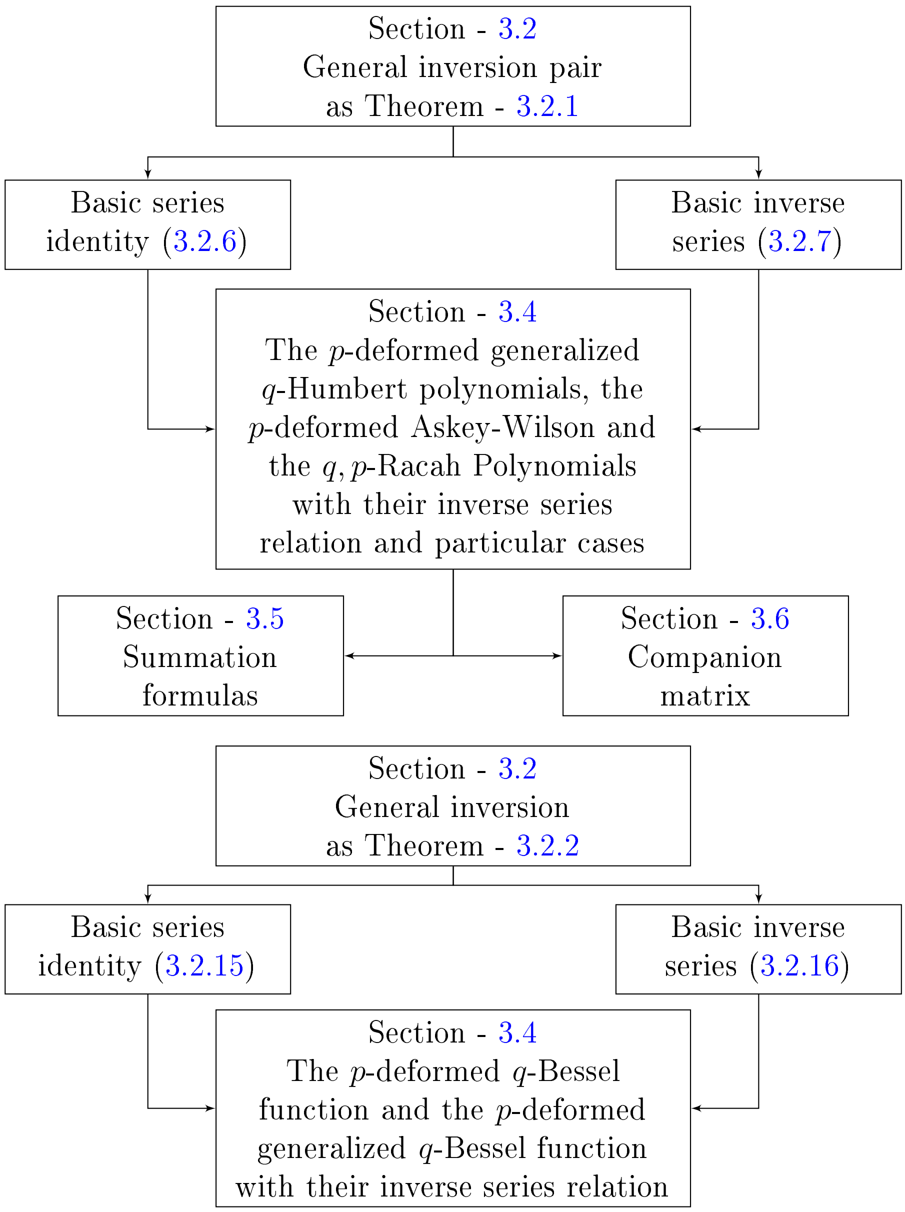
Chapter 3

The p-deformed q-polynomials' system - I



3.1 Introduction

The aim of this chapter to define q-analogue of the p-deformed generalized Humbert polynomials (2.1.14) of Chapter 2 and thereby its particular cases like the p-Humbert polynomials, the p-Kinney polynomial, the p-Pincherle polynomial, the p-Gegenbauer polynomial and the p-Legendre polynomial along with their inverse series relation. A q-analogue of generalized Humbert polynomials and its inverse series relation are given by [9, 11]

$$P_n(m,x,\gamma,s,c|q) = \sum_{k=0}^{\lfloor n/m \rfloor} \gamma^k \frac{(q^{s-n+mk-k+1};q)_\infty (1-q^c)^{s-n+mk-k}(1-q)^{-s+k}}{(q^{s-n+mk+1};q)_\infty (q;q)_{n-mk}(q^{m-1};q^{m-1})_k}((q^m-1)x)^{n-mk} \tag{3.1.1}$$

$\Leftrightarrow$

$$\frac{(q^m-1)^n}{(1-q^c)^n(q;q)_n}x^n = \sum_{k=0}^{\lfloor n/m \rfloor}(-\gamma)^k\left(\frac{1-q}{1-q^c}\right)^{s+k}\frac{q^{(m-1)k(k-1)/2}(1-q^{s-n+mk})}{(q^{s-n+k};q)_\infty} \times\frac{(q^{s-n+1};q)_\infty}{(q^{m-1};q^{m-1})_k}P_{n-mk}(m,x,\gamma,s,c|q). \tag{3.1.2}$$

The specialization of this polynomial are as follow:
The q-Humbert polynomials[11, Eq.(24), p.13] and its inverse series relation[11, Eq.(25), p.14] with $s=-\nu$, $c=1$ and $\gamma=1$, are given by

$$\Pi^{\nu}_{n,m}(x|q) = \sum_{k=0}^{\lfloor n/m \rfloor}\frac{(q^{-\nu-n+mk-k+1};q)_\infty}{(q^{-\nu-n+mk+1};q)_\infty(q;q)_{n-mk}(q^{m-1};q^{m-1})_k} \times\left(\frac{(q^m-1)x}{1-q}\right)^{n-mk} \tag{3.1.3}$$

$\Leftrightarrow$

$$\frac{(q^m-1)^n}{(1-q)^n(q;q)_n}x^n = \sum_{k=0}^{\lfloor n/m \rfloor}(-1)^k q^{(m-1)k(k-1)/2}\frac{(1-q^{-\nu-n+mk})(q^{-\nu-n+1};q)_\infty}{(q^{-\nu-n+k};q)_\infty(q^{m-1};q^{m-1})_k} \times\Pi^{\nu}_{n-mk,m}(x|q). \tag{3.1.4}$$

The q-Kinney polynomial and its inverse series are obtained from (3.1.3) and (3.1.4) by substituting $\nu=-1/m$ as stated below.

$$P_n(m,x|q) = \sum_{k=0}^{\lfloor n/m \rfloor}\frac{(q^{-\frac{1}{m}-n+mk-k+1};q)_\infty}{(q^{-\frac{1}{m}-n+mk+1};q)_\infty(q;q)_{n-mk}(q^{m-1};q^{m-1})_k} \times\left(\frac{(q^m-1)x}{1-q}\right)^{n-mk}$$

$\Leftrightarrow$

$$\frac{(q^m-1)^n}{(1-q)^n(q;q)_n}x^n = \sum_{k=0}^{\lfloor n/m \rfloor}(-1)^k q^{(m-1)k(k-1)/2}\frac{(1-q^{-\frac{1}{m}-n+mk})(q^{-\frac{1}{m}-n+1};q)_\infty}{(q^{-\frac{1}{m}-n+k};q)_\infty(q^{m-1};q^{m-1})_{k,p}} \times P_{n-mk}(m,x|q).$$

The q-Pincherle polynomial ($m = 3$ and $\nu = -1/2$ in (3.1.3) and (3.1.4)) with its inverse series are given by

$$\begin{aligned}\mathcal{P}_{n,q}(x) &= \sum_{k=0}^{\lfloor n/3\rfloor} \frac{(q^2+q+1)^{n-3k}(q^{\frac{1}{2}-n+2k};q)_\infty}{(q^{\frac{1}{2}-n+3k};q)_\infty(q;q)_{n-3k}(q^2;q^2)_k}(-x)^{n-3k} \\ &\Leftrightarrow \\ \frac{(q^2+q+1)^n}{(q;q)_n}(-x)^n &= \sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k q^{k(k-1)}\frac{(1-q^{-\frac{1}{2}-n+3k})(q^{-\frac{1}{2}-n+1};q)_\infty}{(q^{-\frac{1}{2}-n+k};q)_\infty(q^2;q^2)_k} \\ &\quad \times\mathcal{P}_{n-3k,q}(x).\end{aligned}$$

q-Gegenbauer polynomial and its inverse series ($m = 2$ in (3.1.3) and (3.1.4)):

$$C_n^\nu(x|q) = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{-\nu-n+k+1};q)_\infty(1+q)^{n-2k}}{(q^{-\nu-n+2k+1};q)_\infty(q;q)_{n-2k}(q;q)_k}(-x)^{n-2k} \tag{3.1.5}$$

$$\Leftrightarrow$$

$$\frac{(1+q)^n}{(q;q)_n}(-x)^n = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{k(k-1)/2}\frac{(1-q^{-\nu-n+2k})(q^{-\nu-n+1};q)_\infty}{(q^{-\nu-n+k};q)_\infty(q;q)_{k,p}} \times C_{n-2k}^\nu(x|q). \tag{3.1.6}$$

The q-Legendre polynomial with its inverse series ($m = 2$ and $\nu = 1/2$ in (3.1.5) and (3.1.6)):

$$\begin{aligned}P_n(x|q) &= \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{\frac{1}{2}-n+k};q)_\infty(1+q)^{n-2k}}{(q^{\frac{1}{2}-n+2k};q)_\infty(q;q)_{n-2k}(q;q)_k}(-x)^{n-2k} \\ &\Leftrightarrow \\ \frac{(1+q)^n}{(q;q)_n}(-x)^n &= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{k(k-1)/2}\frac{(1-q^{-\frac{1}{2}-n+2k})(q^{\frac{1}{2}-n};q)_\infty}{(q^{-\frac{1}{2}-n+k};q)_\infty(q;q)_{k,p}} \\ &\quad \times P_{n-2k}(x|q).\end{aligned}$$

We first define the p-deformed version of (3.1.1) as

Definition 3.1.1. *For $n \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$, r, s, $c \in \mathbb{C}, p > 0$, and $mr \neq p$*

$$P_{n,p,r}(m,x,\gamma,s,c|q) = \sum_{k=0}^{\lfloor n/m\rfloor}\gamma^k\frac{(q^{s-nr+mrk-kp+p};q)_{\infty,p}(1-q^c)^{s-n+mk-k}}{(q^{s-nr+mkr+p};q)_{\infty,p}(q;q)_{n-mk}} \times\frac{(1-q)^{k-s}}{(q^{mr-p};q^{mr-p})_k}\left((q^m-1)x\right)^{n-mk}. \tag{3.1.7}$$

We shall call this polynomials as the p-deformed generalized q-Humbert polynomials. It extends the p-version of the q-Humbert polynomials, the q-Kinney polynomial, q-Pincherle polynomial, q-Gegenbauer polynomial and q-Legendre polynomial together with their inverse series relation. We also note the inverse pairs of the Askey-Wilson polynomials and the q-Racah polynomials [11, 19]:

$$p_n(cos\theta;a,b,c,d|q) = \frac{(ab;q)_n(ac;q)_n(ad;q)_n}{a^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(abcdq^{n-1};q)_k}{(ab;q)_k(ac;q)_k(ad;q)_k(q;q)_k} \times(ae^{i\theta};q)_k(ae^{-i\theta};q)_k q^k \tag{3.1.8}$$

$$\Leftrightarrow$$

$$\frac{(ae^{i\theta};q)_n(ae^{-i\theta};q)_n}{(ab;q)_n(ac;q)_n(ad;q)_n} = \sum_{k=0}^{n} q^{nk}\frac{(q^{-n};q)_k(1-abcdq^{2k-1})}{(abcdq^{k-1};q)_{n+1}} \times\frac{a^k\ p_k(cos\theta;a,b,c,d|q)}{(ab;q)_k(ac;q)_k(ad;q)_k(q;q)_k} \tag{3.1.9}$$

and

$$R_n(q^{-x}+cdq^{x+1};a,b,c,d|q) = \sum_{k=0}^{n}\frac{(q^{-n};q)_k(abq^{n+1};q)_k(q^{-x};q)_k(cdq^{x+1};q)_k}{(aq;q)_k(bdq;q)_k(cq;q)_k(q;q)_k}q^k \tag{3.1.10}$$

$$\Leftrightarrow$$

$$\frac{(q^{-x};q)_n(cdq^{x+1};q)_n}{(aq;q)_n(bdq;q)_n(cq;q)_n} = \sum_{k=0}^{n} q^{nk}\frac{(q^{-n};q)_k(1-abq^{2k+1})}{(abq^{k+1};q)_{k+1}(q;q)_k} \times R_n(q^{-x}+cdq^{x+1};a,b,c,d|q), \tag{3.1.11}$$

the following pair involving the q-Bessel function $J_\nu^{(1)}(x;q)$ [11]:

$$J_n^{(1)}(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{(q^{n+1};q)_k(q;q)_k}\left(\frac{x}{2}\right)^{n+2k} \tag{3.1.12}$$

$$\Leftrightarrow$$

$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{(1-q^{n+2k})(q;q)_\infty}{(q^{n+k};q)_\infty(q;q)_k}J_{n+2k}(x;q). \tag{3.1.13}$$

We shall obtain the inverse series of the deformed versions of the aforementioned polynomials by establishing a general inversion pair in section - 3.2. Moreover, in section - 3.4, we shall derive the p-version of q-Bessel function (3.1.12) due to Jackson and the generalized q-Bessel function (3.4.14) due to Mansour Mahmoud[44] with their inverse series relation. Next, the summation formulas involving the particular q,p-polynomials are derived in section - 3.5 and lastly, the Companion matrix of monic polynomial derived from the p-deformed generalized q- Humbert polynomial (3.1.7) is given in section - 3.6.

