CHAPTER V

ABSOLUTE CONVERGENCE OF FOURIER SERIES OF ABSOLUTELY CONTINUOUS FUNCTIONS

1. It is known that the Fourier series of an absolutely continuous function is not necessarily absolutely convergent. For example, the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}$$

is the Fourier series of an absolutely continuous function, yet it does not converge absolutely. This necessitates to think in terms of some extra conditions which an absolutely continuous function should satisfy in order that its Fourier series converges absolutely. Various results are obtained in this direction by L. Tonelli[1], G.H. Hardy and J.E. Littlewood[2] and A. Zygmund.[3] Here we intend to prove the following

THEOREM 13. If F be an absolutely continuous and 2π-periodic function with the derivative $F'(x) = f(x)$ such that

(1) $$\omega(\delta) \leq c\,\delta^{\alpha}, \quad \alpha > -\tfrac{1}{2},$$

where ω is the modulus of continuity of f, then the Fourier series of F converges absolutely.

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1) Tonelli [28]
2) Hardy and Littlewood [14]
3) Zygmund [38]

PROOF: If A_n, B_n denote the Fourier coefficients of F, and a_n, b_n denote those of f, then

(2) $$A_n = -b_n/n \quad , \quad B_n = a_n/n ;$$

as also

$$f(x+h) - f(x-h) \sim 2 \sum_{\nu=1}^{\infty} (-a_\nu \sin \nu x + b_\nu \cos \nu x) \sin \nu h .$$

Put

$$T_n(x) \equiv \sum_{k=1}^{n} \lambda_k (\beta_k \cos kx - \alpha_k \sin kx),$$

where $\lambda_k \geq 0$, $\alpha_k = \operatorname{Sign} a_k$ and $\beta_k = \operatorname{Sign} b_k$.
Since $T_n(x)$ is a trigonometric polynomial, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \{ f(x+h) - f(x-h) \} T_n(x)\, dx$$

$$= \frac{1}{2\pi} \left[2 \sum_{\nu=1}^{\infty} \left\{ -a_\nu \int_0^{2\pi} T_n(x) \sin \nu x\, dx + b_\nu \int_0^{2\pi} T_n(x) \cos \nu x\, dx \right\} \sin \nu h \right]$$

$$= \sum_{\nu=1}^{n} \lambda_\nu (|a_\nu| + |b_\nu|) \sin \nu h$$

$$\geq \sum_{\nu=1}^{n} \lambda_\nu \frac{2\nu h}{\pi} (|a_\nu| + |b_\nu|) , \quad 0 \leq \nu h \leq \frac{\pi}{2} .$$

Now putting $h = \frac{\pi}{2n}$, we obtain

$$
*\left\{
\begin{aligned}
&\frac{1}{n}\sum_{\nu=1}^{n} \nu\lambda_\nu \left(|a_\nu| + |b_\nu|\right) \\
&\leq \sum_{\nu=1}^{n} \lambda_\nu \left(|a_n| + |b_\nu|\right) \sin\left(\frac{\nu\pi}{2n}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\{ f\left(x+\frac{\pi}{2n}\right) - f\left(x-\frac{\pi}{2n}\right)\right\} T_n(x)\,dx
\end{aligned}
\right.
$$

$$
\leq \frac{1}{2\pi}\,\omega\left(\frac{\pi}{n}\right)\int_0^{2\pi} |T_n(x)|\,dx
$$

$$
\leq \frac{1}{2\pi}\,\omega\left(\frac{\pi}{n}\right)\left\{\int_0^{2\pi} T_n^2(x)\,dx\right\}^{1/2}\left\{\int_0^{2\pi} dx\right\}^{1/2} .
$$

(by Cauchy-Schwarz inequality)

$$
\leq \omega\left(\frac{\pi}{n}\right)\left\{\sum_{k=1}^{n} \lambda_k^2\left(\alpha_k^2 + \beta_k^2\right)\right\}^{1/2}
$$

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* This inequality was used by Bojanić [6] to give new proofs of theorems (1.1) and (1.5) due to Bernstein and Zygmund respectively.

