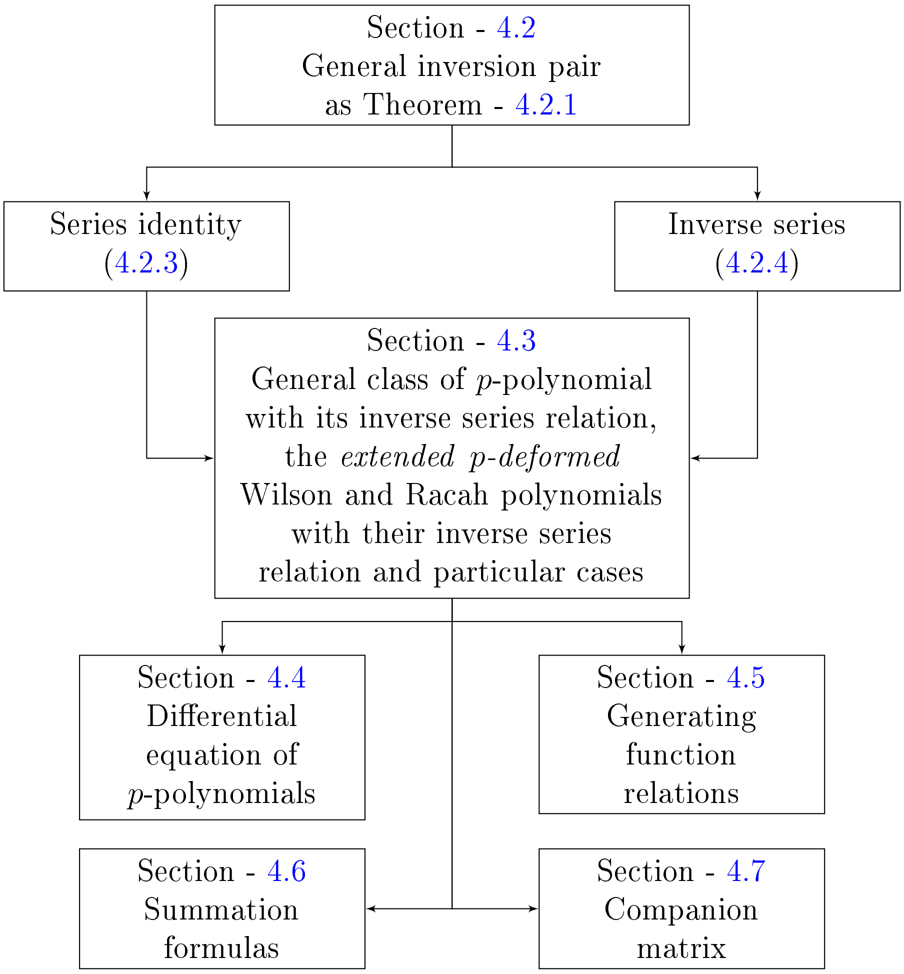
Chapter 4

The p-deformed polynomials' system - II



4.1 Introduction

The objective of this chapter is to provide extension to the p-deformed Wilson polynomials (2.3.18) and the p-deformed Racah polynomials (2.3.19) of chapter 2 by considering the degree $\lfloor n/m \rfloor$ in stead of n. One such polynomial is the extended Jacobi polynomial [41]

$$\mathcal{F}^{(e)}_{n,l,m}[(\alpha);(\beta):x] \;=\; \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(e+n)_{lk}(\alpha_1)_k\cdots(\alpha_c)_k}{(\beta_1)_k\cdots(\beta_d)_k\; k!}\, x^k \tag{4.1.1}$$

due to H. M. Srivastava. Here (α) represents the array of c parameters $\alpha_1, \alpha_2, \ldots, \alpha_c$ and similarly (β) represents the array of d parameters $\beta_1, \beta_2, \ldots, \beta_d$.

The special cases of this polynomials are the Brafman polynomial:

$$B_n^m[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(\alpha_1)_k\cdots(\alpha_c)_k}{(\beta_1)_k\cdots(\beta_d)_k\ k!}\ x^k, \tag{4.1.2}$$

the extended Konhauser polynomial for $s\in\mathbb{N}$:

$$Z_{n,m}^{(\alpha)}(x;s) = \frac{(1+\alpha)_{ns}}{n!}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-n)_{mk}}{(1+\alpha)_{sk}k!}x^{sk} \tag{4.1.3}$$

and the evident case, the extended Laguerre polynomial:

$$L_{n,m}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-n)_{mk}}{(1+\alpha)_k k!}x^k. \tag{4.1.4}$$

Here, we introduce a general class of p-deformed polynomials below.

Definition 4.1.1. *For $a,\ l\in\mathbb{C},\ m\in\mathbb{N}, n\in\mathbb{N}\cup\{0\}$ and $p>0$,*

$$\mathcal{B}_{n,m,p}^{a}(x;l) = \sum_{k=0}^{\lfloor n/m\rfloor}(-n)_{mk}(a+np)_{lk,p}\gamma_k x^k. \tag{4.1.5}$$

in which the floor function $\lfloor r\rfloor = floor\ r$, represents the greatest integer $\leq r$.

This class include the above mentioned polynomials for $p=1$ and for the appropriate choice of a,l and γ_k.

The proposed extensions to the p-deformed Wilson polynomials and the p-deformed Racah polynomials will be obtained by establishing a general inversion pair (GIP) which would besides these two polynomials, also invert the above stated general class of polynomials. The inverse series of the polynomials (4.1.1), (4.1.2), (4.1.3) and (4.1.4) are known in the following forms [9].

$$\frac{(\alpha_1)_n\cdots(\alpha_c)_n(mn)!}{(\beta_1)_n\cdots(\beta_d)_n n!}x^n = \sum_{k=0}^{mn}\frac{(-mn)_k(e+k+lk/m)}{(e+k)_{ln+1}k!}\ \mathcal{F}_{k,l,m}^{(e)}[(\alpha);(\beta):x]$$

see [9, Eq.(4.1.5), p. 80] ; and

$$\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_c)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_d)_n n!}x^n = \sum_{k=0}^{nm}\frac{(-1)^k}{(mn-k)!k!}\ B_k^m[(\alpha);(\beta):x],$$

$$x^{sn} = \sum_{k=0}^{nm} \frac{(-1)^k n!(1+\alpha)_{sn}}{(1+\alpha)_{sk}(mn-k)!} \; Z_{k,m}^{(\alpha)}(x;s),$$

and

$$x^{n} = \sum_{k=0}^{nm} \frac{(-1)^k n!(1+\alpha)_{n}}{(1+\alpha)_{k}(mn-k)!} \; L_{k,m}^{(\alpha)}(x).$$

The inverse series of the general class (4.1.5) will be derived from the GIP which will be proved in section - 4.2. Section - 4.3 contains the inverse series of all the polynomials that are belonging to the GIP. The differential equation of these polynomials are derived in section - 4.4. The generating function relations and the summation formulas are derived in section - 4.5 and section - 4.6 respectively and finally, the Companion matrix representation for the p-deformed monic polynomials will be illustrated in section - 4.7.