3.2 Basic inverse series relations

While proving the general inversion pair, we shall use a result based on the q-Binomial theorem [19, Ex.1.2(vi), p.20]:

Lemma 3.2.1. *For* $|q| < 1$ *and* $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)/2} (q;q)_n}{(q;q)_{n-k}(q;q)_k} z^k = (z;q)_n. \tag{3.2.1}$$

Another result required is the following inverse pair.

Lemma 3.2.2. *There holds the following inverse series relation.*

$$f(n) = \sum_{k=0}^{n} (-1)^k q_1^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1} \frac{1}{(q^{\alpha-ar+np-k(br+p)};q)_{\infty,p}} g(k) \tag{3.2.2}$$

$$\Leftrightarrow$$

$$g(n) = \sum_{k=0}^{n} (-1)^k q_1^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1} (1 - q^{\alpha-ar-kbr})(q^{\alpha-ar+kp-n(br+p)+p};q)_{\infty,p} f(k), \tag{3.2.3}$$

where $q_1 = q^{-br-p}, br \neq -p$ *and* $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q_1} = \dfrac{(q_1;q_1)_n}{(q_1;q_1)_{n-k}(q_1;q_1)_k}$.

Proof. We observe that the diagonal elements of the coefficient matrix of first series are

$$(-1)^i q_1^{ip(1-i)/2}/(q^{\alpha-ar-ibr};q)_{\infty,p}$$

and the diagonal elements of the coefficient matrix of second series are

$$(-1)^i q_1^{ip(i-1)/2}(1 - q^{\alpha-ar-ibr})(q^{\alpha-ar-ibr+p};q)_{\infty,p}$$

which are all non zero, implying that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We shall show that (3.2.3) implies (3.2.2).

We denote the right hand side of (3.2.2) by $\Phi(n)$ and then substitute for $g(k)$ from (3.2.3) to get

$$\begin{aligned} \Phi(j) = & \sum_{k=0}^{n} (-1)^k q_1^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1} \frac{1}{(q^{\alpha-ar+np-k(br+p)};q)_{\infty,p}} \\ & \times \sum_{j=0}^{k} (-1)^j q_1^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_{q_1} (1 - q^{\alpha-ar-jbr})(q^{\alpha-ar+jp-k(br+p)+p};q)_{\infty,p} f(j). \end{aligned}$$

Here applying the double series relation:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} A(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{n-j} A(k+j,j),$$

we further have

$$\begin{aligned}\Phi(n) &= \sum_{j=0}^{n}\sum_{k=0}^{n-j}(-1)^k q_1^{(k+j)(k+j-2n+1)/2+j(j-1)/2}\begin{bmatrix} n \\ k+j\end{bmatrix}_{q_1}\begin{bmatrix} k+j \\ j\end{bmatrix}_{q_1} \\ &\times\frac{(1-q^{\alpha-ar-jbr})(q^{\alpha-ar+jp-(k+j)(br+p)+p};q)_{\infty,p}}{(q^{\alpha-ar+np-(k+j)(br+p)};q)_{\infty,p}}f(j) \\ &= f(n)+\sum_{j=0}^{n-1}\sum_{k=0}^{n-j}(-1)^k q_1^{(k+j)(k+j-2n+1)/2+j(j-1)/2}\begin{bmatrix} n \\ k+j\end{bmatrix}_{q_1}\begin{bmatrix} k+j \\ j\end{bmatrix}_{q_1} \\ &\times\frac{(1-q^{\alpha-ar-jbr})(q^{\alpha-ar+jp-(k+j)(br+p)+p};q)_{\infty,p}}{(q^{\alpha-ar+np-(k+j)(br+p)};q)_{\infty,p}}f(j) \\ &= f(n)+\sum_{j=0}^{n-1}\begin{bmatrix} n \\ j\end{bmatrix}_{q_1}(1-q^{\alpha-ar-jbr})f(j)\sum_{k=0}^{n-j}(-1)^k q_1^{(k+j)(k+j-2n+1)/2+j(j-1)/2} \\ &\times\begin{bmatrix} n-j \\ k\end{bmatrix}_{q_1}\frac{(q^{\alpha-ar+jp-(k+j)(br+p)+p};q)_{\infty,p}}{(q^{\alpha-ar+np-(k+j)(br+p)};q)_{\infty,p}}. \end{aligned} \tag{3.2.4}$$

Here, the ratio of two q,p-gamma functions represents a polynomial of degree $n-j-1$ in k, that is

$$\frac{(q^{\alpha-ar+jp+(k+j)(-br-p)+p};q)_{\infty,p}}{(q^{\alpha-ar+np+(k+j)(-br-p)};q)_{\infty,p}} = \sum_{l=0}^{n-j-1} C_l\, q_1^{kl},$$

say, hence from (3.2.4), we have

$$\begin{aligned}\Phi(n) &= f(n)+\sum_{j=0}^{n-1}\begin{bmatrix} n \\ j\end{bmatrix}_{q_1}(1-q^{\alpha-ar-jbr})f(j)\sum_{k=0}^{n-j}(-1)^k q_1^{(k+j)(k+j-2n+1)/2+j(j-1)/2} \\ &\times\begin{bmatrix} n-j \\ k\end{bmatrix}_{q_1}\sum_{l=0}^{n-j-1} C_l\, q_1^{kl} \\ &= f(n)+\sum_{j=0}^{n-1}\begin{bmatrix} n \\ j\end{bmatrix}_{q_1} q_1^{j(j-n)}(1-q^{\alpha-ar-jbr})f(j)\sum_{k=0}^{n-j}(-1)^k q_1^{k(k-2n+2j+1)/2} \\ &\times\begin{bmatrix} n-j \\ k\end{bmatrix}_{q_1}\sum_{l=0}^{n-j-1} C_l\, q_1^{kl} \\ &= f(n)+\sum_{j=0}^{n-1}\begin{bmatrix} n \\ j\end{bmatrix}_{q_1} q_1^{j(j-n)}(1-q^{\alpha-ar-jbr})f(j)\sum_{l=0}^{n-j-1} C_l \end{aligned}$$

$$\times \sum_{k=0}^{n-j} (-1)^k q_1^{k(k-2n+2j+1)/2} \begin{bmatrix} n-j \\ k \end{bmatrix}_{q_1} q_1^{kl}$$

$$= f(n) + \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q_1} q_1^{j(j-n)} (1 - q^{\alpha - ar - jbr}) f(j) \sum_{l=0}^{n-j-1} C_l$$

$$\times \sum_{k=0}^{n-j} (-1)^k q_1^{k(k-1)/2} \begin{bmatrix} n-j \\ k \end{bmatrix}_{q_1} q_1^{k(l-n+j+1)}.$$

On making use of Lemma - 3.2.1, we get

$$\begin{aligned}
\Phi(n) &= f(n) + \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q_1} q_1^{j(j-2n)/2} (1 - q^{\alpha - ar - jbr}) f(j) \sum_{l=0}^{n-j-1} C_l \\
&\quad \times (q_1^{(l-n+j+1)}; q_1)_{n-j} \\
&= f(n) + \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q_1} q_1^{j(j-2n)/2} (1 - q^{\alpha - ar - jbr}) f(j) \sum_{l=0}^{n-j-1} C_l \\
&\quad \times (1 - q_1^{(l-n+j+1)})(1 - q_1^{(l-n+j+2)}) \cdots (1 - q_1^{(l-n+j+1+n-j-1)}) \\
&= f(n) + \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q_1} q_1^{j(j-2n)/2} (1 - q^{\alpha - ar - jbr}) f(j) \sum_{l=0}^{n-j-1} C_l \\
&\quad \times (1 - q_1^{(l-n+j+1)})(1 - q_1^{(l-n+j+2)}) \cdots (1 - q_1^{l}) \\
&= f(n).
\end{aligned}$$

This completes the proof of (3.2.2)⇔(3.2.3). □

This lemma yields the series orthogonality relation when $f(r)$ or $g(r)$ is chosen to be $\begin{bmatrix} 0 \\ r \end{bmatrix}_{q_1}$. In fact, we have

Corollary 3.2.1. *In the notations of the above lemma, there holds the series orthogonal relation:*

$$\begin{bmatrix} 0 \\ n \end{bmatrix}_{q_1} = \sum_{k=0}^{n} (-1)^k q_1^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q_1} (1 - q^{\alpha - ar - kbr}) \frac{(q^{\alpha - ar + kp - n(br+p) + p}; q)_{\infty,p}}{(q^{\alpha - ar + kp}; q)_{\infty,p}}. \quad (3.2.5)$$

Proof. The proof follows by substituting $g(r) = \begin{bmatrix} 0 \\ r \end{bmatrix}_{q_1}$ in the lemma. □

There are two p-deformed general inverse pairs in this section namely, p-deformed general inverse pair-1 and p-deformed general inverse pair-2, in brief, GIP-1 and GIP-2 respectively. Each of GIP-1 and -2 have two versions; one involving the finite series and the other involving the infinite series. The GIP-1 for $br \neq -p$ is given by

$$\begin{aligned} F(a) &= \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q_1;q_1)_k} G(a+bk) \\ &\Leftrightarrow \\ G(a) &= \sum_{k=0}^{N} (-\gamma)^k q_1^{k(k-1)/2} \frac{(1-q^{\alpha-ar-brk})}{(q^{\alpha-ar+kp};q)_{\infty,p}(q_1;q_1)_k} F(a+bk). \end{aligned}$$

In this, we choose a to be a non negative integer n. If we choose b to be (i) negative integer $-m$, then we have the finite series version and if b is taken to be (ii) positive integer then we have the pair involving the infinite series. In case (i), we take $N = \lfloor n/m \rfloor$, whereas in (ii), $N = \infty$. The general inverse pair-1 or GIP-1 is a q-analogue of the inverse pair:

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} \frac{\gamma^k}{k!\Gamma_p(p+\alpha-ar-brk-kp)} v(a+bk) \\ &\Leftrightarrow \\ v(a) &= \sum_{k=0}^{N} \frac{(-\gamma)^k(\alpha-ar-brk)\Gamma_p(\alpha-ar+kp)}{k!} u(a+bk) \end{aligned}$$

proved in Chapter 2 with $N = \lfloor n/m \rfloor$ or $N = \infty$. The GIP-2 for $br = -p$ is

$$\begin{aligned} F(a) &= \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q;q)_k} G(a+bk) \\ &\Leftrightarrow \\ G(a) &= \sum_{k=0}^{N} (-\gamma)^k q^{k(k-1)/2} \frac{F(a+bk)}{(q^{p+\alpha-ar+kp};q)_{\infty,p}(q;q)_k}. \end{aligned}$$

Here also we have the cases as stated above. Thus, a is a non negative integer n, and b is either (i) negative integer $-m$ or (ii) a positive integer m. In the case (i) $N = \lfloor n/m \rfloor$., whereas in (ii) $N = \infty$.
We now prove the first main result as

Theorem 3.2.1. *If* $0 < q < 1,\ \alpha,\ r,\ \gamma \in \mathbb{C},\ a = n \in \mathbb{N}, q_2 = q^{mr-p}$, *$mr \neq p$ and $p > 0$, then*

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \gamma^k \frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q_2;q_2)_k} G(n-mk) \tag{3.2.6}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} F(n-mk), \tag{3.2.7}$$

where $q_2 = q^{mr-p}, mr \neq p$.