If $\lambda_k = 1$; $\alpha_k^2 = \beta_k^2 = 1$, then

$$\mu_n \equiv \frac{1}{n} \sum_{\nu=1}^{n} \nu\left(|a_\nu| + |b_\nu|\right) \leq \omega\left(\frac{\pi}{n}\right)\sqrt{2n} \, ;$$

and hence from (2)

$$(3) \quad \mu_n = \frac{1}{n} \sum_{\nu=1}^{n} \nu^2\left(|A_\nu| + |B_\nu|\right) \leq \omega\left(\frac{\pi}{n}\right)\sqrt{2n} \, .$$

This gives

$$\begin{aligned}
|A_1| + |B_1| &= \mu_1 \\
|A_2| + |B_2| &= \frac{2}{2^2}\mu_2 - \frac{1}{2^2}\mu_1 \\
&\cdots\cdots\cdots \\
|A_n| + |B_n| &= \frac{n}{n^2}\mu_n - \frac{n-1}{n^2}\mu_{n-1} \, ;
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{\nu=1}^{n}\left(|A_\nu| + |B_\nu|\right) &= \frac{1}{n}\mu_n + \sum_{\nu=1}^{n-1}\left\{\frac{1}{\nu} - \frac{\nu}{(\nu+1)^2}\right\}\mu_\nu \\
&\leq \frac{1}{n}\,\omega\left(\frac{\pi}{n}\right)\sqrt{2n} + \sum_{\nu=1}^{n-1}\frac{2\nu+1}{\nu(\nu+1)^2}\,\omega\left(\frac{\pi}{\nu}\right)\sqrt{2\nu} \\
&= \sqrt{2}\left[\frac{1}{n^{1/2}}\,\omega\left(\frac{\pi}{n}\right) + \sum_{\nu=1}^{n-1}\frac{2\nu+1}{\nu^{1/2}(\nu+1)^2}\,\omega\left(\frac{\pi}{\nu}\right)\right] \\
&= O\left(\frac{1}{n^{\alpha+1/2}} + \sum_{\nu=1}^{n-1}\frac{1}{\nu^{3/2+\alpha}}\right),
\end{aligned}$$

Therefore, taking limit, as $N \to \infty$,

$$\sum_{\nu=1}^{\infty} (|A_\nu| + |B_\nu|) < \infty \quad \text{, for } \alpha > -\tfrac{1}{2}.$$

The proof is complete.

We shall also prove the following routine generalization of Theorem 13:

THEOREM 14. <u>If</u> $0 < \beta \leq 1$ <u>and</u> f <u>satisfies the condition</u> (1) <u>then</u>

$$(4) \qquad \sum_{\nu=1}^{\infty} (|A_\nu|^\beta + |B_\nu|^\beta) < \infty,$$

<u>for</u> $\beta > 2/(2\alpha+3)$.

<u>PROOF:</u> Taking $B_\nu = 0$ in (3), we have

$$\sum_{\nu=1}^{n} \nu^2 |A_\nu| \leq n\, \omega\left(\frac{\pi}{n}\right)\sqrt{2\pi}.$$

But by Hölder's inequality

$$\sum_{\nu=1}^{n} \nu^2 |A_\nu|^\beta \leq \left(\sum_{\nu=1}^{n} \nu^2\right)^{1-\beta} \left(\sum_{\nu=1}^{n} \nu^2 |A_\nu|\right)^\beta$$

$$\leq c \cdot n^{3(1-\beta)} \cdot n^{3\beta/2} \cdot \omega^\beta\left(\frac{\pi}{n}\right)$$

$$= c \cdot n^{3-3\beta/2} \cdot \omega^\beta\left(\frac{\pi}{n}\right);$$

and similarly

$$\sum_{\nu=1}^{n} \nu^2 |B_\nu|^\beta \leq c \, . \, n^{3-3\beta/2} \omega^\beta\left(\frac{\pi}{n}\right).$$