4.2 Inverse series relations

The general inversion pair (GIP) which will be proved here uses the following lemma in its proof.

Lemma 4.2.1. *For $M \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $p > 0$,*

$$\begin{aligned} g(M) &= \sum_{k=0}^{M} (-1)^k \binom{M}{k} (\alpha + k\lambda + mj\lambda - kp - mjp) \\ &\quad \times \Gamma_p(\alpha + (M+mj)\lambda - kp - mjp) f(k) \end{aligned} \tag{4.2.1}$$

$$\Leftrightarrow$$

$$f(M) = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \frac{1}{\Gamma_p(\alpha + k\lambda + mj\lambda + p - (M+mj)p)} \; g(k). \tag{4.2.2}$$

Proof. We first note that the diagonal elements of the coefficient matrix of the first series are $(-1)^i(\alpha + i\lambda + mj\lambda - ip - mjp)\Gamma_p(\alpha + (i+mj)\lambda - kp - mjp)$ and those of the second series are

$$(-1)^i \frac{1}{\Gamma_p(\alpha + i\lambda + mj\lambda + p - (i+mj)p)}.$$

Since these elements are all non zero; it follows that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We prefer to show that (4.2.1) implies (4.2.2). For that we denote the right hand side of (4.2.2) by $\Phi(M)$ and substitute for $g(k)$ from (4.2.1) to get

$$\begin{aligned}
\Phi(M) &= \sum_{k=0}^{M}(-1)^k\binom{M}{k}\frac{1}{\Gamma_p(\alpha+k\lambda+mj\lambda+p-(M+mj)p)}\sum_{i=0}^{k}(-1)^i\binom{k}{i} \\
&\quad\times(\alpha+i\lambda+mj\lambda-ip-mjp)\Gamma_p(\alpha+(k+mj)\lambda-ip-mjp)f(i) \\
&= \sum_{k=0}^{M}\sum_{i=0}^{k}(-1)^{k+i}\binom{M}{k}\binom{k}{i}\frac{\Gamma_p(\alpha+(k+mj)\lambda-ip-mjp)}{\Gamma_p(\alpha+k\lambda+mj\lambda+p-(M+mj)p)} \\
&\quad\times(\alpha+i\lambda+mj\lambda-ip-mjp)f(i) \\
&= \sum_{i=0}^{M}\sum_{k=0}^{M-i}(-1)^{k}\binom{M}{k+i}\binom{k+i}{i}\frac{\Gamma_p(\alpha+(k+i+mj)\lambda-ip-mjp)}{\Gamma_p(\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p)} \\
&\quad\times(\alpha+i\lambda+mj\lambda-ip-mjp)f(i) \\
&= \sum_{i=0}^{M}\binom{M}{i}(\alpha+i\lambda+mj\lambda-ip-mjp)f(i)\sum_{k=0}^{M-i}(-1)^k\binom{M-i}{k} \\
&\quad\times\frac{\Gamma_p(\alpha+(k+i+mj)\lambda-ip-mjp)}{\Gamma_p(\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p)}.
\end{aligned}$$

Here, the ratio

$$\frac{\Gamma_p(\alpha+(k+i+mj)\lambda-ip-mjp)}{\Gamma_p(\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p)} = \sum_{l=0}^{M-i-1} A_l\, k^l$$

say, which represents a polynomial of degree $M-i-1$ in k, hence we further have

$$\begin{aligned}
\Phi(M) &= f(M)+\sum_{i=0}^{M}\binom{M}{i}(\alpha+i\lambda+mj\lambda-ip-mjp)f(i)\sum_{l=0}^{M-i-1} A_l \\
&\quad\times\sum_{k=0}^{M-i}(-1)^k\binom{M-i}{k}k^l.
\end{aligned}$$

Now, if $P(a+bk)$ is a polynomial in k of degree less than N, then

$$\sum_{k=0}^{N}(-1)^k\binom{N}{k}P(a+bk) = 0.$$

Thus, we get $\Phi(M)=f(M)$. This completes the proof of the inverse pair. □

This lemma gives rise to the *orthogonality relation*. In fact, the substitution $\binom{0}{M}$ for either $f(M)$ or $g(M)$ yields this property. One of the orthogonality relations which we later need, is recorded as

Corollary 4.2.1. *For $0 \leq j \leq n,\ m \in \mathbb{N},\ \lambda \in \mathbb{C}$ and $p > 0$,*

$$\binom{0}{M} = \sum_{k=0}^{M}(-1)^k \binom{M}{k} \frac{(\alpha + k\lambda + mj\lambda - kp - mjp)}{\Gamma_p(\alpha + mj\lambda + p - kp - mjp)} \Gamma_p(\alpha + mn\lambda - kp - mjp).$$

Proof. In (4.2.1), the substitution $g(k) = \binom{0}{k}$ gives $f(k) = 1/(\Gamma_p(\alpha + mj\lambda + p - kp - mjp))$, and with these $f(k)$ and $g(k)$, (4.2.2) yields the series orthogonality relation. $\square$

As a main result, we now establish the GIP as

Theorem 4.2.1. *For $\lambda \in \mathbb{C},\ \alpha \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N}$ and $p > 0$,*

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \frac{1}{\Gamma_p(\alpha + mk\lambda + p - np)(n - mk)!} G(k) && (4.2.3) \\ &\Rightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} F(k), && (4.2.4) \end{aligned}$$

and conversely, the series in (4.2.4) implies the series (4.2.3) if for $n \neq mr,\ r \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + n\lambda - kp)}{(n - k)!} F(k) = 0. \qquad (4.2.5)$$

Proof. We first show that (4.2.3)⇒(4.2.4). We denote the right hand side of (4.2.4) by $V(n)$ and then substitute for $F(k)$ from (4.2.3) to get

$$\begin{aligned} V(n) &= \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} F(k) \\ &= \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} \\ &\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj} \frac{1}{\Gamma_p(\alpha + mj\lambda + p - kp)(k - mj)!} G(j). \end{aligned}$$