Proof. We first prove (3.2.6)⇒(3.2.7). For that, we denote the right hand side of (3.2.7) by $\Phi(n)$ and then substitute for $F(n-mk)$ from (3.2.6) with $\lfloor n/m \rfloor = N$ as follows.

$$\begin{aligned}
\Phi(n) &= \sum_{k=0}^{N} (-\gamma)^k \; q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} F(n-mk) \\
&= \sum_{k=0}^{N} (-\gamma)^k \; q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} \\
&\quad \times \sum_{j=0}^{N-k} \gamma^j \frac{(q^{p+\alpha-nr+mrk+mrj-jp};q)_{\infty,p}}{(q_2;q_2)_j} \; G(n-mk-mj) \\
&= \sum_{k=0}^{N} \sum_{j=0}^{N-k} (-1)^k \gamma^{j+k} \; q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} \\
&\quad \times \frac{(q^{p+\alpha-nr+mrk+mrj-jp};q)_{\infty,p}}{(q_2;q_2)_j} \; G(n-mk-mj).
\end{aligned}$$

We apply the double series relation:

$$\sum_{k=0}^{N} \sum_{j=0}^{N-k} A(k,j) \;=\; \sum_{j=0}^{N} \sum_{k=0}^{j} A(k,j-k) \tag{3.2.8}$$

to get

$$\begin{aligned}
\Phi(n) &= \sum_{j=0}^{N} \sum_{k=0}^{j} (-1)^k \gamma^{j-k+k} q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} \\
&\quad \times \frac{(q^{p+\alpha-nr+mrk+mrj-mrk-jp+kp};q)_{\infty,p}}{(q_2;q_2)_{j-k}} G(n-mk-mj+mk) \\
&= G(n) + \sum_{j=1}^{N} \sum_{k=0}^{j} (-1)^k \gamma^j q_2^{k(k-1)/2} \frac{(1-q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_2;q_2)_k} \\
&\quad \times \frac{(q^{p+\alpha-nr+mrj-jp+kp};q)_{\infty,p}}{(q_2;q_2)_{j-k}} G(n-mj) \\
&= G(n) + \sum_{j=1}^{N} \frac{\gamma^j}{(q_2;q_2)_j} G(n-mj) \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_2} \\
&\quad \times \frac{(1-q^{\alpha-nr+mrk})(q^{p+\alpha-nr+mrj-jp+kp};q)_{\infty,p}}{(q^{\alpha-nr+kp};q)_{\infty,p}}.
\end{aligned} \tag{3.2.9}$$

Next, we re-write the inverse series relation (3.2.2) and (3.2.3) by replacing n by j, $a = n$ and $b = -m$ as follows.

$$f(j) = \sum_{k=0}^{j} (-1)^k q_2^{k(k-2j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_2} \frac{1}{(q^{\alpha-nr-k(-mr+p)+jp};q)_{\infty,p}} g(k) \tag{3.2.10}$$

$$\Leftrightarrow$$

$$g(j) = \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_2} (1-q^{\alpha-nr+kmr})(q^{\alpha-nr-j(-mr+p)+kp+p};q)_{\infty,p} f(k). \tag{3.2.11}$$

Now the substitution $g(j) = \begin{bmatrix} 0 \\ j \end{bmatrix}_{q_2}$ in (3.2.10) and (3.2.11) yields $f(j) = \frac{1}{(q^{\alpha-nr+jp};q)_{\infty,p}}$ and the inverse pair

$$\frac{1}{(q^{\alpha-nr+jp};q)_{\infty,p}} = \sum_{k=0}^{j} (-1)^k q_2^{k(k-2j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_2} \frac{1}{(q^{\alpha-nr-k(-mr+p)+jp};q)_{\infty,p}} \begin{bmatrix} 0 \\ k \end{bmatrix}_{q_2} \tag{3.2.12}$$

$$\Leftrightarrow$$

$$\begin{bmatrix} 0 \\ j \end{bmatrix}_{q_2} = \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_2} \frac{(1-q^{\alpha-nr+kmr})}{(q^{\alpha-nr+kp};q)_{\infty,p}} (q^{\alpha-nr-j(-mr+p)+kp+p};q)_{\infty,p}. \tag{3.2.13}$$

On making use of (3.2.13) in (3.2.9), we obtain

$$\begin{aligned} \Phi(n) &= G(n) + \sum_{j=1}^{N} \frac{\gamma^j}{(q_2;q_2)_j} G(n-mj) \begin{bmatrix} 0 \\ j \end{bmatrix}_{q_2} \\ &= G(n) \end{aligned}$$

Thus, (3.2.6)⇒(3.2.7). For the converse part, we denote the right hand side of (3.2.6) by $\Theta(n)$ and then substitute (3.2.7) to get

$$\begin{aligned} \Theta(n) &= \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q_2;q_2)_k} G(n-mk) \\ &= \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q_2;q_2)_k} \\ &\quad \times \sum_{j=0}^{N-k} (-\gamma)^j q_2^{j(j-1)/2} \frac{(1-q^{\alpha-nr+mrk+mrj})}{(q^{\alpha-nr+mrk+jp};q)_{\infty,p}(q_2;q_2)_j} F(n-mk-mj). \end{aligned}$$

Here also applying the double series identity (3.2.8), we obtain

$$\begin{aligned}
\Theta(n) &= \sum_{j=0}^{N}\sum_{k=0}^{j}(\gamma)^{j-k+k}\frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q_2;q_2)_k}(-1)^{j-k}q_2^{(j-k)(j-k-1)/2} \\
&\quad\times\frac{(1-q^{\alpha-nr+mrk+mrj-mrk})}{(q^{\alpha-nr+mrk+jp-kp};q)_{\infty,p}(q_2;q_2)_{j-k}}F(n-mk-mj+mk) \\
&= \sum_{j=0}^{N}q_2^{j(j-1)/2}\frac{(-\gamma)^j(1-q^{\alpha-nr+mrj})}{(q_2;q_2)_j}F(n-mj) \\
&\quad\times\sum_{k=0}^{j}(-1)^k q_2^{k(k+1-2j)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_2}\frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q^{\alpha-nr+mrk+jp-kp};q)_{\infty,p}} \\
&= F(n)+\sum_{j=1}^{N}q_1^{j(j-1)/2}\frac{(-\gamma)^j(1-q^{\alpha-nr+mrj})}{(q_2;q_2)_j}F(n-mj) \\
&\quad\times\sum_{k=0}^{j}(-1)^k q_2^{k(k+1-2j)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_2}\frac{(q^{p+\alpha-nr+mrk-kp};q)_{\infty,p}}{(q^{\alpha-nr+mrk+jp-kp};q)_{\infty,p}}. \qquad (3.2.14)
\end{aligned}$$

But,

$$\begin{aligned}
\frac{(q^{p+\alpha-nr+k(mr-p)};q)_{\infty,p}}{(q^{\alpha-nr+k(mr-p)+jp};q)_{\infty,p}} &= \prod_{l=1}^{j-1}(1-q^{\alpha-nr+(mr-p)k+lp}) \\
&= \sum_{l=0}^{j-1}A_l\, q_2^{kl},
\end{aligned}$$

say, hence from (3.2.14), we further get

$$\begin{aligned}
\Theta(n) &= F(n)+\sum_{j=1}^{N}q_2^{j(j-1)/2}\frac{(-\gamma)^j(1-q^{\alpha-nr+mrj})}{(q_2;q_2)_j}F(n-mj) \\
&\quad\times\sum_{k=0}^{j}(-1)^k q_2^{k(k+1-2j)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_2}\sum_{l=0}^{j-1}A_l q_2^{kl} \\
&= F(n)+\sum_{j=1}^{N}q_2^{j(j-1)/2}\frac{(-\gamma)^j(1-q^{\alpha-nr+mrj})}{(q_2;q_2)_j}F(n-mj) \\
&\quad\times\sum_{l=0}^{j-1}A_l\sum_{k=0}^{j}(-1)^k q_2^{k(k+1-2j)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_2}q_2^{kl}.
\end{aligned}$$

In this last expression, applying Lemma - 3.2.1, we find

$$\Theta(n) = F(n)+\sum_{j=1}^{N}q_2^{j(j-1)/2}\frac{(-\gamma)^j(1-q^{\alpha-nr+mrj})}{(q_2;q_2)_j}F(n-mj)\sum_{l=0}^{j-1}A_l(q_2^{(l-j+1)};q)_j$$

$$
\begin{aligned}
&= F(n) + \sum_{j=1}^{N} q_2^{j(j-1)/2} \frac{(-\gamma)^j (1 - q^{\alpha - nr + mrj})}{(q_2; q_2)_j} F(n - mj) \\
&\quad \times \sum_{l=0}^{j-1} A_l (1 - q_2^{l-j+1})(1 - q_2^{l-j+2}) \cdots (1 - q_2^{l-j+1+j-1}) \\
&= F(n).
\end{aligned}
$$

Thus the converse part, and hence the theorem. □

Now corresponding to the case (ii), that is, when $N = \infty$, we prove the inverse pair as

Theorem 3.2.2. *If $p > 0,\ 0 < q < 1,\ \alpha,\ r \in \mathbb{C}$ and $a,\ b \in \mathbb{N}$, then*

$$
F(a) = \sum_{k=0}^{\infty} \gamma^k \frac{(q^{p+\alpha-ar-brk-kp}; q)_{\infty,p}}{(q_3; q_3)_k} G(a + bk) \tag{3.2.15}
$$

$$\Leftrightarrow$$

$$
G(a) = \sum_{k=0}^{\infty} (-\gamma)^k \, q_3^{k(k-1)/2} \frac{(1 - q^{\alpha - ar - brk})}{(q^{\alpha - ar + kp}; q)_{\infty,p} (q_3; q_3)_k} F(a + bk), \tag{3.2.16}
$$

where $q_3 = q^{-br-p},\ br \neq -p$.

Proof. First, we prove (3.2.15)⇒(3.2.16). We begin as in the above proof, with the notation

$$
\begin{aligned}
\Delta(a) &= \sum_{k=0}^{\infty} (-\gamma)^k q_3^{k(k-1)/2} \frac{(1 - q^{\alpha - ar - brk})}{(q^{\alpha - ar + kp}; q)_{\infty,p} (q_3; q_3)_k} F(a + bk) \\
&= \sum_{k=0}^{\infty} (-\gamma)^k q_3^{k(k-1)/2} \frac{(1 - q^{\alpha - ar - brk})}{(q^{\alpha - ar + kp}; q)_{\infty,p} (q_3; q_3)_k} \\
&\quad \times \sum_{j=0}^{\infty} \gamma^j \frac{(q^{p+\alpha-ar-brk-brj-jp}; q)_{\infty,p}}{(q_3; q_3)_j} G(a + bk + bj) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k (\gamma)^{k+j} q_3^{k(k-1)/2} \\
&\quad \times \frac{(1 - q^{\alpha - ar - brk})(q^{p+\alpha-ar-brk-brj-jp}; q)_{\infty,p}}{(q^{\alpha - ar + kp}; q)_{\infty,p} (q_3; q_3)_k (q_3; q_3)_j} G(a + bk + bj).
\end{aligned}
$$

The double series relation:

$$
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} A(k, j) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} A(k, j - k) \tag{3.2.17}
$$

gives us

$$\begin{aligned}\Delta(a) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^k\ \gamma^{k+j-k}\ q_3^{k(k-1)/2}\ (1-q^{\alpha-ar-brk})\\ &\quad\times\frac{(q^{p+\alpha-ar-brk-brj+brk-jp+kp};q)_{\infty,p}}{(q^{\alpha-ar+kp};q)_{\infty,p}(q_3;q_3)_k(q_3;q_3)_{j-k}}\ G(a+bk+bj-bk)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^k\ \gamma^{j}\ q_3^{k(k-1)/2}\ (1-q^{\alpha-ar-brk})\\ &\quad\times\frac{(q^{p+\alpha-ar-brj-jp+kp};q)_{\infty,p}}{(q^{\alpha-ar+kp};q)_{\infty,p}(q_3;q_3)_k(q_3;q_3)_{j-k}}\ G(a+bj)\\ &= \sum_{j=0}^{\infty}\gamma^j\sum_{k=0}^{j}(-1)^k q_3^{k(k-1)/2}\ (1-q^{\alpha-ar-brk})\\ &\quad\times\frac{(q^{p+\alpha-ar-brj-jp+kp};q)_{\infty,p}}{(q^{\alpha-ar+kp};q)_{\infty,p}(q_3;q_3)_k(q_3;q_3)_{j-k}}\ G(a+bj)\\ &= G(a)+\sum_{j=1}^{\infty}\frac{\gamma^j}{(q_3;q_3)_j}\ G(a+bj)\sum_{k=0}^{j}(-1)^k q_3^{k(k-1)/2}\begin{bmatrix} j\\ k\end{bmatrix}_{q_3}\ (1-q^{\alpha-ar-brk})\\ &\quad\times\frac{(q^{p+\alpha-ar-brj-jp+kp};q)_{\infty,p}}{(q^{\alpha-ar+kp};q)_{\infty,p}}.\end{aligned}$$

Now, using (3.2.2) and (3.2.3) with n is replaced by j, and setting $g(j)=\begin{bmatrix} 0\\ j\end{bmatrix}_{q_3}$, then in view of the Corollary - 3.2.5, we have $f(j)=\dfrac{1}{(q^{\alpha-ar+jp};q)_{\infty,p}}$. Thus,

$$\begin{aligned}\frac{1}{(q^{\alpha-ar+jp};q)_{\infty,p}} &= \sum_{k=0}^{j}(-1)^k q_3^{k(k-2j+1)/2}\begin{bmatrix} j\\ k\end{bmatrix}_{q_3}\frac{1}{(q^{\alpha-ar-k(br+p)+jp};q)_{\infty,p}}\begin{bmatrix} 0\\ k\end{bmatrix}_{q_3}\\ \Leftrightarrow\quad&\\ \begin{bmatrix} 0\\ j\end{bmatrix}_{q_3} &= \sum_{k=0}^{j}(-1)^k q_3^{k(k-1)/2}\begin{bmatrix} j\\ k\end{bmatrix}_{q_3}\frac{(1-q^{\alpha-ar-brk})(q^{p+\alpha-ar-brj-jp+kp};q)_{\infty,p}}{(q^{\alpha-ar+kp};q)_{\infty,p}}.\end{aligned}$$

