > 1/p$.

of (3)remains bounded, as $N \to \infty$, because of Lemma 1 under the condition $p(\alpha+1-\delta)>2$; while the second, because of Lemma 2 under the condition $(\delta+\alpha)q>1$. This completes the proof of the theorem.

AN APPLICATION OF THEOREM 7.

We can also obtain Young's continuous functions with the help of ~~the~~ Theorem 7 which can be stated in an equivalent form as follows:

Let $0<\alpha\leq 1$, $1<p\leq 2$ *and* $t>0$. *If*

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

and

$$\int_0^{2\pi} |f(x+t)-f(x)|^p dx = O\{t^{\delta}(\log t^{-1})^{-1-\alpha p}\},$$

where $\delta = 1+p(1-\beta)/\beta$, *then*

$$\sum_{-\infty}^{\infty}{}' |c_n|^{\beta}(\log|n|)^{T} < \infty,$$

for all $\beta > p(T+1)/(1+\alpha p)$; *but not necessarily for* $\beta = p(T+1)/(1+\alpha p)$. ($\sum'$ denotes a summation for n in which the term corresponding to $n=0$ is omitted.)

In fact, we shall need the following lemma which can be deduced as a corollary of this theorem by taking $\beta = p$.

LEMMA 3. <u>Let</u> $0 < \alpha \leq 1$, $1 < p \leq 2$ <u>and</u> $t > 0$. <u>If</u>

$$(4) \qquad \int_0^{2\pi} |f(x+t) - f(x)|^p dx = O\left\{t^{2-p} (\log t^{-1})^{-1-\alpha p}\right\},$$

<u>then</u>

$$\sum_{-\infty}^{\infty}{}' \, |C_n (\log |n|)^T|^p < \infty,$$

<u>for</u> $T < \alpha$; <u>but not necessarily for</u> $T = \alpha$.

We shall prove the following

THEOREM 12. <u>Let</u> $0 < \alpha \leq 1$, $1 < p \leq 2$ <u>and</u> $t > 0$. <u>If</u> f <u>satisfies the condition</u> (4) <u>and</u> $g \in Lip(\frac{1}{q}, q)$, <u>where</u> $p(1-\alpha) < 1$ <u>and</u> q <u>is given by</u> $p^{-1} + q^{-1} = 1$, <u>then the function</u> h <u>given by</u> (1) <u>is a Young's continuous function.</u>

<u>PROOF:</u> We first prove that the integral

$$\int_0^{2\pi} f(x+t) g(t) \, dt$$

does exist under the hypotheses of the theorem. It follows from the condition (4) that $f \in Lip(\frac{2-p}{p}, p)$; and hence that $f \in L_p$. Also $g \in L_q$. Therefore the integral in question will exist by Hölder's inequality. Again, in order to prove the theorem we shall prove that the Fourier series of h is absolutely convergent.

If

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad g \sim \sum_{-\infty}^{\infty} d_n e^{inx},$$

then

$$h \sim \sum_{-\infty}^{\infty} c_n d_n e^{inx}.$$

Now

$$(5) \quad \sum_{-N}^{N}{}' |c_n d_n| \leq \Big\{\sum_{-N}^{N}{}' |c_n (\log|n|)^{\tau}|^{p}\Big\}^{1/p} \Big\{\sum_{-N}^{N}{}' |d_n (\log|n|)^{-\tau}|^{q}\Big\}^{1/q},$$

where τ is chosen such that $\tau < \alpha$ and $\tau q > 1$. This choice of τ is possible because of the hypothesis $p(1-\alpha) < 1$.* The first sum on the right of (5) remains bounded, as $N \to \infty$, by Lemma 3 under the condition $\tau < \alpha$, and the second, by Lemma 2 under the condition $\tau q > 1$. This completes the proof of the theorem.

It remains to prove the second part of Lemma 1. For this, consider the function:

$$f(x) = |x|^{-a}, \quad \text{where} \quad (a+1)p > 1, \; 1 < p \leq 2.$$

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* For, τ should be such that $\frac{1}{q} < \tau < \alpha$. Therefore such a τ will exist if $\frac{1}{q} < \alpha$, or which means the same as, if $p(1-\alpha) < 1$.

It is not difficult to see that $f \in Lip(\alpha, p)$, for $\alpha = \frac{1}{p} - a$. Also, its Fourier coefficients c_n are of exact order $|n|^{a-1}$. Therefore the series

$$\sum_{-\infty}^{\infty} |c_n|^{\beta} |n|^{\tau} \leq A \sum_{-\infty}^{\infty} |n|^{\beta(a-1)+\tau}$$

$$= A \sum_{-\infty}^{\infty} |n|^{p(\tau+1)(a-1)/\{p(\alpha+1)-1\}+\tau}$$

$$= A \sum_{-\infty}^{\infty} |n|^{-1}$$

diverges.