Here making use of the double series relation [62]:

$$\sum_{k=0}^{mn} \sum_{j=0}^{\lfloor k/m \rfloor} A(k, j) = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k + mj, j),$$

we further get

$$\begin{aligned}
V(n) &= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}(-1)^k\frac{(\alpha+(k+mj)\lambda-kp-mjp)\Gamma_p(\alpha+mn\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)(mn-k-mj)!k!}\\
&\quad\times G(j)\\
&= \sum_{j=0}^{n}\frac{G(j)}{(mn-mj)!}\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{\Gamma_p(\alpha+mn\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}\\
&\quad\times(\alpha+(k+mj)\lambda-kp-mjp)\\
&= \sum_{j=0}^{n-1}\frac{G(j)}{(mn-mj)!}\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{\Gamma_p(\alpha+mn\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}\\
&\quad\times(\alpha+(k+mj)\lambda-kp-mjp)+G(n). \qquad (4.2.6)
\end{aligned}$$

We now show that the inner series in this last expression vanishes. For that we replace $1/\Gamma_p(\alpha+mj\lambda+p-kp-mjp)$ by $f(k)$ and denote the inner series by $g(mn-mj)$, then we have

$$\begin{aligned}
g(mn-mj) &= \sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}(\alpha+k\lambda+mj\lambda-kp-mjp)\\
&\quad\times\Gamma_p(\alpha+mn\lambda-kp-mjp)f(k). \qquad (4.2.7)
\end{aligned}$$

The inverse series of this series follows from Lemma 4.2.1 in the form:

$$f(mn-mj)=\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{1}{\Gamma_p(\alpha+k\lambda+mj\lambda+p-mnp)}\,g(k). \qquad (4.2.8)$$

According to Corollary - 4.2.1, we set $g(k)=\binom{0}{k}$ in series (4.2.8), we then get $f(k)=1/\Gamma_p(\alpha+mj\lambda+p-kp-mjp)$ back, and with these $f(k)$ and $g(k)$, the series orthogonality relation occurs from (4.2.9) as given below.

$$\begin{aligned}
\binom{0}{mn-mj} &= \sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{(\alpha+k\lambda+mj\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}\\
&\quad\times\Gamma_P(\alpha+mn\lambda-kp-mjp). \qquad (4.2.9)
\end{aligned}$$

Using this in (4.2.6), we get

$$V(n)=G(n)+\sum_{j=0}^{n-1}\frac{G(j)}{(mn-mj)!}\binom{0}{mn-mj}=G(n).$$

Thus, (4.2.3)⇒(4.2.4). Our next aim is to show that (4.2.3)⇒(4.2.5). For that let $R(n)$ denote the right hand side of (4.2.5) that is,

$$R(n) \quad = \quad \sum_{k=0}^{n}(-1)^k \frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!}\; F(k). \qquad (4.2.10)$$

Proceeding as before, that is, substituting for $F(k)$ from (4.2.3), we have

$$\begin{aligned} R(n) \quad &= \quad \sum_{k=0}^{n}(-1)^k \frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!} \\ &\quad \times \sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}\frac{1}{\Gamma_p(\alpha+mj\lambda+p-kp)(k-mj)!}\; G(j) \\ &= \sum_{j=0}^{\lfloor n/m\rfloor}\sum_{k=0}^{n-mj}(-1)^k \frac{\Gamma_p(\alpha+n\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)(n-mj-k)!\;k!} \\ &\quad \times(\alpha+k\lambda+mj\lambda-kp-mjp)\; G(j) \\ &= \sum_{j=0}^{\lfloor n/m\rfloor}\frac{G(j)}{(n-mj)!}\sum_{k=0}^{n-mj}(-1)^k\binom{n-mj}{k}\frac{\Gamma_p(\alpha+n\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}. \\ &\quad \times(\alpha+k\lambda+mj\lambda-kp-mjp). \qquad (4.2.11) \end{aligned}$$

We see that the inner series on the right hand side in this last expression differs slightly from the one occurring in (4.2.6); that is, instead of $mn-mj$, it is $n-mj$ here. Accordingly, the series orthogonality relation occurs in the form:

$$\begin{aligned} \sum_{k=0}^{n-mj}(-1)^k\binom{n-mj}{k}\frac{(\alpha+k\lambda+mj\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}\Gamma_p(\alpha+n\lambda-kp-mjp) \\ = \binom{0}{n-mj}. \end{aligned}$$

This leads us to

$$R(n) \quad = \quad \sum_{j=0}^{\lfloor n/m\rfloor}\frac{G(j)}{(n-mj)!}\binom{0}{n-mj}.$$

If $n \neq mr, r \in \mathbb{N}$, then the right hand member in (4.2.11) vanishes and thus (4.2.3)⇒(4.2.5); which completes the proof of the first part. For the converse part, assume that (4.2.4) and (4.2.5) both hold true. In view of (4.2.5),

$$R(n) = 0,\;\; n \neq mr,\;\; r \in \mathbb{N}, \qquad (4.2.12)$$

and also,

$$R(mn) = G(n) \tag{4.2.13}$$

by comparing (4.2.4) with (4.2.10). Now, from the inverse pair (4.2.7) and (4.2.8), taking $j = 0$ and $m = 1$, we find that

$$\begin{aligned} R(n) &= \sum_{k=0}^{n} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + n\lambda - kp)}{(n-k)!} F_k \\ \Rightarrow & \\ F_n &= \sum_{k=0}^{n} (-1)^k \frac{1}{\Gamma_p(\alpha + k\lambda + p - np)(n-k)!} R(k). \end{aligned}$$

Hence, in view of the relations (4.2.12) and (4.2.13), we arrive at

$$\begin{aligned} R(mn) &= \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn-k)!} F_k \\ \Rightarrow & \\ F_n &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \frac{1}{\Gamma_p(\alpha + mk\lambda + p - np)(n-mk)!} R(mk). \end{aligned}$$

Thus, the series in (4.2.4) with $R(n) = 0, n \neq mr$ for $r \in \mathbb{N}$, implies the series in (4.2.3). This proves the converse part and hence the theorem. □

4.3 Particular cases

We first deduce the general class (4.1.5) and its inverse series from the theorem. For that the GIP of the theorem needs to be transformed sequentially as follows. While obtaining such transformed pairs, we shall assume that the condition (4.2.5) holds true which will not be mentioned in each of the inverse pairs below.