This pair leads us to

$$\begin{aligned}\Delta(a) &= G(a)+\sum_{j=1}^{\infty}\frac{(\gamma)^j}{(q_3;q_3)_j}\begin{bmatrix} 0\\ j\end{bmatrix}_{q_3} G(a+bj)\\ &= G(a)\end{aligned}$$

thus, (3.2.15)⇒(3.2.16). For the converse part, we proceed by taking

$$\Upsilon(a) = \sum_{k=0}^{\infty}\gamma^k\frac{(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q_3;q_3)_k}G(a+bk)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \gamma^k \frac{(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q_3;q_3)_k} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(-\gamma)^j q_3^{pj(j-1)/2}(1-q^{\alpha-ar-brk-brj})}{(q^{\alpha-ar-brk+jp};q)_{\infty,p}(q_3;q_3)_j} F(a+bj+bk) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (-1)^j \gamma^{k+j} q_3^{j(j-1)/2} \\
&\quad \times \frac{(1-q^{\alpha-ar-brk-brj})(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q^{\alpha-ar-brk+jp};q)_{\infty,p}(q_3;q_3)_j(q_3;q_3)_k} F(a+bj+bk).
\end{aligned}
$$

From the double series relation (3.2.17), this changes to

$$
\begin{aligned}
\Upsilon(a) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-1)^{j-k}\gamma^{k+j-k} q_3^{(j-k)(j-k-1)/2} \\
&\quad \times \frac{(1-q^{\alpha-ar-brk-brj+brk})(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q^{\alpha-ar-brk+jp-kp};q)_{\infty,p}(q_3;q_3)_{j-k}(q_3;q_3)_k} F(a+bj-bk+bk) \\
&= F(a) + \sum_{j=1}^{\infty}\sum_{k=0}^{j} (-1)^{j-k}\gamma^j q_3^{j(j-1)/2} q_3^{k(k-2j+1)/2}\ (1-q^{\alpha-ar-brj}) \\
&\quad \times \frac{(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q^{\alpha-ar-brk+jp-kp};q)_{\infty,p}(q_3;q_3)_{j-k}(q_3;q_3)_k} F(a+bj) \\
&= F(a) + \sum_{j=1}^{\infty} (-1)^j\gamma^j q_3^{j(j-1)/2} \frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j} F(a+bj) \\
&\quad \times \sum_{k=0}^{j} \frac{(-1)^k q_3^{k(k-2j+1)/2}(q_3;q_3)_j(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q^{\alpha-ar-brk+jp-kp};q)_{\infty,p}(q_3;q_3)_{j-k}(q_3;q_3)_k} \\
&= F(a) + \sum_{j=1}^{\infty} (-1)^j\gamma^j q_3^{j(j-1)/2} \frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j} F(a+bj) \\
&\quad \times \sum_{k=0}^{j} (-1)^k q_3^{k(k-2j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{q_3} \frac{(q^{p+\alpha-ar-brk-kp};q)_{\infty,p}}{(q^{\alpha-ar-brk+jp-kp};q)_{\infty,p}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{(q^{p+\alpha-ar+k(-br-p)};q)_{\infty,p}}{(q^{\alpha-ar+k(-br-p)+jp};q)_{\infty,p}} &= \prod_{i=1}^{j-1}(1-q^{\alpha-ar+k(-br-p)+ip}) \\
&= \sum_{i=0}^{j-1} A_i\ q_3^{ki},
\end{aligned}
$$

say, we further have

$$\begin{aligned}
\Upsilon(a) &= F(a) + \sum_{j=1}^{\infty}(-1)^j\gamma^j q_3^{j(j-1)/2}\frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j}F(a+bj) \\
&\quad \times \sum_{k=0}^{j}(-1)^k q_3^{k(k-2j+1)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_3}\sum_{i=0}^{j-1}A_i q_3^{ki} \\
&= F(a) + \sum_{j=1}^{\infty}(-1)^j\gamma^j q_3^{j(j-1)/2}\frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j}F(a+bj) \\
&\quad \times \sum_{i=0}^{j-1}A_i\sum_{k=0}^{j}(-1)^k q_3^{k(k-2j+1)/2}\begin{bmatrix} j \\ k \end{bmatrix}_{q_3} q_3^{ki}.
\end{aligned}$$

Now, apply Lemma - 3.2.1 to inner series in this last expression, we find

$$\begin{aligned}
\Upsilon(a) &= F(a) + \sum_{j=1}^{\infty}(-1)^j\gamma^j q_3^{j(j-1)/2}\frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j}F(a+bj) \\
&\quad \times \sum_{i=0}^{j-1}A_i(q_3^{(i-j+1)};q_3)_j \\
&= F(a) + \sum_{j=1}^{\infty}(-1)^j\gamma^j q_3^{j(j-1)/2}\frac{(1-q^{\alpha-ar-brj})}{(q_3;q_3)_j}F(a+bj) \\
&\quad \times \sum_{i=0}^{j-1}A_i(1-q_3^{(i-j+1)})(1-q_3^{(i-j+2)})\cdots(1-q_3^{(i-j+1+j-1)}) \\
&= F(a).
\end{aligned}$$

This completes the proof of converse part and hence the theorem. □

Here, one can observe that $br \neq -p$ in Theorem - 3.2.1 and Theorem - 3.2.2. The two inverse pairs corresponding to $br = -p$ are proved in the forms of the following theorems.

Theorem 3.2.3. *if* $p > 0, 0 < q < 1,\ a = n \in \mathbb{N}, mr = p, m \in \mathbb{N}$ *and* $r \in \mathbb{C}$*, then*

$$F(n) = \sum_{k=0}^{N}\gamma^k\frac{(q^{p+\alpha-nr};q)_{\infty,p}}{(q;q)_k}G(n-mk) \tag{3.2.18}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{N}(-\gamma)^k q^{k(k-1)/2}\frac{F(n-mk)}{(q^{p+\alpha-nr+kp};q)_{\infty,p}(q;q)_k}, \tag{3.2.19}$$

where $N = \lfloor n/m \rfloor$.

Proof. First we prove (3.2.18)⇒(3.2.19), we began with notation $\Omega(n)$ to right hand side of (3.2.19) and then we put value of (3.2.18) as follow,

$$\begin{aligned}
\Omega(n) &= \sum_{k=0}^{N}(-\gamma)^k \frac{q^{k(k-1)/2}F(n-mk)}{(q^{p+\alpha-nr+kp};q)_{\infty,p}(q;q)_k} \\
&= \sum_{k=0}^{N}(-\gamma)^k \frac{q^{k(k-1)/2}}{(q^{p+\alpha-nr+kp};q)_{\infty,p}(q;q)_k} \\
&\quad \times \sum_{j=0}^{N-k}\gamma^j \frac{(q^{p+\alpha-nr+kp};q)_{\infty,p}}{(q;q)_j}G(n-mk-mj) \\
&= \sum_{k=0}^{N}\sum_{j=0}^{N-k}(-1)^k\gamma^{k+j}\frac{q^{k(k-1)/2}(q^{p+\alpha-nr+kp};q)_{\infty,p}}{(q^{p+\alpha-nr+kp};q)_{\infty,p}(q;q)_k(q;q)_j}G(n-mk-mj) \\
&= \sum_{k=0}^{N}\sum_{j=0}^{N-k}(-1)^k\gamma^{k+j}\frac{q^{k(k-1)/2}}{(q;q)_k(q;q)_j}G(n-mk-mj). \qquad (3.2.20)
\end{aligned}$$

On making use of double series relation (3.2.8) in (3.2.20), we get

$$\begin{aligned}
\Omega(n) &= \sum_{j=0}^{N}\sum_{k=0}^{j}(-1)^k\gamma^{k+j-k}\frac{q^{k(k-1)/2}}{(q;q)_k(q;q)_{j-k}}G(n-mk-mj+mk) \\
&= G(n)+\sum_{j=1}^{N}\sum_{k=0}^{j}(-1)^k\gamma^j q^{k(k-1)/2}\frac{1}{(q;q)_k(q;q)_{j-k}}G(n-mj) \\
&= G(n)+\sum_{j=1}^{N}\gamma^j\frac{G(n-mj)}{(q;q)_j}\sum_{k=0}^{j}(-1)^k q^{k(k-1)/2}\begin{bmatrix} j \\ k \end{bmatrix}_q. \qquad (3.2.21)
\end{aligned}$$

On applying Lemma - 3.2.1 in (3.2.21), we get

$$\begin{aligned}
\Omega(n) &= G(n)+\sum_{j=1}^{N}\gamma^j\frac{(1;q)_j}{(q;q)_j}G(n-mj) \\
&= G(n)
\end{aligned}$$

Thus, we proved ⇒ part. For converse part, we denote right hand side of (3.2.18) by $\Xi(n)$ and then we put value of (3.2.19) as follow

$$\begin{aligned}
\Xi(n) &= \sum_{k=0}^{N}\gamma^k\frac{(q^{p+\alpha-nr};q)_{\infty,p}}{(q;q)_k}G(n-mk) \\
&= \sum_{k=0}^{N}\gamma^k\frac{(q^{p+\alpha-nr};q)_{\infty,p}}{(q;q)_k}\sum_{j=0}^{\lfloor n/m-k\rfloor}(-\gamma)^j q^{j(j-1)/2}\frac{F(n-mk-mj)}{(q^{p+\alpha-nr+kp+jp};q)_{\infty,p}(q;q)_j}
\end{aligned}$$

$$= \sum_{k=0}^{N} \sum_{j=0}^{\lfloor n/m-k \rfloor} (-1)^j \gamma^{k+j} \frac{q^{j(j-1)/2}(q^{p+\alpha-nr};q)_{\infty,p}}{(q^{p+\alpha-nr+kp+jp};q)_{\infty,p}(q;q)_j(q;q)_k} \times F(n-mk-mj). \tag{3.2.22}$$

On making use of double series relation (3.2.8) in (3.2.22), we get

$$\begin{aligned}
\Xi(n) &= \sum_{j=0}^{N} \sum_{k=0}^{j} (-1)^{j-k} \gamma^{k+j-k} \frac{q^{(j-k)(j-k-1)/2}(q^{p+\alpha-nr};q)_{\infty,p}}{(q^{p+\alpha-nr+kp+jp-kp};q)_{\infty,p}(q;q)_{j-k}(q;q)_k} \\
&\quad \times F(n-mk-mj+mk) \\
&= \sum_{j=0}^{N} \sum_{k=0}^{j} (-1)^{j-k} \gamma^{j} \frac{q^{(j-k)(j-k-1)/2}(q^{p+\alpha-nr};q)_{\infty,p}}{(q^{p+\alpha-nr+jp};q)_{\infty,p}(q;q)_{j-k}(q;q)_k} F(n-mj) \\
&= F(n) + \sum_{j=1}^{N} (-1)^j \sum_{k=0}^{j} \gamma^j q^{j(j-1)/2} (q^{p+\alpha-nr};q)_{j,p} \\
&\quad \times \frac{(-1)^k q^{k(k+1-2j)/2} F(n-mj)}{(q;q)_{j-k}(q;q)_k} \\
&= F(n) + \sum_{j=0}^{N} (-1)^j \gamma^j \frac{q^{j(j-1)/2}(q^{p+\alpha-nr};q)_{j,p}}{(q;q)_j} F(n-mj) \\
&\quad \times \sum_{k=0}^{j} (-1)^k q^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k(1-j)}.
\end{aligned} \tag{3.2.23}$$

We applied Lemma - 3.2.1 to (3.2.23) to obtain,

$$\begin{aligned}
\Xi(n) &= F(n) + \sum_{j=1}^{N} (-1)^j \gamma^j q^{j(j-1)/2} \frac{(q^{p+\alpha-nr};q)_{j,p}}{(q;q)_j} (q^{1-j};q)_j F(n-mj) \\
&= F(n) + \sum_{j=1}^{N} (-1)^j \gamma^j q^{j(j-1)/2} \frac{(q^{p+\alpha-nr};q)_{j,p}}{(q;q)_j} \\
&\quad \times (1-q^{1-j})(1-q^{2-j}) \cdots (1-q^{1-j+j-1}) F(n-mj) \\
&= F(n).
\end{aligned}$$

This completes the proof of Theorem - 3.2.3. □

Next, we prove inverse series relation

Theorem 3.2.4. *if $p > 0,\ 0 < q < 1,\ a,\ b \in \mathbb{N}$ and $br = -p$, then*

$$F(a) = \sum_{k=0}^{\infty} \gamma^k \frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q;q)_k} G(a+bk) \tag{3.2.24}$$

$$\Leftrightarrow$$

$$G(a) = \sum_{k=0}^{\infty}(-\gamma)^k q^{k(k-1)/2}\frac{F(a+bk)}{(q^{p+\alpha-ar+kp};q)_{\infty,p}(q;q)_k}. \tag{3.2.25}$$