Therefore

$$\frac{1}{n} \sum_{\nu=1}^{n} \nu^2 \left(|A_\nu|^\beta + |B_\nu|^\beta\right) \leq c \, . \, n^{2-3\beta/2} \omega^\beta\left(\frac{\pi}{n}\right).$$

Now putting

$$\mu_n \equiv \frac{1}{n} \sum_{\nu=1}^{n} \nu^2 \left(|A_\nu|^\beta + |B_\nu|^\beta\right),$$

we get

$$\sum_{\nu=1}^{n} \left(|A_\nu|^\beta + |B_\nu|^\beta\right)$$

$$= \frac{\mu_n}{n} + \sum_{\nu=1}^{n-1} \left\{ \frac{1}{\nu} - \frac{\nu}{(1+\nu)^2} \right\} \mu_\nu$$

$$\leq c \, . \, n^{1-3\beta/2} \omega^\beta\left(\frac{\pi}{n}\right) + \sum_{\nu=1}^{n-1} \frac{c.(2\nu+1)\nu^{2-3\beta/2}}{\nu(\nu+1)^2} \omega^\beta\left(\frac{\pi}{\nu}\right)$$

$$= O\left[\frac{1}{n^{\alpha\beta+3\beta/2-1}} + \sum_{\nu=1}^{n-1} \frac{1}{\nu^{3\beta/2+\alpha\beta}} \right]$$

Therefore

$$\sum_{\nu=1}^{\infty} (|A_\nu|^\beta + |B_\nu|^\beta) < \infty \text{ , for } \beta > 2/(2\alpha+3).$$

This completes the proof.

2. In this section we intend to establish two more theorems regarding the absolute convergence of the Fourier series of functions of bounded variation which satisfy certain additional conditions. Our theorems are in a sequel to certain theorems of R. Mohanty[1] and S. Izumi.[2] Our first theorem is

THEOREM 15. Let

$$f \sim \sum_{n=1}^{\infty} b_n \sin nx .$$

If f is such that

(i) f is of bounded variation in $(0, \pi)$

(ii) $\int_0^\delta |d\phi(x,h)| = O(|h|^\alpha), \quad \alpha > 0$, where

$\phi(x,h) = f(x+h) - f(x-h)$ and $\delta > 0$; and

(iii) the sequence $\{n^\gamma \Delta(n b_n)\}$ is of bounded variation for some $\gamma > 0$, then

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1) Mohanty [19] 2) Izumi [15]

$$\sum_{n=1}^{\infty} |b_n| < \infty .$$

We have also the cosine series analogue:

THEOREM 16. Let

$$g \sim \sum_{n=1}^{\infty} a_n \cos nx .$$

If

(i) g is of bounded variation in $(0, \pi)$

(ii) $\int_0^{\delta} |d\psi(x,h)| = O(|h|^{\alpha})$, $\alpha > 0$, where

$\psi(x,h) = g(x+h) - g(x-h)$ and $\delta > 0$; and

(iii) the sequence $\{n^{\gamma} \Delta (n a_n)\}$ is of bounded variation for some $\gamma > 0$, then

$$\sum_{n=1}^{\infty} |a_n| < \infty .$$

We shall only prove Theorem 15; the proof of Theorem 16 will follow similarly.