The first transformed version is deduced by taking $\alpha = -c$, $\lambda = -(r-m)p/m$ and replacing $G(n)$ by $G(n)\Gamma_p(-c - rnp + p)$ in Theorem- 4.2.1 to get

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{n-mk} \frac{\Gamma_p(p - c - rkp)}{\Gamma_p(p - c - rkp - np - mkp)\ (n-mk)!} G(k), \tag{4.3.1}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} \frac{(c + rkp/m)\ \Gamma_p(p - c - (r-m)np - kp)}{(c + (r-m)np + kp)\ \Gamma_p(p - c - (r-m)np - mnp)(mn-k)!} F(k). \tag{4.3.2}$$

On making use of the property (1.3.6) with appropriate values of n, k and z in (4.3.2) leads us to

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(c+rkp)_{n-mk,p}}{(n-mk)!} G(k), \tag{4.3.3}$$
$$\Leftrightarrow$$
$$G(n) = \sum_{k=0}^{mn} \frac{(-c-rkp/m)\ (p-c-rnp)_{mn-k-1,p}}{(mn-k)!} F(k), \tag{4.3.4}$$

or in more simplified form:

$$G(n) = \sum_{k=0}^{mn} \frac{(-1)^{mn-k}(c+rkp/m)(c)_{rn,p}}{(c)_{(r-m)n+k+1,p}\ (mn-k)!} F(k). \tag{4.3.5}$$

Now, if we put $r = l + m$, replace $G(n)$ by $(-1)^{mn}(c)_{ln+mn,p}G(n)$ and $F(n)$ by $F(n)/n!$ in (4.3.3), (4.3.5), then we get

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk}\ n!\ (c)_{lk+mk,p}\ (c+lkp+mkp)_{n-mk,p}}{(n-mk)!} G(k),$$
$$\Leftrightarrow$$
$$G(n) = \sum_{k=0}^{mn} \frac{(-1)^{k}\ (c+kp+lkp/m)}{(c)_{ln+k+1,p}\ (mn-k)!\ k!} F(k).$$

More elegant form of this pair is given by

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-n)_{mk}\ (c)_{n+lk,p}\ G(k); \quad G(n) = \sum_{k=0}^{mn} \frac{(-mn)_k\ (c+kp+lkp/m)}{(c)_{ln+k+1,p}\ (mn)!\ k!} F(k).$$

Finally, replacing $F(n)$ by $(c)_{n,p}F(n)$, this gives

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-n)_{mk}\ (c+np)_{lk,p}\ G(k); \tag{4.3.6}$$
$$\Leftrightarrow$$
$$G(n) = \sum_{k=0}^{mn} \frac{(-mn)_k\ (c+kp+lkp/m)}{(c+kp)_{ln+1,p}\ (mn)!\ k!} F(k). \tag{4.3.7}$$

This inverse pair is used to derive the inverse series of the general class (4.1.5) and its particular cases. Moreover, this pair provides the extension of p-deformed Wilson polynomials and p-deformed Racah polynomials along with their inverse series relation in p-gamma and Pochhammer p-symbol[17]. Now, put $c = a$ and $G(n) = \gamma_n x^n$ in (4.3.7), then $F(n) = \mathcal{B}^{a}_{n,m,p}(x; l)$ yields (4.1.5) and its inverse series:

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{(-mn)_k \ (a+kp+lkp/m)}{(a+kp)_{ln+1,p} \ (mn)! \ k!} \mathcal{B}^a_{k,m,p}(x;l). \tag{4.3.8}$$

The p-deformation of the extended Jacobi polynomial and its inverse series can be obtained from (4.1.5) and (4.3.8) by taking $\gamma_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}$ $/((\beta_1)_{n,p}\cdots \ (\beta_d)_{n,p} \ n!)$. The inverse pair thus formed, is given by

$$\mathcal{F}^{(a)}_{n,l,m,p}[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(a+np)_{lk,p}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p} \ k!} x^k \tag{4.3.9}$$

$$\Leftrightarrow$$

$$\frac{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p} \ (mn)!}{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p} \ n!} x^n = \sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p} k!} \mathcal{F}^{(a)}_{k,l,m,p}[(\alpha);(\beta):x]. \tag{4.3.10}$$

The extended Jacobi polynomial $\mathcal{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]$ and its inverse series occur when $p=1$ [9, Eqs. (4.1.4) and (4.1.5), p. 80]. In this pair, putting $l=0$, we obtain the p-deformed Brafman polynomial and its inverse series as given below.

$$B^m_{n,p}[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p} \ k!} x^k, \tag{4.3.11}$$

$$\Leftrightarrow$$

$$\frac{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}(mn)!}{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p} n!} x^n = \sum_{k=0}^{mn} \frac{(-mn)_k}{k!} B^m_{k,p}[(\alpha);(\beta):x]. \tag{4.3.12}$$

Let us consider a generalized p-deformed extended Jacobi polynomial given by (cf. [53, p.254] with $m=l=p=1$)

$$P^{(\alpha,\beta)}_{n,l,m,p}(x) = \frac{(1+\alpha)_{n,p}}{n!} \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(\alpha+\beta+np+1)_{lk,p}}{(1+\alpha)_{k,p} \ k!} \left(\frac{1-x}{2}\right)^k. \tag{4.3.13}$$

This is a particular case $a=\alpha+\beta+1$, $\gamma_n = 1/((1+\alpha)_{n,p}n!)$ and x is replaced by $(1-x)/2$ of (4.1.5). Then from (4.3.8), we immediately obtain its inverse series:

$$\frac{(1-x)^n}{(1+\alpha)_{n,p}2^n n!} = \sum_{k=0}^{mn} \frac{(-mn)_k(\alpha+\beta+kp+lkp/m+1)}{(\alpha+\beta+kp+1)_{ln+1,p}(1+\alpha)_{k,p}(mn)!} P^{(\alpha,\beta)}_{k,l,m,p}(x). \tag{4.3.14}$$

The extended p-deformed Konhauser polynomial and its inverse series can be obtained from (4.1.5) and (4.3.8) by taking $l = 0$, $\gamma_n = 1/((p+\alpha)_{sn,p}\ n!)$ and replacing x by $x^s, s \in \mathbb{N}$. Then $\mathcal{B}^a_{n,m,p}(x;0) = n! Z^{(\alpha)}_{n,m,p}(x;s)/(p+\alpha)_{sn,p}$ yields

$$\begin{aligned} Z^{(\alpha)}_{n,m,p}(x;s) &= \frac{(p+\alpha)_{sn,p}}{n!} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{sk,p}\ k!}\ x^{sk}, && (4.3.15) \\ \Leftrightarrow & \\ x^{sn} &= \sum_{k=0}^{mn} \frac{(-1)^k (p+\alpha)_{sn,p}\ n!}{(p+\alpha)_{sk,p}\ (mn-k)!}\ Z^{(\alpha)}_{k,m,p}(x;s). && (4.3.16) \end{aligned}$$