Proof. First we prove (3.2.24)⇒(3.2.25), we assign $\Psi(a)$ notation to right hand side of (3.2.25) and then we put value of (3.2.24) as follow:

$$\begin{aligned}\Psi(a) &= \sum_{k=0}^{\infty}(-\gamma)^k q^{k(k-1)/2}\frac{F(a+bk)}{(q^{p+\alpha-ar+kp};q)_{\infty,p}(q;q)_k}\\ &= \sum_{k=0}^{\infty}(-\gamma)^k q^{k(k-1)/2}\frac{1}{(q^{p+\alpha-ar+kp};q)_{\infty,p}(q;q)_k}\\ &\quad\times\sum_{j=0}^{\infty}\gamma^j\frac{(q^{p+\alpha-ar+kp};q)_{\infty,p}}{(q;q)_j}G(a+bk+bj)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^k\gamma^{j+k}q^{k(k-1)/2}\frac{(q^{p+\alpha-ar+kp};q)_{\infty,p}}{(q^{p+\alpha-ar+kp};q)_{\infty,p}(q;q)_k(q;q)_j}\\ &\quad\times G(a+bk+bj).\end{aligned} \tag{3.2.26}$$

On making use of double series relation (3.2.17) in (3.2.26), we obtained

$$\begin{aligned}\Psi(a) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^k\gamma^{j-k+k}q^{k(k-1)/2}\frac{1}{(q;q)_k(q;q)_{j-k}}G(a+bk+bj-bk)\\ &= G(a)+\sum_{j=1}^{\infty}\gamma^j\frac{1}{(q;q)_j}G(a+bj)\sum_{k=0}^{j}(-1)^k q^{k(k-1)/2}\begin{bmatrix} j\\ k\end{bmatrix}_q.\end{aligned} \tag{3.2.27}$$

Now apply result Lemma - 3.2.1 in (3.2.27) to obtain,

$$\begin{aligned}\Psi(a) &= G(a)+\sum_{j=1}^{\infty}\gamma^j\frac{(1;q)_j}{(q;q)_j}G(a+bj)\\ &= G(a).\end{aligned}$$

Thus, we proved ⇒ part. Now for converse part, we denote right hand side of (3.2.24) by $\Lambda(n)$ and then put value of (3.2.25) as follow

$$\begin{aligned}\Lambda(a) &= \sum_{k=0}^{\infty}\gamma^k\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q;q)_k}G(a+bk)\\ &= \sum_{k=0}^{\infty}\gamma^k\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q;q)_k}\sum_{j=0}^{\infty}(-\gamma)^j q^{j(j-1)/2}\frac{F(a+bk+bj)}{(q^{p+\alpha-ar+pk+jp};q)_{\infty,p}(q;q)_j}\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\gamma^{k+j}q^{j(j-1)/2}\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q^{p+\alpha-ar+pk+jp};q)_{\infty,p}(q;q)_j(q;q)_k}\end{aligned}$$

$$\times F(a+bk+bj). \tag{3.2.28}$$

On making use of double series relation (3.2.17) in (3.2.28), we obtained

$$\begin{aligned}
\Lambda(a) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j-k}\gamma^{k+j-k}q^{(j-k)(j-k-1)/2} \\
&\quad\times\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q^{p+\alpha-ar+pk+jp-kp};q)_{\infty,p}(q;q)_{j-k}(q;q)_k}F(a+bk+bj-bk) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j-k}\gamma^{j}q^{j(j-1)/2}q^{k(k+1-2j)/2} \\
&\quad\times\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q^{p+\alpha-ar+jp};q)_{\infty,p}(q;q)_{j-k}(q;q)_k}F(a+bj) \\
&= F(a)+\sum_{j=1}^{\infty}(-1)^{j}\gamma^{j}q^{j(j-1)/2}\frac{F(a+bj)}{(q;q)_j} \\
&\quad\times\sum_{k=0}^{j}(-1)^{k}q^{k(k+1-2j)/2}\begin{bmatrix} j \\ k\end{bmatrix}_q\frac{(q^{p+\alpha-ar};q)_{\infty,p}}{(q^{p+\alpha-ar+jp};q)_{\infty,p}} \\
&= F(a)+\sum_{j=1}^{\infty}(-1)^{j}\gamma^{j}q^{j(j-1)/2}(q^{p+\alpha-ar};q)_{j,p}\frac{F(a+bj)}{(q;q)_j} \\
&\quad\times\sum_{k=0}^{j}(-1)^{k}q^{k(k+1-2j)/2}\begin{bmatrix} j \\ k\end{bmatrix}_q.
\end{aligned} \tag{3.2.29}$$

On making use of Lemma - 3.2.1 in (3.2.29), we obtained

$$\begin{aligned}
\Lambda(a) &= F(a)+\sum_{j=1}^{\infty}(-1)^{j-k}\gamma^{j}q^{j(j-1)/2}(q^{p+\alpha-ar};q)_{j,p}\frac{(q^{1-j};q)_j}{(q;q)_j}F(a+bj) \\
&= F(a)+\sum_{j=1}^{\infty}(-1)^{j-k}\gamma^{j}q^{j(j-1)/2}(q^{p+\alpha-ar};q)_{j,p}\frac{1}{(q;q)_j} \\
&\quad\times\prod_{i=0}^{j-1}(1-q^{1-j+i})F(a+bj) \\
&= F(a)+\sum_{j=1}^{\infty}(-1)^{j-k}\gamma^{j}q^{j(j-1)/2}(q^{p+\alpha-ar};q)_{j,p}\frac{1}{(q;q)_j} \\
&\quad\times(1-q^{1-j})(1-q^{2-j})\ldots(1-q^{1-j+j-1})F(a+bj) \\
&= F(a).
\end{aligned}$$

This completes the proof of converse part and Theorem - 3.2.4. □

3.3 Alternative inverse pairs

We obtain the p-deformation of the Askey-Wilson polynomials (3.1.8) and the q -Racah polynomials (3.1.10) with the help of the alternative form of Theorem - 3.2.1. We first consider the reducibility of this theorem by taking $a = n, \gamma = 1, b = -1, m = 1$ and $q_4 = q^{r-p}, r \neq p$. We then get

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} \frac{(q^{p+\alpha-nr+kr-kp};q)_{\infty,p}}{(q_4;q_4)_k}\, G(n-k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} (-1)^k q_4^{k(k-1)/2} \frac{(1-q^{\alpha-nr+kr})}{(q^{\alpha-nr+kp};q)_{\infty,p}(q_4;q_4)_k}\, F(n-k). \end{aligned}$$

Reversing these inverse, gives

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} \frac{(q^{p+\alpha-nr+r(n-k)-(n-k)p};q)_{\infty,p}}{(q_4;q_4)_{n-k}} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} (-1)^{k+n} q_4^{(n-k)(n-k-1)/2} \frac{(1-q^{\alpha-nr+r(n-k)})}{(q^{\alpha-nr+(n-k)p};q)_{\infty,p}(q_4;q_4)_{n-k}} F(k), \end{aligned}$$

that is,

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} \frac{(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(q_4;q_4)_{n-k}} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} (-1)^{k+n} q_4^{\binom{n}{2}+\binom{k}{2}+k-nk} \frac{(1-q^{\alpha-kr})}{(q^{\alpha-nr+np-kp};q)_{\infty,p}(q_4;q_4)_{n-k}} F(k), \end{aligned}$$

that is,

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} \frac{(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(q_4;q_4)_{n-k}} G(k) \\ &\Leftrightarrow \\ \frac{(-1)^n G(n)}{q_4^{\binom{n}{2}}} &= \sum_{k=0}^{n} (-1)^k \frac{q_4^{\binom{k}{2}+k-nk}(1-q^{\alpha-kr})}{(1-q^{\alpha-nr+np-kp})(q^{\alpha-nr+np-kp+p};q)_{\infty,p}(q_4;q_4)_{n-k}} F(k). \end{aligned}$$

On replacing $G(n)$ by $(-1)^n q_4^{\binom{n}{2}} G(n)$, this pair reduces to

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} (-1)^k q_4^{\binom{k}{2}} \frac{(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(q_4;q_4)_{n-k}} G(k) \\ &\Leftrightarrow \end{aligned}$$

$$G(n) = \sum_{k=0}^{n}(-1)^k \frac{q_4^{\binom{k}{2}+k-nk}(1-q^{\alpha-kr})}{(1-q^{\alpha-nr+np-kp})(q^{\alpha-nr+np-kp+p};q)_{\infty,p}(q_4;q_4)_{n-k}} F(k)$$

By making an appeal to the formula (1.5.10) with the substitutions $a=1,\ p=1$ and replacing q by q_4, that is, using

$$(q_4;q_4)_{n-k} = (-1)^k q_4^{k(k+1)/2-nk-k}\frac{(q_4;q_4)_n}{(q_4^{-n};q_4)_k} = (-1)^k q_4^{\binom{k}{2}-nk}\frac{(q_4;q_4)_n}{(q_4^{-n};q_4)_k},$$

the above pair changes to

$$\begin{aligned} F(n) &= \sum_{k=0}^{n}(-1)^k q_4^{\binom{k}{2}} \frac{(q_4^{-n};q_4)_k(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(-1)^k q_4^{\binom{k}{2}-nk}(q_4;q_4)_n} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n}(-1)^k \frac{q_4^{\binom{k}{2}+k-nk}(1-q^{\alpha-kr})}{(1-q^{\alpha-nr+np-kp})(q^{\alpha-nr+np-kp+p};q)_{\infty,p}} \\ &\quad \times \frac{(q_4^{-n};q_4)_k}{(-1)^k q_4^{\binom{k}{2}-nk}(q_4;q_4)_n} F(k), \end{aligned}$$

or alternatively,

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} q_4^{nk} \frac{(q_4^{-n};q_4)_k(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(q_4;q_4)_n} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} \frac{q_4^{k}(1-q^{\alpha-kr})(q_4^{-n};q_4)_k}{(1-q^{\alpha-nr+np-kp})(q^{\alpha-nr+np-kp+p};q)_{\infty,p}(q_4;q_4)_n} F(k). \end{aligned}$$

We replace here $G(n)$ by $G(n)/(q^{p+\alpha-nr};q)_{\infty,p}$ to get

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} q_4^{nk} \frac{(q_4^{-n};q_4)_k(q^{p+\alpha-np-kr+kp};q)_{\infty,p}}{(q^{p+\alpha-kr};q)_{\infty,p}(q_4;q_4)_n} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} \frac{q_4^{k}(1-q^{\alpha-kr})(q_4^{-n};q_4)_k(q^{p+\alpha-nr};q)_{\infty,p}}{(1-q^{\alpha-nr+np-kp})(q^{\alpha-nr+np-kp+p};q)_{\infty,p}(q_4;q_4)_n} F(k), \end{aligned}$$

that is,

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} q_4^{nk} \frac{(q_4^{-n};q_4)_k}{(q^{p+\alpha-kr};q)_{k-n,p}(q_4;q_4)_n} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} \frac{q_4^{k}(1-q^{\alpha-kr})(q_4^{-n};q_4)_k(q^{p+\alpha-nr};q)_{n-k,p}}{(1-q^{\alpha-nr+np-kp})(q_4;q_4)_n} F(k). \end{aligned}$$

Now from the formula (1.5.11), we have

$$\begin{aligned}
(q^{p+\alpha-kr};q)_{k-n,p} &= (q^{p+\alpha-kr};q)_{-(n-k),p} \\
&= (-1)^{n+k}q^{p(n-k)(n-k-1)/2-(p+\alpha-kr)(n-k)}\frac{1}{(q^{p-\alpha-p+kr};q)_{n-k,p}} \\
&= (-1)^{n+k}q^{p(n-k)(n-k-1)/2-(p+\alpha-kr)(n-k)}\frac{1}{(q^{-\alpha+kr};q)_{n-k,p}} \\
&= (-1)^{n+k}q^{p\binom{n}{2}+p\binom{k}{2}-nkp+kp-(p+\alpha-kr)(n-k)}\frac{1}{(q^{-\alpha+kr};q)_{n-k,p}},
\end{aligned}$$

and

$$\begin{aligned}
(q^{p+\alpha-nr};q)_{n-k,p} &= (q^{p+\alpha-nr};q)_{-(k-n),p}, \\
&= (-1)^{n+k}q^{p(k-n)(k-n-1)/2-(p+\alpha-nr)(k-n)}\frac{1}{(q^{p-\alpha-p+nr};q)_{k-n,p}} \\
&= (-1)^{n+k}q^{p(k-n)(k-n-1)/2-(p+\alpha-nr)(k-n)}\frac{1}{(q^{-\alpha+nr};q)_{k-n,p}} \\
&= (-1)^{n+k}q^{p\binom{k}{2}+p\binom{n}{2}-nkp+np-(p+\alpha-nr)(k-n)}\frac{1}{(q^{-\alpha+nr};q)_{k-n,p}},
\end{aligned}$$

hence, the above inverse pair assumes the form:

$$\begin{aligned}
F(n) &= \sum_{k=0}^{n} q_4^{nk}\frac{(q_4^{-n};q_4)_k(q^{-\alpha+kr};q)_{n-k,p}}{(-1)^{n+k}q^{p\binom{n}{2}+p\binom{k}{2}-nkp+kp-(p+\alpha-kr)(n-k)}(q_4;q_4)_n}G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n}\frac{q_4^k(1-q^{\alpha-kr})(q_4^{-n};q_4)_k(-1)^{n+k}q^{p\binom{k}{2}+p\binom{n}{2}-nkp+np-(p+\alpha-nr)(k-n)}}{(1-q^{\alpha-nr+np-kp})(q^{-\alpha+nr};q)_{k-n,p}(q_4;q_4)_n}F(k).
\end{aligned}$$