PROOF OF THEOREM 15: We first observe that

$$\phi(x,h) \sim 2 \sum_{n=1}^{\infty} (b_n \sin nh) \cos nx .$$

Therefore

$$2 b_n \sin nh$$

$$= \frac{1}{\pi}\int_0^{2\pi} \phi(x,h) \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\phi(x,h)\,\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} \sin nx \, d\phi(x,h)$$

$$= -\frac{2}{\pi}\int_0^{\pi} \sin nx \, d\phi(x,h);$$

and hence

$$-\pi b_n \sin nh = \frac{1}{n}\int_0^{\pi} \sin nx \, d\phi(x,h)$$

$$= \frac{1}{n}\int_0^{\pi/n^{\gamma'}} \sin nx \, d\phi(x,h) +$$

$$+ \frac{1}{n}\int_{\pi/n^{\gamma'}}^{\pi} \sin nx \, d\phi(x,h),$$

where $\gamma' = \gamma'_n$ is chosen such that $n^{1-\gamma'}$ is an integer and

$$\gamma' = \frac{\gamma}{4} + O\left(\frac{1}{n^{1-\gamma/4}\log n}\right).^{*}$$

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* A γ' satisfying these conditions can be chosen; see Izumi, loc. cit.

Therefore

$$\pi \sum_{n=1}^{\infty} |b_n \sin nh|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \sin nx \, d\phi(x,h) \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi/n^{\gamma'}} \sin nx \, d\phi(x,h) \right| +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{\pi/n^{\gamma'}}^{\pi} \sin nx \, d\phi(x,h) \right|$$

Putting $h = \frac{\pi}{2n}$, we get

$$\pi \sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi/n^{\gamma'}} \sin nx \, d\phi\left(x, \frac{\pi}{2n}\right) \right| +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{\pi/n^{\gamma'}}^{\pi} \sin nx \, d\phi\left(x, \frac{\pi}{2n}\right) \right|$$

$$= I + J, \text{ say.}$$

Now, since

$$d\phi(x,h) \sim -2 \sum_{k=1}^{\infty} (k b_k \sin kh) \sin kx,$$

we have

$$\int_{\pi/n^{\gamma'}}^{\pi} \sin nx\, d\phi\left(x, \frac{\pi}{2n}\right) = -2\sum_{k=1}^{\infty} \int_{\pi/n^{\gamma'}}^{\pi} \left(k\, l_k \sin\frac{k\pi}{2n}\right) \times$$

$$\times \sin kx \sin nx\, dx.$$

Hence

$$\left|\int_{\pi/n^{\gamma'}}^{\pi} \sin nx\, d\phi\left(x, \frac{\pi}{2n}\right)\right| \leq 2\left|\sum_{k=1}^{\infty} k\, l_k \int_{\pi/n^{\gamma'}}^{\pi} \sin kx \sin nx\, dx\right|$$

From this it follows, under the hypothesis (iii), that $J < \infty$.[1]

Again, choosing $N = N(\gamma, \delta)$ such that $\frac{\pi}{n^{\gamma'}} < \delta$, for $n \geq N$, we have

$$I = \sum_{n=1}^{N} \frac{1}{n}\left|\int_0^{\pi/n^{\gamma'}} \sin nx\, d\phi\left(x, \frac{\pi}{2n}\right)\right| +$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{n}\left|\int_0^{\pi/n^{\gamma'}} \sin nx\, d\phi\left(x, \frac{\pi}{2n}\right)\right|.$$

Since

$$\sum_{n=N+1}^{\infty} \frac{1}{n}\left|\int_0^{\pi/n^{\gamma'}} \sin nx\, d\phi\left(x, \frac{\pi}{2n}\right)\right|$$

$$\leq \sum_{n=N+1}^{\infty} \frac{1}{n}\int_0^{\pi/n^{\gamma'}} \left|d\phi\left(x, \frac{\pi}{2n}\right)\right|$$

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1) Izumi, loc. cit.

$$\leq A \sum_{n=N+1}^{\infty} \frac{1}{n^{1+\alpha}} < \infty ,$$

it follows that $I < \infty$.

This completes the proof of the theorem.

<u>REMARK:</u> It is easy to see that the condition (ii) can be replaced by the weaker condition

$$\int_0^{\delta} |d\phi(x,h)| = O\left(\frac{1}{\log(1/h)}\right)^{\eta}, \quad \eta > 1 .$$