In this last inverse pair, taking $s = 1$, we readily get the extended p-deformed Laguerre polynomial(cf. [53, p. 201, 207]) and its inverse in the forms:

$$\begin{aligned} L^{(\alpha)}_{n,m,p}(x) &= \frac{(p+\alpha)_{n,p}}{n!} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{k,p}\ k!}\ x^{k}, && (4.3.17) \\ \Leftrightarrow & \\ x^{n} &= \sum_{k=0}^{mn} \frac{(-1)^k (p+\alpha)_{n,p}\ n!}{(p+\alpha)_{k,p}\ (mn-k)!}\ L^{(\alpha)}_{k,m,p}(x). && (4.3.18) \end{aligned}$$

The well known generalized orthogonal polynomials possessing the higher order hypergeometric function form ${}_4F_3[*]$ are the Wilson polynomials [35, 67, 69] and the Racah polynomials (or Racah coefficient or 6-j symbols) [19, Eq.(7.2.16), p. 165] (also [35, 67]). These polynomials encompass several particular polynomials such as the polynomials of Jacobi, Hahn, continuous Hahn, continuous dual Hahn, Meixner, Meixner-Pollaczek, Krawtchouk and Charlier (see [35, Askey-Scheme, p.23] for interconnections chart). It is interesting to see that both these polynomials along with their inverse series relation are contained in the inverse pair (4.3.6) and (4.3.7) with $p = 1$. In fact, It provides p-deformation to both these polynomials together with their inverse series.

The *extended p-deformed* Wilson polynomials and its inverse occur by replacing c by $a+b+c+d-p$ and choosing

$$G(n) = (a+ix)_{n,p}\ (a-ix)_{n,p}/((a+b)_{n,p}\ (a+c)_{n,p}\ (a+d)_{n,p} n!).$$

Then

$$F(n) = W_{n,l,m,p}(x^2;a,b,c,d)/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p})$$

in the inverse pair (4.3.6) and (4.3.7) gives rise to the pair:

$$\frac{W_{n,l,m,p}(x^2;a,b,c,d)}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}} = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(a+b+c+d+np-p)_{lk,p}}{(a+b)_{k,p}(a+c)_{k,p}}$$

$$\times \frac{(a+ix)_{k,p}(a-ix)_{k,p}}{(a+d)_{k,p}k!}, \tag{4.3.19}$$

$$\Leftrightarrow$$

$$\frac{(a+ix)_{n,p}(a-ix)_{n,p}(mn)!}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!} = \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+c+d-p+kp+lkp/m)}{(a+b+c+d-p+kp)_{ln+1,p}} \times \frac{W_{k,l,m,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!}. \tag{4.3.20}$$

Likewise, the *extended p-deformed* Racah polynomials and its inverse series are obtainable from (4.3.6) and (4.3.7) by replacing c by $a+b+p$ and making the substitution

$$G(n) = (-x)_{n,p}(x+c+d+p)_{n,p}/((a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}n!).$$

Then with $F(n) = R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)$, yields

$$R_{n,l,m,p}(x(x+c+d+p);a,b,c,d) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}\ (a+b+np+p)_{lk,p}\ (-x)_{k,p}}{(a+p)_{k,p}\ (b+d+p)_{k,p}}, \times \frac{(x+c+d+p)_{k,p}}{(c+p)_{k,p}\ k!}, \tag{4.3.21}$$

$$\Leftrightarrow$$

$$\frac{(-x)_{n,p}(x+c+d+p)_{n,p}(mn)!}{(a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}\ n!} = \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}k!} \times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d). \tag{4.3.22}$$

Thus, the p-Wilson polynomials and the p-Racah polynomials extend all the above mentioned particular polynomials in p-gamma function and p-Pochhammer symbol.
As mentioned in the beginning of this section, we remind that the condition (4.2.5) holds for each of the above inverse pairs.
The series identities (4.3.6) and (4.3.7) are used to derive certain generating function relations and some summation formulas involving p-polynomials respectively.

4.4 Differential equations

With the help of (1.3.11), we derive the differential equation satisfied by the special case of (4.1.5) corresponding to the choice $\gamma_k = 1/((b_1)_{k,p}(b_2)_{k,p}\cdots(b_r)_{k,p}k!)$ and $l \in \mathbb{N}$. The particular polynomial thus obtained is denoted by $V^m_{n,p}(x;l)$ whose explicit series representation is given by

$$\begin{aligned} V^m_{n,p}(x;l) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\left\{ \prod_{i=1}^{m} \left(\frac{-np+ip-p}{m} \right)_{k,p} \prod_{j=1}^{l} \left(\frac{a+np+jp-p}{l} \right)_{k,p} \right\}}{(b_1)_{k,p}(b_2)_{k,p}\cdots(b_r)_{k,p}\ k!} \\ &\quad \times \left(m^m l^l x p^{-m}\right)^k \\ &= {}_{m+l}F_r\left((\triangle_p(m;-np), \triangle_p(l;a+np)), p, (b_1, b_2, \cdots b_r), p\right)\left(m^m l^l x p^{-m}\right). \end{aligned}$$

Now comparing this with (1.3.11), we obtain in straight forward manner, the differential equation:

$$\left[D\left\{ \prod_{i=1}^{r} (pD + b_i - p) \right\} - x \prod_{j=1}^{m} \prod_{k=1}^{l} \left\{ (mD - n + j - 1)(lpD + a + np + kp - p) \right\} \right] \times V^m_{n,p}(x;l) = 0,$$

This may be further reduced to the differential equations satisfied by the p-deformed extended Jacobi polynomial and the extended p-deformed Konhauser polynomial along with their particular cases. The explicit representation of the p-deformed extended Jacobi polynomial (4.3.9) is given by

$$\begin{aligned} \mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x] &= \sum_{k=0}^{\lfloor n/m \rfloor} \prod_{i=1}^{m} \left(\frac{-np+ip-p}{m} \right)_{k,p} \prod_{j=1}^{l} \left(\frac{e+np+jp-p}{l} \right)_{k,p} \\ &\quad \times \frac{(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\ k!} \left(m^m l^l x p^{-m}\right)^k \end{aligned}$$

which satisfies differential equation

$$\begin{aligned} \Bigg[D\left\{ \prod_{g=1}^{d} (pD + \beta_g - p) \right\} - x \prod_{i=1}^{m} \prod_{j=1}^{l} \prod_{h=1}^{c} \Bigg\{ (mD - n + i - 1) \\ \times (lpD + e + np + jp - p)(pD + \alpha_h) \Bigg\} \Bigg] y = 0. \end{aligned} \tag{4.4.1}$$