Also,

$$\frac{(1-q^{\alpha-kr})}{(1-q^{\alpha+np-kp})} = q^{nr-kr-np+kp}\frac{(1-q^{-\alpha+kr})}{(1-q^{-\alpha+nr-np+kp})},$$

consequently, we get

$$\begin{aligned}
F(n) &= \sum_{k=0}^{n} q_4^{nk}\frac{(q_4^{-n};q_4)_k(q^{-\alpha+kr};q)_{n-k,p}}{(-1)^{n+k}q^{p\binom{n}{2}+p\binom{k}{2}-nkp+kp-(p+\alpha-kr)(n-k)}(q_4;q_4)_n}G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n}\frac{q_4^k q^{nr-kr-np+kp}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k}{(1-q^{-\alpha+nr-np+kp})(q^{-\alpha+nr};q)_{k-n,p}(q_4;q_4)_n} \\
&\quad\times(-1)^{n+k}q^{p\binom{k}{2}+p\binom{n}{2}-nkp+np-(p+\alpha-nr)(k-n)}F(k),
\end{aligned}$$

that is,

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{n} q_4^{nk} \frac{(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{n-k,p}}{(q_4;q_4)_n} \\
&\quad \times (-1)^{n+k} q^{-p\binom{n}{2}-p\binom{k}{2}+nkp-kp-(p+\alpha-kr)(k-n)} G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n} \frac{q_4^k q^{nr-kr-np+kp}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k}{(q^{-\alpha+nr};q)_{k-n+1,p}(q_4;q_4)_n} \\
&\quad \times (-1)^{n+k} q^{p\binom{k}{2}+p\binom{n}{2}-nkp+np-(p+\alpha-nr)(k-n)} F(k).
\end{aligned}
$$

Here on replacing $F(n)(-1)^n q^{-p\binom{n}{2}+(p+\alpha)n}$ and $G(n)(-1)^n q^{p\binom{n}{2}+np-(\alpha+p)n}$ by $F(n)$ and $G(n)$ respectively, we get simplified form:

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{n} q_4^{nk} \frac{q^{nkp-kr(n-k)}(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{n-k,p}}{(q_4;q_4)_n} G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n} \frac{q_4^k q^{nr-kr-np+kp-nkp-nr(n-k)}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k q^{-nkp-nr(n-k)}}{(q^{-\alpha+nr};q)_{k-n+1,p}(q_4;q_4)_n} F(k).
\end{aligned}
$$

Rewriting the powers of q, we get

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{n} q^{nk(r-p)+nkp-kr(n-k)} \frac{(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{n-k,p}}{(q_4;q_4)_n} G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n} \frac{q^{k(r-p)+nr-kr-np+kp-nkp-nr(n-k)}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k}{(q^{-\alpha+nr};q)_{k-n+1,p}(q_4;q_4)_n} F(k);
\end{aligned}
$$

which gives

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{n} q^{k^2 r} \frac{(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{n-k,p}}{(q_4;q_4)_n} G(k) \\
&\Leftrightarrow \\
G(n) &= \sum_{k=0}^{n} \frac{q^{nr-np} q^{-nkp-n^2r+nkr}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k}{(q^{-\alpha+nr};q)_{k-n+1,p}(q_4;q_4)_n} F(k).
\end{aligned}
$$

Next, replacing $G(n)$ by $q^{nr-np}G(n)/q^{n^2r}$, gives

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{n} q^{kr-kp} \frac{(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{n-k,p}}{(q_4;q_4)_n} G(k) \\
&\Leftrightarrow
\end{aligned}
$$

$$G(n) = \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k}{(q^{-\alpha+nr};q)_{k-n+1,p}(q_4;q_4)_n} F(k).$$

Now using the formula (1.5.7), we get

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} q^{kr-kp} \frac{(q_4^{-n};q_4)_k (q^{-\alpha+kr};q)_{\infty,p}}{(q^{-\alpha+kr+np-kp};q)_{\infty,p}(q_4;q_4)_n} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k (q^{-\alpha+nr+kp-np+p};q)_{\infty,p}}{(q^{-\alpha+nr};q)_{\infty,p}(q_4;q_4)_n} F(k). \end{aligned}$$

If $G(n)$ is replaced by $G(n)/(q^{-\alpha+nr};q)_{\infty,p}$, then we get

$$F(n) = \sum_{k=0}^{n} q^{kr-kp} \frac{(q_4^{-n};q_4)_k}{(q^{-\alpha+kr+np-kp};q)_{\infty,p}(q_4;q_4)_n} G(k) \tag{3.3.1}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-q^{-\alpha+kr})(q_4^{-n};q_4)_k (q^{-\alpha+nr+kp-np+p};q)_{\infty,p}}{(q_4;q_4)_n} F(k). \tag{3.3.2}$$

This pair is used to provides p-version of Askey Wilson polynomials and q-Racah polynomials.

3.4 Particular cases

The polynomials (3.1.7) when compared with the series (3.2.6) of Theorem -3.2.1 suggests the substitutions $\alpha = s$,

$$G(n) = (q^m - 1)^n x^n / ((q^{s-nr+p};q)_{\infty,p}(1-q^c)^{n-s}(q;q)_n)$$

and the replacement of γ by $(1-q)\gamma/(1-q^c)$. Then $F(n) = P_{n,p}(m,x,\gamma,s,c|q)$ yields the p-deformed generalized q-Humbert polynomials. The inverse series is then obtained from (3.2.16) in the form:

$$\begin{aligned} &\frac{(q^m-1)^n}{(1-q^c)^n(q;q)_n} x^n \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k \left(\frac{1-q}{1-q^c}\right)^{s+k} \frac{q^{(mr-p)k(k-1)/2}(1-q^{s-nr+mrk})(q^{s-nr+p};q)_{\infty,p}}{(q^{s-nr+kp};q)_{\infty,p}(q^{mr-p};q^{mr-p})_k} \\ &\quad \times P_{n-mk,p,r}(m,x,\gamma,s,c|q) \end{aligned} \tag{3.4.1}$$

The p-deformed q-Humbert polynomials and its inverse series follows by taking $s=-\nu,\ c=1$ and $\gamma=1$ in (3.1.7) and (3.4.1):

$$\Pi^{\nu}_{n,m,p,r}(x|q) = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(q^{-\nu-nr+mrk-kp+p};q)_{\infty,p}(1-q)^{-n+mk}}{(q^{-\nu-nr+mkr+p};q)_{\infty,p}(q;q)_{n-mk}} \times\frac{((q^m-1)x)^{n-mk}}{(q^{mr-p};q^{mr-p})_k} \tag{3.4.2}$$

$$\Leftrightarrow$$

$$\frac{(q^m-1)^n}{(1-q)^n(q;q)_n}x^n = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k q^{(mr-p)k(k-1)/2}\frac{(1-q^{-\nu-nr+mrk})}{(q^{-\nu-nr+kp};q)_{\infty,p}} \times\frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}\Pi^{\nu}_{n-mk,m,p,r}(x|q). \tag{3.4.3}$$

The p-deformed q-Kinney polynomial and its inverse series is obtained by taking $\nu=1/m$ in (3.4.2) and (3.4.3) as given below

$$P_{n,p,r}(m,x|q) = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(q^{-\frac{1}{m}-nr+mrk-kp+p};q)_{\infty,p}(1-q)^{-n+mk}}{(q^{-\frac{1}{m}-nr+mkr+p};q)_{\infty,p}(q;q)_{n-mk}} \times\frac{((q^m-1)x)^{n-mk}}{(q^{mr-p};q^{mr-p})_k}$$

$$\Leftrightarrow$$

$$\frac{(q^m-1)^n}{(1-q)^n(q;q)_n}x^n = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k q^{(mr-p)k(k-1)/2}\frac{(1-q^{-\frac{1}{m}-nr+mrk})}{(q^{-\frac{1}{m}-nr+kp};q)_{\infty,p}} \times\frac{(q^{-\frac{1}{m}-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}P_{n-mk,p,r}(m,x|q).$$

Take $m=3$ and $\nu=1/2$ in (3.4.2) and (3.4.3), then we obtain the p-deformed q-Pincherle polynomial and its inverse:

$$\mathcal{P}_{n,p,r}(x|q) = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{(q^{-\frac{1}{2}-nr+3rk-kp+p};q)_{\infty,p}}{(q^{-\frac{1}{2}-nr+3kr+p};q)_{\infty,p}(q;q)_{n-3k}(q^{3r-p};q^{3r-p})_k} \times\left(-(q^2+q+1)x\right)^{n-3k}$$

$$\Leftrightarrow$$

$$\frac{(q^2+q+1)^n}{(q;q)_n}(-x)^n = \sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k q^{(3r-p)k(k-1)/2}\frac{(1-q^{-\frac{1}{2}-nr+3rk})(q^{-\frac{1}{2}-nr+p};q)_{\infty,p}}{(q^{-\frac{1}{2}-nr+kp};q)_{\infty,p}(q^{3r-p};q^{3r-p})_k} \times\mathcal{P}_{n-3k,p,r}(x|q)$$

The p-deformed q-Gegenbauer polynomial and its inverse series are obtained by taking $m = 2$ in (3.4.2) and (3.4.3) as given below.

$$\begin{aligned} C^{\nu}_{n,p,r}(x|q) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q^{-\nu-nr+2rk-kp+p};q)_{\infty,p}(1+q)^{n-2k}}{(q^{-\nu-nr+2kr+p};q)_{\infty,p}(q;q)_{n-2k}(q^{2r-p};q^{2r-p})_k} \\ &\quad \times(-x)^{n-2k} \end{aligned} \quad (3.4.4)$$

$$\Leftrightarrow$$

$$\begin{aligned} \frac{(1+q)^n}{(q;q)_n}(-x)^n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(2r-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+2rk})(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{-\nu-nr+kp};q)_{\infty,p}(q^{2r-p};q^{2r-p})_k} \\ &\quad \times C^{\nu}_{n-2k,p,r}(x|q). \end{aligned} \quad (3.4.5)$$

The p-deformed q-Legendre polynomial and its inverse series occur when $\nu = 1/2$ in (3.4.4) and (3.4.5). The pair is

$$\begin{aligned} P_{n,p,r}(x|q) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q^{-\frac{1}{2}-nr+2rk-kp+p};q)_{\infty,p}(1+q)^{n-2k}}{(q^{-\frac{1}{2}-nr+2kr+p};q)_{\infty,p}(q;q)_{n-2k}(q^{2r-p};q^{2r-p})_k} \\ &\quad \times(-x)^{n-2k} \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} \frac{(1+q)^n}{(q;q)_n}(-x)^n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(2r-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{2}-nr+2rk})(q^{-\frac{1}{2}-nr+p};q)_{\infty,p}}{(q^{-\frac{1}{2}-nr+kp};q)_{\infty,p}(q^{2r-p};q^{2r-p})_k} \\ &\quad \times P_{n-2k,p,r}(x|q). \end{aligned}$$

Note 3.4.1. *When $p = 1$, the polynomial (3.1.7) and its inverse series (3.4.1) get reduced to the polynomial (3.1.1) and its inverse (3.1.2). Moreover, (3.1.7) and its inverse series (3.4.1) approach to (2.1.1) and its inverse series relation (2.3.3) as $q \to 1$ with $p = 1$ and $r = p$.*

The inverse series pair (3.3.1) and (3.3.2) when specialized suitably, provides us inversion pairs corresponding to the p-deformed Askey-Wilson polynomials and the p-deformed q-Racah polynomials. In fact, if $\alpha = -a-b-c-d+p$, and

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{r-p};q^{r-p})_n)$$

then

$$F(n) = \frac{(abcdq^{-p};q)_{n,p}a^n \; p_{n,p,r}(cos\theta;a,b,c,d|q)}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(abcdq^{-p};q)_{\infty,p}(q^{r-p};q^{r-p})_n}$$

gives the p-deformed bibasic Askey-Wilson polynomials and the inverse series:

$$\frac{(abcdq^{-p};q)_{n,p}a^n \; p_{n,p,r}(cos\theta;a,b,c,d|q)}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{r-p};q^{r-p})_n}$$

$$= \sum_{k=0}^{n} q^{kr-kp} \frac{(q^{-n(r-p)};q^{r-p})_k (abcdq^{-p};q)_{\infty,p} (ae^{i\theta};q)_{k,p} (ae^{-i\theta};q)_{k,p}}{(abcdq^{-p+kr+np-kp};q)_{\infty,p} (ab;q)_{k,p} (ac;q)_{k,p} (ad;q)_{k,p} (q^{r-p};q^{r-p})_k}$$
$$\times \frac{1}{(q^{r-p};q^{r-p})_n}$$
$$\Leftrightarrow$$
$$\frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{r-p};q^{r-p})_n}$$
$$= \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-abcdq^{-p+kr})(q^{-n(r-p)};q^{r-p})_k (abcdq^{-p+nr+kp-np+p};q)_{\infty,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(abcdq^{-p};q)_{\infty,p}(q^{r-p};q^{r-p})_k (q^{r-p};q^{r-p})_n}$$
$$\times (abcdq^{-p};q)_{k,p} a^k \; p_{k,p,r}(cos\theta;a,b,c,d|q).$$