We know that the case $l = 0$ is the p-deformed Brafman polynomial (4.3.11) whose differential equation occurs form (4.4.1) by taking $l = 0$, given by

$$\left[D\left\{ \prod_{s=1}^{d} (pD + \beta_s - p) \right\} - x \prod_{i=1}^{m} \prod_{r=1}^{c} \left\{ (mD - n + i - 1)(pD + \alpha_r) \right\} \right] \times B^m_{n,p}(\alpha);(\beta):x] = 0.$$

We note that the explicit representation of the extended p-deformed Konhauser polynomial(4.3.15) with the help of (1.3.7), is given by

$$\begin{aligned} Z^{(\alpha)}_{n,m,p}(x;l) &= \frac{(p+\alpha)_{ln,p}}{n!}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{\left\{\prod_{i=1}^{m}\left(\frac{-np+ip-p}{m}\right)_{k,p}\right\}}{\left\{\prod_{j=1}^{l}\left(\frac{\alpha+jp}{l}\right)_{k,p}k!\right\}}\left(\frac{m^m x^l p^{-m}}{l^l}\right)^k \\ &= \frac{(p+\alpha)_{ln,p}}{n!}{}_mF_l\left((\triangle_p(m;-np),p,(\triangle_p(l;\alpha+p),p)\left(\frac{m^m x^l p^{-m}}{l^l}\right),\right. \end{aligned}$$

whose p-deformed differential equation is obtained by making use of (1.3.11) as stated below.

$$\left[D\left\{\prod_{j=1}^{l}(lpD+\alpha+jp-lp)\right\}-x^l\prod_{i=1}^{m}\left\{(mD-n+i-1)\right\}\right]\frac{Z^{(\alpha)}_{n,m,p}(x;l)n!}{(p+\alpha)_{ln,p}}=0.$$

This is equivalent to

$$\left[D\left\{\prod_{j=1}^{l}(lpD+\alpha+jp-lp)\right\}-x^l\left\{\prod_{i=1}^{m}(mD-n+i-1)\right\}\right]Z^{(\alpha)}_{n,m,p}(x;l)=0.$$

Taking $l=1$, this reduces to

$$\left[D\left\{(pD+\alpha)\right\}-xp^{-m}\prod_{i=1}^{m}(pmD-np+ip-p)\right]L^{(\alpha)}_{n,m,p}(x)=0.$$

If we choose $m=1$ in this last equation, then we get the differential equation of p-deformed Laguerre polynomial as follow

$$\left[D\left\{(pD+\alpha)\right\}-x\left(D-n\right)\right]L^{(\alpha)}_{n,m,p}(x) \;=\; 0.$$

This further reduces to a differential equation of Laguerre polynomial[53] when $p=1$ as stated below.

$$x\left[\frac{d^2L^{(\alpha)}_n(x)}{dx^2}\right]+(\alpha+1-x)\frac{dL^{(\alpha)}_n(x)}{dx}+nL^{(\alpha)}_n(x) \;=\; 0.$$

4.5 Generating function relations

We recall the first series of the inverse pair (4.3.6) and derive the generating function relations (or GFR) of p-deformed general class of polynomials

$\{\mathcal{B}^{a}_{n,m,p}(x;l); n=0,1,2,\ldots\}$ defined by (4.1.5). We require the generalized p-Write function due to K. Gehlot et al.[20] which is defined by

$$ {}_q\Psi^p_r\left[\begin{array}{ll}(a_i,\alpha_i)_{1,q}; & z\\ & \\ (b_j,\beta_j)_{1,r}; & \end{array}\right] = \sum_{k=0}^{\infty}\frac{\left\{\prod_{i=1}^{q}\Gamma_p(a_i+\alpha_i k)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_p(b_j+\beta_j k)\right\}k!}z^k, \tag{4.5.1}$$

where $z\in\mathbb{C}$, $p>0$, $\alpha_i,\ \beta_j\in\mathbb{R}\backslash\{0\}$ and $a_i+\alpha_i k,\ b_j+\beta_j k\in\mathbb{C}\backslash p\mathbb{Z}^-$ for $1\le i\le q$ and $1\le j\le r$. In the notations

$$\begin{aligned}\Delta &= \sum_{j=1}^{r}\frac{\beta_j}{p}-\sum_{i=1}^{q}\frac{\alpha_i}{p}\ ;\ \delta=\left\{\prod_{j=1}^{r}\left|\frac{\beta_j}{p}\right|^{\frac{\beta_j}{p}}\right\}\left\{\prod_{i=1}^{q}\left|\frac{\alpha_i}{p}\right|^{-\frac{\alpha_i}{p}}\right\}\\ \mu &= \sum_{j=1}^{r}\frac{b_j}{p}-\sum_{i=1}^{q}\frac{a_i}{p}+\frac{q-r}{2},\end{aligned}$$

the series converges for all $z\in\mathbb{C}$ if $\Delta>-1$. If $\Delta=-1$ then the converges absolutely for $|z|<\delta$ and if $|z|=\delta$, then $\Re(\mu)>1/2$. We shall also use the generalized p-Mittag-Leffler function:

$$E^{\gamma,\,\tau}_{p,\,\alpha,\,\beta}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_{\tau k,p}}{\Gamma_p(\alpha k+\beta)k!}\,z^k, \tag{4.5.2}$$

given by R. K. Saxena et al. [59], wherein $p\in\mathbb{R}$, $\alpha,\ \beta,\ \gamma\in\mathbb{C}$; $\Re(\alpha,\beta)>0$ and $\tau\in\mathbb{C}$. This will be needed while deriving the generating function relation of the extended p-deformed Konhauser polynomial. Now in order to derive the first GFR, we take $c=a$, multiply it by $(a)_{n,p}\ t^n/n!$ with $|t|<1/p$ in (4.3.6) and then take the summation n from 0 to ∞ to get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a)_{n,p}}{n!}F(n)\ t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-n)_{mk}(a)_{n,p}(a+np)_{lk,p}}{n!}\,G(k)\ t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^{mk}\ n!\ (a)_{n+lk,p}}{(n-mk)!\ n!}\,G(k)\ t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{mk}(a)_{n+mk+lk,p}}{n!}\,G(k)\ t^{n+mk}\\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(a+mkp+lkp)_{n,p}}{n!}t^n\right)(a)_{mk+lk,p}\ G(k)\ (-t)^{mk}\end{aligned}$$

$$= \sum_{k=0}^{\infty}(1-tp)^{\frac{-a-mkp-lkp}{p}} (a)_{mk+lk,p}\ G(k)\ (-t)^{mk}$$

$$= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} (a)_{mk+lk,p}\ G(k) \left(\frac{(-t)^m}{(1-tp)^{m+l}}\right)^k .$$

Here if $G(n) = \gamma_n x^n$ then $F(n) = \mathcal{B}^a_{n,m,p}(x;l)$ leads us to the GFR of (4.1.5) in the form:

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\ \mathcal{B}^a_{n,m,p}(x;l)\ t^n = (1-tp)^{\frac{-a}{p}} \sum_{k=0}^{\infty}(a)_{mk+lk,p}\gamma_k \left(\frac{x(-t)^m}{(1-tp)^{m+l}}\right)^k . \quad (4.5.3)$$

From this, the GFR of the p-deformed extended Jacobi polynomial occurs by substituting $\gamma_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}/((\beta_1)_{n,p}\cdots\ (\beta_d)_{n,p}\ n!)$ and using the generalized p-Write function (4.5.1) in the form:

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\ \mathcal{F}^a_{n,l,m,p}[(\alpha);(\beta):x]\ t^n$$

$$= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{(m+l)k,p}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}k!} \left(\frac{x(-t)^m}{(1-tp)^{m+l}}\right)^k$$

$$= \frac{(1-tp)^{-\frac{a}{p}}\left\{\prod_{i=1}^{d}\Gamma_p(\beta_i)\right\}}{\Gamma_p(a)\left\{\prod_{j=1}^{c}\Gamma_p(\alpha_c)\right\}}$$

$$\times\ {}_{c+1}\Psi^p_d\left[\begin{array}{c} (a, mp+lp), (\alpha_1,p),\ldots,(\alpha_c,p)\,; \quad \dfrac{x(-t)^m}{(1-tp)^{m+l}} \\ (\beta_1,p),\ldots,(\beta_d,p)\,; \end{array}\right].$$

If $l \in \mathbb{N}\bigcup\{0\}$, then this further reduces to

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\ \mathcal{F}^{(a)}_{n,l,m,p}[(\alpha);(\beta):x]\ t^n$$

$$= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty}\left\{\prod_{j=1}^{m+l}\left(\frac{a+jp-p}{m+l}\right)_{k,p}\right\}\frac{(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}k!}$$

$$\times\left(\frac{x(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}}\right)^k$$

$$= (1-tp)^{-\frac{a}{p}}$$

$$\times\ {}_{m+l+c}F_d\left((\triangle_p(m+l,a),\alpha_1,\ldots,\alpha_c),p,(\beta_1,\ldots,\beta_d),p\right)\left(\frac{x(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}}\right).$$

For $l = 0$, this yields the GFR of the p-deformed Brafman polynomial as follows.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\, B^{m}_{n,p}[(\alpha);(\beta):x]\, t^n \\
&= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{mk,p}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}k!} \left(\frac{x(-t)^m}{(1-tp)^m}\right)^k \\
&= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{a+jp-p}{m}\right)_{k,p} \right\} \frac{(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}k!} \left(\frac{xm^m(-t)^m}{(1-tp)^m}\right)^k \\
&= (1-tp)^{-\frac{a}{p}}\, {}_{m+c}F_d\left(\left(\triangle_p(m,a),(\alpha)\right),p,(\beta)\right),p\right) \left(\frac{x(-mt)^m}{(1-tp)^m}\right).
\end{aligned}$$

Next, the GFR of the extended p-deformed Konhauser polynomial can be derived from (4.5.3) by taking $l = 0, \gamma_n = 1/((p+\alpha)_{ns,p}n!)$ and replacing x by $x^s, s \in \mathbb{N}$. It occurs in the form:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{sn,p}}\, Z^{(\alpha)}_{n,m,p}(x;s)\, t^n \\
&= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{mk,p}}{(p+\alpha)_{sk,p}k!} \left(\frac{x^s\,(-t)^m}{(1-tp)^m}\right)^k \\
&= (1-tp)^{-\frac{a}{p}} \frac{\Gamma_p(p+\alpha)}{\Gamma_p(a)}\, {}_1\Psi^p_1 \left[\begin{matrix} (a,mp); & \dfrac{x^s\,(-t)^m}{(1-tp)^m} \\ (p+\alpha,sp); & \end{matrix} \right].
\end{aligned}$$

Alternatively, incorporating p-Mittag-Leffler function (4.5.2), this assumes the form:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{sn,p}}\, Z^{(\alpha)}_{n,m,p}(x;s)\, t^n &= \Gamma_p(p+\alpha)(1-tp)^{-\frac{a}{p}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(a)_{mk,p}}{\Gamma_p(p+\alpha+skp)\ k!} \left(\frac{x^s\,(-t)^m}{(1-tp)^m}\right)^k \\
&= \Gamma_p(p+\alpha)\ (1-tp)^{-\frac{a}{p}} E^{a,m}_{p,\,sp,\,\alpha+p}\left(\frac{x^s(-t)^m}{(1-tp)^m}\right),
\end{aligned}$$

that is,

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{\Gamma_p(p+\alpha+snp)}\, Z^{(\alpha)}_{n,m,p}(x;s)\, t^n = (1-tp)^{-\frac{a}{p}} E^{a,m}_{p,\,sp,\,\alpha+p}\left(\frac{x^s(-t)^m}{(1-tp)^m}\right).$$

Since $s \in \mathbb{N}\bigcup\{0\}$, then this GFR can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{sn,p}}\ Z^{(\alpha)}_{n,m,p}(x;s)\ t^n \\
= &\ (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{\left\{\prod_{j=1}^{m}\left(\frac{a+jp-p}{m}\right)\right\}}{\left\{\prod_{j=1}^{s}\left(\frac{\alpha+jp}{s}\right)\right\} k!} \left(\frac{x^s(-mt)^m}{s^s(1-tp)^m}\right)^k \\
= &\ (1-tp)^{-\frac{a}{p}}\,{}_mF_s\left(\left(\triangle_p(m,a)\right),p,\left(\triangle_p(s,\alpha+p)\right),p\right)\left(\frac{x^s(-mt)^m}{s^s(1-tp)^m}\right).
\end{aligned}$$