A further simplified form is

$$\frac{a^n \; p_{n,p,r}(cos\theta;a,b,c,d|q)}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}$$
$$= \sum_{k=0}^{n} q^{kr-kp} \frac{(q^{-n(r-p)};q^{r-p})_k (abcdq^{np-p};q)_{\frac{kr}{p}-k,p} (ae^{i\theta};q)_{k,p} (ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{r-p};q^{r-p})_k} \tag{3.4.6}$$

$$\Leftrightarrow$$
$$\frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}$$
$$= \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-abcdq^{-p+kr})(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k}$$
$$\times a^k \; p_{k,p,r}(cos\theta;a,b,c,d|q). \tag{3.4.7}$$

If $\alpha = -a-b-p$, and

$$G(n) = (q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{r-p};q^{r-p})_n)$$

then

$$F(n) = (abq^p;q)_{n,p} R_{n,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q)/((abq^p;q)_{\infty,p}(q^{r-p};q^{r-p})_n)$$

in (3.3.1) and (3.3.2), yields the p-deformed q-Racah polynomials along with the inverse series as follows.

$$\frac{(abq^p;q)_{n,p} R_{n,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q)}{(q^{r-p};q^{r-p})_n}$$
$$= \sum_{k=0}^{n} q^{kr-kp} \frac{(q^{-n(r-p)};q^{r-p})_k (q^{-x};q)_{k,p} (cdq^{x+p};q)_{k,p} (abq^p;q)_{\infty,p}}{(abq^{kr+np-kp+p};q)_{\infty,p} (aq^p;q)_{k,p} (bdq^p;q)_{k,p} (cq^p;q)_{k,p}}$$

$$\times \frac{1}{(q^{r-p};q^{r-p})_k (q^{r-p};q^{r-p})_n}$$
$$\Leftrightarrow$$
$$\frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{r-p};q^{r-p})_n}$$
$$= \sum_{k=0}^{n} q^{nkr-nkp} \frac{(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k (abq^{nr+kp-np+2p};q)_{\infty,p}(abq^p;q)_{k,p}}{(abq^p;q)_{\infty,p}(q^{r-p};q^{r-p})_k (q^{r-p};q^{r-p})_n}$$
$$\times R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q).$$

Taking $q^{-x}+cdq^{x+p}=\mu(x)$, this may be put in an elegant form:

$$R_{n,p,r}(\mu(x);a,b,c,N|q) = \sum_{k=0}^{n} q^{kr-kp} \frac{(q^{-n(r-p)};q^{r-p})_k (abq^{np+p};q)_{\frac{kr}{p}-k,p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}} \times \frac{(cdq^{x+p};q)_{k,p}(q^{-x};q)_{k,p}}{(q^{r-p};q^{r-p})_k}. \quad (3.4.8)$$
$$\Leftrightarrow$$
$$\frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}} = \sum_{k=0}^{n} \frac{q^{nkr-nkp}(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} \times R_{k,p,r}(\mu(x);a,b,c,N|q). \quad (3.4.9)$$

If $p=1$, then the polynomials and their inverse series get reduced to those obtained in [11]. The particular polynomials belonging to the above polynomials are the p-deformed q-Hahn polynomial, the p-deformed little as well as big q-Jacobi polynomials, the p-deformed q-Gegenbauer polynomial, the p-deformed q-Legendre polynomial, the p-deformed q-Chebyshev polynomial together their inverse series relation. Next, the p-deformation of the q-Bessel function (due to Jackson) (3.1.12) and its inverse series occur from theorem - 3.2.2 when $\gamma=-1$, $a=n$, $\alpha=0$, $b=2$, $r\neq -p/2$ and $G(n)=(x/2)^n$. With this, $F(n)=J^{(1)}_{n,p,r}(x;q)(q^p;q)_{\infty,p}$ provides the pair:

$$J^{(1)}_{n,p,r}(x;q) = \frac{(q^{p-nr};q)_{\infty,p}}{(q^p;q)_{\infty,p}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(q^{p-nr};q)_{-2\frac{rk}{p}-k,p}(q^{-2r-p};q^{-2r-p})_k} \times \left(\frac{x}{2}\right)^{n+2k} \quad (3.4.10)$$
$$\Leftrightarrow$$
$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} q^{(-2r-p)k(k-1)/2} \frac{(1-q^{-nr-2rk})(q^p;q)_{\infty,p}}{(q^{-nr+kp};q)_{\infty,p}(q^{-2r-p};q^{-2r-p})_k} \times J^{(1)}_{n+2k,p,r}(x;q). \quad (3.4.11)$$

We call (3.4.10) as the p-deformed q-Bessel function which provides q-analogues of the p-deformed Bessel function (2.3.12) of Chapter 2 for $r = -p$. Here one can observe that (3.4.11) is a p-deformation of the q-Neumann's expansion (when $r = -1$ and $p = 1$)[11]:

$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(1-q^{n+2k})(q;q)_\infty}{(q^{n+k};q)_\infty (q;q)_k} J^{(1)}_{n+2k}(x;q), \tag{3.4.12}$$

whose inverse series is the q-Bessel function [19, Ex. 1.24, p.25]:

$$J^{(1)}_n(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(q^{n+1};q)_k (q;q)_k} \left(\frac{x}{2}\right)^{n+2k}. \tag{3.4.13}$$

There is a generalization of the q-Bessel function which is due to Mansour Mahmoud [44]. It is given by

$$J_n(x;a,q) = \frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^{(a+1)k} q^{ak(k+n)/2}}{(q^{n+1};q)_k (q;q)_k} \left(\frac{x}{2}\right)^{n+2k}. \tag{3.4.14}$$

The series on the right hand side converges absolutely for all x when $a \in \mathbb{Z}^+$ and for $|x| < 2$, if $a = 0$. Mansour showed that one can obtain the q-Bessel functions of 1st, 2nd and 3rd kind from (3.4.14) by making use of $a = 0, 2$ and 1 respectively. Also q-Bessel function of 4th kind given by Mansour Mahmoud [45] occur from (3.4.14) when $a = 3$. We obtain the inverse series of this function with the help of Theorem - 3.2.2. For that, we first put $a = n,\ \alpha = 0,\ b = 2,\ q_5 = q^{-2r-p}, r \neq -p/2$ and $\gamma = (-1)^{a+1}$. Then we find

$$\begin{aligned} F(n) &= \sum_{k=0}^{\infty} (-1)^{k(a+1)} \frac{(q^{p-nr-2rk-kp};q)_{\infty,p}}{(q_5;q_5)_k} G(n+2k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{\infty} (-1)^{ak} q_5^{\binom{k}{2}} \frac{(1-q^{-nr-2rk})}{(q^{-nr+kp};q)_{\infty,p} (q_5;q_5)_k} F(n+2k). \end{aligned}$$

Here replacing $F(n)$ and $G(n)$ by $q^{an^2/8}F(n)$ and $q^{an^2/8}G(n)$ respectively, we get

$$\begin{aligned} q^{\frac{an^2}{8}} F(n) &= \sum_{k=0}^{\infty} (-1)^{k(a+1)} \frac{(q^{p-nr-2rk-kp};q)_{\infty,p}}{(q_5;q_5)_k} q^{\frac{a(n+2k)^2}{8}} G(n+2k) \\ &\Leftrightarrow \\ q^{\frac{an^2}{8}} G(n) &= \sum_{k=0}^{\infty} (-1)^{ak} q_5^{\binom{k}{2}} \frac{(1-q^{-nr-2rk})}{(q^{-nr+kp};q)_{\infty,p} (q_5;q_5)_k} q^{\frac{a(n+2k)^2}{8}} F(n+2k); \end{aligned}$$

Which may be simplified to get

$$
\begin{aligned}
q^{\frac{an^2}{8}}F(n) &= \sum_{k=0}^{\infty}(-1)^{k(a+1)}\frac{(q^{p-nr-2rk-kp};q)_{\infty,p}}{(q_5;q_5)_k}q^{\frac{a(n^2+4kn+4k^2)}{8}}G(n+2k)\\
&\Leftrightarrow\\
q^{\frac{an^2}{8}}G(n) &= \sum_{k=0}^{\infty}(-1)^{ak}q_5^{\binom{k}{2}}\frac{(1-q^{-nr-2rk})}{(q^{-nr+kp};q)_{\infty,p}(q_5;q_5)_k}q^{\frac{a(n^2+4kn+4k^2)}{8}}F(n+2k),
\end{aligned}
$$

or alternatively,

$$
\begin{aligned}
F(n) &= \sum_{k=0}^{\infty}(-1)^{k(a+1)}q^{\frac{ak(n+k)}{2}}\frac{(q^{p-nr-2rk-kp};q)_{\infty,p}}{(q_5;q_5)_k}G(n+2k)\\
&\Leftrightarrow\\
G(n) &= \sum_{k=0}^{\infty}(-1)^{ak}q_5^{\binom{k}{2}}q^{\frac{ak(n+k)}{2}}\frac{(1-q^{-nr-2rk})}{(q^{-nr+kp};q)_{\infty,p}(q_5;q_5)_k}F(n+2k).
\end{aligned}
$$

In this pair, if we put $G(n)=(x/2)^n$ then $F(n)=J_{n,p,r}(x;a,q)(q^p;q)_{n,p}(q^{p-nr};q)_{\infty,p}$ provides us the p-deformation of the function (3.4.14) along with its inverse series, in the form:

$$
\begin{aligned}
J_{n,p,r}(x;a,q) &= \frac{1}{(q^p;q)_{n,p}(q^{p-nr};q)_{\infty,p}}\sum_{k=0}^{\infty}(-1)^{(a+1)k}q^{ak(n+k)/2}\frac{(q^{p-nr-2rk-kp};q)_{\infty,p}}{(q^{-2r-p};q^{-2r-p})_k}\\
&\quad\times\left(\frac{x}{2}\right)^{n+2k}\\
&\Leftrightarrow\\
\left(\frac{x}{2}\right)^n &= \sum_{k=0}^{\infty}(-1)^{ak}q^{(-2r-p)\binom{k}{2}}q^{\frac{ak(n+k)}{2}}\frac{(1-q^{-nr-2rk})}{(q^{-nr+kp};q)_{\infty,p}(q^{-2r-p};q^{-2r-p})_k}\\
&\quad\times J_{n+2k,p,r}(x;a,q)(q^p;q)_{n,p}(q^{p+np};q)_{\infty,p},
\end{aligned}
$$

alternatively,

$$
J_{n,p,r}(x;a,q) = \frac{1}{(q^p;q)_{n,p}}\sum_{k=0}^{\infty}\frac{(-1)^{(a+1)k}q^{ak(n+k)/2}}{(q^{p-nr};q)_{-\frac{2rk}{p}-k,p}(q^{-2r-p};q^{-2r-p})_k}\left(\frac{x}{2}\right)^{n+2k} \tag{3.4.15}
$$

$$
\Leftrightarrow
$$

$$
\begin{aligned}
\left(\frac{x}{2}\right)^n &= \sum_{k=0}^{\infty}(-1)^{ak}q^{(-2r-p)\binom{k}{2}}q^{ak(n+k)/2}\frac{(1-q^{-nr-2rk})(q^p;q)_{\infty,p}}{(q^{-nr+kp};q)_{\infty,p}(q^{-2r-p};q^{-2r-p})_k}\\
&\quad\times J_{n+2k,p,r}(x;a,q).
\end{aligned} \tag{3.4.16}
$$

The inverse series of (3.4.14) now follows from (3.4.16) when $p=1, r=-1$.

3.5 Summation formulas

In this section, we obtain certain summation formulas involving the inverse series of the p-deformed generalized q-Humbert polynomials (3.1.7), the p-deformed Askey-Wilson polynomials (3.4.6) and the q,p-Racah polynomials (3.4.8) given by (3.4.1), (3.4.7) and (3.4.9) obtained in above section. We state the q-Binomial theorem[19, Eq(1.3.2), page 7] and two q-exponential functions $e_q(x)$[19, Eq(1.3.15), page 9], $E_q(x)$[19, Eq(1.3.16), page 9] below.
For $|z|<1$ and $|q|<1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.5.1}$$

The q-exponential function

$$\sum_{k=0}^{\infty}\frac{(0;q)_k}{(q;q)_k}\,x^k=\frac{1}{(x;q)_\infty}:=e_q(x), \quad (|x|<1,\ |q|<1) \tag{3.5.2}$$

and the q-Exponential function

$$\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{(q;q)_k}\,x^k=(-z;q)_\infty:=E_q(x), \quad (x\in\mathbb{C},\ |q|<1) \tag{3.5.3}$$

which will be abbreviated here as qeF and qEF respectively.

Next, rearranging coefficient of x^n in (3.4.1) and multiply it by $(a;q)_n/(q;q)_n$ and then taking sum from $n=0$ to ∞ yields the summation formulas, by using (3.5.1), involving the p-deformed generalized q-Humbert polynomials as follows.

$$\sum_{n=0}^{\infty}\frac{(1-q^c)^n(a;q)_n}{(q^m-1)^n}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k\left(\frac{1-q}{1-q^c}\right)^{s+k}\frac{q^{(mr-p)k(k-1)/2}}{(q^{s-np+kp};q)_{\infty,p}}$$
$$\times\frac{(1-q^{s-nr+mrk})(q^{s-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}P_{n-mk,p,r}(m,x,\gamma,s,c|q) = \frac{(ax;q)_\infty}{(x;q)_\infty} \tag{3.5.4}$$

The summation formula

$$\sum_{n=0}^{\infty}\frac{(1-q^c)^n}{(q^m-1)^n}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k\left(\frac{1-q}{1-q^c}\right)^{s+k}\frac{q^{(mr-p)k(k-1)/2}}{(q^{s-np+kp};q)_{\infty,p}}$$
$$\times\frac{(1-q^{s-nr+mrk})(q^{s-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}P_{n-mk,p,r}(m,x,\gamma,s,c|q) = e_q(x) \tag{3.5.5}$$

can be obtained by rearranging coefficient of x^n in (3.4.1) and using (3.5.2). By rearranging coefficient of x^n in (3.4.1) and multiply it by $q^{n(n-1)/2}$ then taking sum from $n = 0$ to ∞ yields summation formulas involving the p-deformed generalized q-Humbert polynomials:

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(1-q^c)^n}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k \left(\frac{1-q}{1-q^c}\right)^{s+k} \frac{q^{(mr-p)k(k-1)/2}}{(q^{s-np+kp};q)_{\infty,p}} \times \frac{(1-q^{s-nr+mrk})(q^{s-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} P_{n-mk,p,r}(m,x,\gamma,s,c|q) = E_q(x). \quad (3.5.6)$$

Similarly, the substitutions $s = -\nu,\ c = 1$ and $\gamma = 1$ in (3.5.4), (3.5.5) and (3.5.6) yield

$$\sum_{n=0}^{\infty} \frac{(1-q)^n (a;q)_{n,p}}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+mrk})}{(q^{-\nu-nr+kp};q)_{\infty,p}} \times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} \Pi^{\nu}_{n-mk,m,p,r}(x|q) = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad (3.5.7)$$

$$\sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+mrk})}{(q^{-\nu-nr+kp};q)_{\infty,p}} \times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} \Pi^{\nu}_{n-mk,m,p,r}(x|q) = e_q(x), \quad (3.5.8)$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(1-q)^n}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+mrk})}{(q^{-\nu-nr+kp};q)_{\infty,p}} \times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} \Pi^{\nu}_{n-mk,m,p,r}(x|q) = E_q(x), \quad (3.5.9)$$

respectively.

For $\nu = 1/m$, then (3.5.7), (3.5.8) and (3.5.9) get reduced to the sum involving the p-deformed q-Kinney polynomial:

$$\sum_{n=0}^{\infty} \frac{(1-q)^n (a;q)_{n,p}}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{m}-nr+mrk})}{(q^{-\frac{1}{m}-nr+kp};q)_{\infty,p}} \times \frac{(q^{-\frac{1}{m}-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} P_{n-mk,p,r}(m,x|q) = \frac{(ax;q)_\infty}{(x;q)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{m}-nr+mrk})}{(q^{-\frac{1}{m}-nr+kp};q)_{\infty,p}} \times \frac{(q^{-\frac{1}{m}-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} P_{n-mk,p,r}(m,x|q) = e_q(x),$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(1-q)^n}{(q^m-1)^n} \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k q^{(mr-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{m}-nr+mrk})}{(q^{-\frac{1}{m}-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\frac{1}{m}-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k} P_{n-mk,p,r}(m,x|q) \;=\; E_q(x).$$

Here $m=3$ and $\nu=1/2$ further reduce these summation formulas involving the p-deformed q-Pincherle polynomial as follows.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(a;q)_{n,p}}{(q^2+q+1)^n} \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k q^{(3r-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{2}-nr+3rk})}{(q^{-\frac{1}{2}-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\frac{1}{2}-nr+p};q)_{\infty,p}}{(q^{3r-p};q^{3r-p})_k} \mathcal{P}_{n-3k,p,r}(x|q) \;=\; \frac{(ax;q)_\infty}{(x;q)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(q^2+q+1)^n} \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k q^{(3r-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{2}-nr+3rk})}{(q^{-\frac{1}{2}-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\frac{1}{2}-nr+p};q)_{\infty,p}}{(q^{3r-p};q^{3r-p})_k} \mathcal{P}_{n-3k,p,r}(x|q) \;=\; e_q(x),$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q^2+q+1)^n} \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k q^{(3r-p)k(k-1)/2} \frac{(1-q^{-\frac{1}{2}-nr+3rk})}{(q^{-\frac{1}{2}-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\frac{1}{2}-nr+p};q)_{\infty,p}}{(q^{3r-p};q^{3r-p})_k} \mathcal{P}_{n-3k,p,r}(x|q) \;=\; E_q(x)$$

If we take $m=2$ in (3.5.7), (3.5.8) and (3.5.9), then we get summation formulas involving the p-deformed q-Gegenbauer polynomial as follows.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(a;q)_{n,p}}{(q+1)^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(2r-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+2rk})}{(q^{-\nu-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{2r-p};q^{2r-p})_k} C^{\nu}_{n-2k,p,r}(x|q) \;=\; \frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{3.5.10}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(q+1)^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(2r-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+2rk})}{(q^{-\nu-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{2r-p};q^{2r-p})_k} C^{\nu}_{n-2k,p,r}(x|q) \;=\; e_q(x), \tag{3.5.11}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-1)^n}{(q+1)^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(2r-p)k(k-1)/2} \frac{(1-q^{-\nu-nr+2rk})}{(q^{-\nu-nr+kp};q)_{\infty,p}}$$
$$\times \frac{(q^{-\nu-nr+p};q)_{\infty,p}}{(q^{2r-p};q^{2r-p})_k} C^{\nu}_{n-2k,p,r}(x|q) \;=\; E_q(x). \tag{3.5.12}$$

Similarly the choice $\nu = 1/2$ in (3.5.10), (3.5.11) and (3.5.12) gives rise to the summation formulas involving the p-deformed q-Legendre polynomial.

For deducing such sums involving the p-deformed Askey-Wilson polynomials and the p-deformed q-Racah polynomials, we need p-deformation of the q-Gauss sum and the corresponding q-Chu Vandermonde's sum [19, Eq(1.5.1) and (1.5.2), p. 10-11]:

$$
{}_2\phi_1\left(a, b; c; q, \frac{c}{ab}\right) = \frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty}
$$

and

$$
{}_2\phi_1\left(q^{-n}, b; c; q, \frac{cq^n}{b}\right) = \frac{(c/b;q)_n}{(c;q)_n},
$$

respectively. For that we notice the relation $(a;q)_{n,p} = (a;q^p)_n,\ p > 0$, and thereby transform these sums to the forms:

$$
{}_2\phi_1\left(a, b; c; q^p, \frac{c}{ab}\right) = \frac{(c/a;q)_{\infty,p}\,(c/b;q)_{\infty,p}}{(c;q)_{\infty,p}\,(c/ab;q)_{\infty,p}}. \tag{3.5.13}
$$

and

$$
{}_2\phi_1\left(q^{-np}, b; c; q^p, \frac{cq^{np}}{b}\right) = \frac{(c/b;q)_{n,p}}{(c;q)_{n,p}}. \tag{3.5.14}
$$

Now multiply both sides of (3.4.7) by

$$
\frac{q^{n(a+b-(a+cos\theta+isin\theta+a+cos\theta-isin\theta))}}{(q^p;q)_{n,p}} = \frac{q^{n(b-a-2cos\theta)}}{(q^p;q)_{n,p}}
$$

and then taking sum from $n = 0$ to ∞, we get

$$
\sum_{n=0}^{\infty} \frac{(ac;q)_{n,p}(ad;q)_{n,p}}{(q^p;q)_{n,p}} q^{n(b-a-2cos\theta)} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abcdq^{-p+kr})(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}}
$$
$$
\times \frac{a^k\ p_{k,p,r}(cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} = \sum_{n=0}^{\infty} \frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}} q^{n(b-a-2cos\theta)}.
$$

Here applying (3.5.13), this becomes

$$
\sum_{n=0}^{\infty} \frac{(ac;q)_{n,p}(ad;q)_{n,p}}{(q^p;q)_{n,p}} q^{n(b-a-2cos\theta)} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abcdq^{-p+kr})(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}}
$$
$$
\times \frac{a^k\ p_{k,p,r}(cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} = \frac{\left(be^{-i\theta};q\right)_{\infty,p}\left(be^{i\theta};q\right)_{\infty,p}}{(ab;q)_{\infty,p}\left(q^{b-a-2cos\theta};q\right)_{\infty,p}}.
$$

Next, re-writing (3.4.7) in the form:

$$\frac{(ae^{i\theta};q)_{n,p}}{(ab;q)_{n,p}} = \frac{(ac;q)_{n,p}(ad;q)_{n,p}}{(ae^{-i\theta};q)_{n,p}} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abcdq^{-p+kr})(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}} \times \frac{a^k\ p_{k,p,r}(cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k},$$

and then multiply by $(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n/((q^p;q)_{n,p})$ and taking the sum from $n=0$ to j, we find

$$\sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}}(q^{jp}be^{-i\theta})^n \sum_{k=0}^{n} \frac{q^{nk(r-p)}(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}} \times \frac{(1-abcdq^{-p+kr})a^k\ p_{k,p,r}(cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} = \sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(ae^{i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}(q^{jp}be^{-i\theta})^n.$$

This in view of (3.5.14) takes the form:

$$\sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}}(q^{jp}be^{-i\theta})^n \sum_{k=0}^{n} \frac{q^{nk(r-p)}(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}} \times \frac{(1-abcdq^{-p+kr})a^k\ p_{k,p,r}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} = \frac{\left(be^{-i\theta};q\right)_{j,p}}{(ab;q)_{j,p}}.$$

Now, from the inverse series (3.4.9) of p-deformed q-Racah polynomials, we have

$$\frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}} = (bdq^p;q)_{n,p}(cq^p;q)_{n,p} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}} \times \frac{(q^{-n(r-p)};q^{r-p})_k}{(q^{r-p};q^{r-p})_k} R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q);$$

which on multiplication by $q^{n(a-c-d)}/(q^p;q)_{n,p}$ and then taking sum from $n=0$ to ∞, yields the form:

$$\sum_{n=0}^{\infty} \frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(q^p;q)_{n,p}} q^{n(a-c-d)} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k} \times R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q) = \sum_{n=0}^{\infty} \frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(q^p;q)_{n,p}} q^{n(a-c-d)}.$$

Once again using (3.5.13), we obtain

$$\sum_{n=0}^{\infty} \frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(q^p;q)_{n,p}} q^{n(a-c-d)} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k}$$
$$\times R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q) = \frac{(aq^{x+p};q)_{\infty,p}\left(aq^{-c-d-x};q\right)_{\infty,p}}{(aq^p;q)_{\infty,p}\left(aq^{-c-d};q\right)_{\infty,p}}.$$

Re-writing the inverse series (3.4.9) as

$$\frac{(q^{-x};q)_{n,p}}{(aq^p;q)_{n,p}} = \frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(cdq^{x+p};q)_{n,p}} \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k}$$
$$\times R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q),$$

and multiplying by $(q^{-jp};q)_{n,p}(axq^{jp+p})^n/((q^p;q)_{n,p})$, and then taking sum from $n=0$ to j, yields

$$\sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}}(axq^{jp+p})^n \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}}$$
$$\times \frac{(q^{-n(r-p)};q^{r-p})_k}{(q^{r-p};q^{r-p})_k} R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q)$$
$$= \sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(q^{-x};q)_{n,p}}{(aq^p;q)_{n,p}(q^p;q)_{n,p}}(axq^{jp+p})^n.$$

The formula (3.5.14) simplifies this in the form:

$$\sum_{n=0}^{j} \frac{(q^{-jp};q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}}(axq^{jp+p})^n \sum_{k=0}^{n} \frac{q^{nk(r-p)}(1-abq^{p+kr})}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}}$$
$$\times \frac{(q^{-n(r-p)};q^{r-p})_k}{(q^{r-p};q^{r-p})_k} R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q) = \frac{(aq^{x+p};q)_{j,p}}{(aq^p;q)_{j,p}}.$$

Similarly one can derive other summation formulas using inverse series relation of these p-polynomials and its particular cases.

3.6 Companion matrix

Taking $\lfloor n/m \rfloor = N$ in (3.1.7) and converting it to the monic form, denoted by $\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)$, we get

$$\widetilde{P}_{n,p}(m,x,\gamma,s,c|q) = \sum_{k=0}^{N} \delta_k \; x^{n-mk},$$

where

$$\delta_k = \gamma^k \frac{(q^{mr-p};q^{mr-p})_n (q^{s-nr+mrk-kp+p};q)_{\infty,p} (1-q^c)^{mk-k} (1-q)^k}{(q^{s-nr+mrk+p};q)_{\infty,p} (q^{mr-p};q^{mr-p})_{n-mk} (q^{mr-p};q^{mr-p})_k} (q^m-1)^{-mk}.$$

With this δ_k, $C\left(\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)\right)$ assumes the form as stated in Definition 1.3.1. The eigen values of this matrix will be the zeros of $\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)$, (see [48, p. 39]).

We shall revisit some alternative forms of the general inversion pair of this chapter for the purpose of deducing the p-version of basic versions of Riordan's inverse pairs belonging to the table 1.7 to table 1.13 (which are listed in chapter 1) in chapter 9.

BASIC POLYNOMIALS' REDUCIBILITY