The instance $s=1$ in all these three GFRs of the p-deformed Konhauser polynomial yield the GFR of the extended p-deformed Laguerre polynomial respectively in the forms:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{n,p}}\ L^{(\alpha)}_{n,m,p}(x)\ t^n \\
= &\ (1-tp)^{\frac{-a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{mk,p}}{(p+\alpha)_{k,p}k!}\left(\frac{x(-t)^m}{(1-tp)^m}\right)^k \\
= &\ (1-tp)^{\frac{-a}{p}} \frac{\Gamma_p(p+\alpha)}{\Gamma_p(a)}\,{}_1\Psi^p_1\left[\begin{array}{cc} (a,mp); & \frac{x(-t)^m}{(1-tp)^m} \\ (p+\alpha,p); & \end{array}\right],
\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{\Gamma_p(p+\alpha+np)}\ L^{(\alpha)}_{n,m,p}(x)\ t^n \ = \ (1-tp)^{-\frac{a}{p}} E^{a,m}_{p,\,p,\,\alpha+p}\left(\frac{x(-t)^m}{(1-tp)^m}\right)$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{n,p}}\ L^{(\alpha)}_{n,m,p}(x)\ t^n &= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{mk,p}}{(p+\alpha)_{k,p}k!}\left(\frac{x(-t)^m}{(1-tp)^m}\right)^k \\
&= (1-tp)^{-\frac{a}{p}} \\
&\quad \times {}_mF_1\left(\left(\triangle_p(m,a)\right),p,\left(\alpha+p\right),p\right)\left(\frac{x(-mt)^m}{(1-tp)^m}\right),
\end{aligned}$$

where $m=1,2$ for convergence. For other values of m, this provides the divergent generating function relations. For $m=1$, we have

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{n,p}}\ L^{(\alpha)}_{n,1,p}(x)\ t^n \ = \ (1-tp)^{-\frac{a}{p}}\,{}_1F_1\left(\left(a\right),p,\left(\alpha+p\right),p\right)\left(\frac{-xt}{(1-tp)}\right).$$

Here, $a = p + \alpha$ gives

$$\sum_{n=0}^{\infty} L_{n,1,p}^{(\alpha)}(x)t^n \;=\; (1-tp)^{\frac{-p-\alpha}{p}} exp\left(\frac{-xt}{(1-tp)}\right)$$

(cf. [53, Eq.(4), p.202] with $p = 1$). For $m = 2$, we have with $|4xt^2/(1-pt)^2| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{(p+\alpha)_{n,p}} L_{n,2,p}^{(\alpha)}(x)\; t^n \;=\; (1-tp)^{-\frac{a}{p}} {}_2F_1\left(\left(\triangle_p(2,a)\right), p, (\alpha+p), p\right)\left(\frac{4xt^2}{(1-tp)^2}\right).$$

It may be noted that if $p = 1$ then this reduces to the GFR given in [53, Eq.(3), p.202].

In this last GFR, the substitution $a = 2\alpha + p$ gives an elegant form:

$$\sum_{n=0}^{\infty} \frac{(2\alpha+p)_{n,p}}{(p+\alpha)_{n,p}} L_{n,2,p}^{(\alpha)}(x)\; t^n \;=\; (1-pt)^{-(2\alpha+p)/p}\left(1-\frac{4pxt^2}{(pt-1)^2}\right)^{-(2\alpha+p)/2p}.$$

Next, the GFR of the *extended p-deformed* Wilson polynomials occurs from (4.3.6) by replacing c by $a+b+c+d-p$ and choosing

$$G(n) = (a+ix)_{n,p}(a-ix)_{n,p}/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!).$$

Then $F(n) = W_{n,l,m,p}(x^2; a, b, c, d)/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p})$ gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+b+c+d-p)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!} W_{n,l,m,p}(x^2; a, b, c, d)\; t^n \\
&= (1-tp)^{\frac{-a-b-c-d+p}{p}} \sum_{k=0}^{\infty} \frac{(a+b+c+d-p)_{mk+lk,p}}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}k!} \\
&\times (a+ix)_{k,p}(a-ix)_{k,p}\left(\frac{(-t)^m}{(1-tp)^{m+l}}\right)^k \\
&= (1-tp)^{\frac{-a-b-c-d+p}{p}} \frac{\Gamma_p(a+b)\Gamma_p(a+c)\Gamma_p(a+d)}{\Gamma_p(a+b+c+d-p)\Gamma_p(a+ix)\Gamma_p(a-ix)} \\
&\times {}_3\Psi_3^p\left[\begin{array}{ll} (a+b+c+d-p, mp+lp), (a+ix, p), (a-ix, p); & \dfrac{(-t)^m}{(1-tp)^{m+l}} \\ (a+b, p), (a+c, p), (a+d, p); & \end{array}\right],
\end{aligned}$$

in which we have used the generalized p-Write function given in (4.5.1). If $l \in \mathbb{N}\bigcup\{0\}$, then this reduces to

$$\sum_{n=0}^{\infty} \frac{(a+b+c+d-p)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!} W_{n,l,m,p}(x^2; a, b, c, d)t^n$$

$$
\begin{aligned}
&= (1-tp)^{\frac{-a-b-c-d+p}{p}} \\
&\quad \times {}_{m+l+2}F_3\left(\left(\triangle_p\left(m+l, a+b+c+d-p\right), a+ix, a-ix\right), p,\right. \\
&\qquad\qquad\qquad\qquad \left.\left(a+b, a+c, a+d\right), p\right)\left(\frac{(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}}\right),
\end{aligned}
$$

wherein $m+l=1,2$ for convergence.

In a similar manner, the GFR of the *extended p-deformed* Racah polynomials is obtained by replacing c by $a+b+p$ and taking

$$G(n) = (-x)_{n,p}(x+c+d+p)_{n,p}/((a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}n!),$$

in (4.3.6), then $F(n) = R_{n,l,m,p}(x(x+c+d+p)$ leads us to the GFR:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+b+p)_{n,p}}{n!}\ R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)\ t^n \\
&= (1-tp)^{\frac{-a-b-p}{p}} \sum_{k=0}^{\infty} \frac{(a+b+p)_{mk+lk,p}(-x)_{k,p}(x+c+d+p)_{k,p}}{(a+p)_{k,p}(b+d+p)_{k,p}(c+p)_{k,p}k!}\left(\frac{(-t)^m}{(1-tp)^{m+l}}\right)^k \\
&= (1-tp)^{\frac{-a-b-p}{p}} \frac{\Gamma_p(a+p)\ \Gamma_p(b+d+p)\ \Gamma_p(c+p)}{\Gamma_p(a+b+p)\ \Gamma_p(-x)\ \Gamma_p(x+c+d+p)} \\
&\quad \times {}_3\Psi_3^p\left[\begin{array}{cc} (a+b+p, mp+lp), (-x,p), (x+c+d+p,p); & \dfrac{(-t)^m}{(1-tp)^{m+l}} \\ (a+p,p), (b+d+p,p), (c+p,p); & \end{array}\right].
\end{aligned}
$$

If $l \in \mathbb{N}\bigcup\{0\}$, then this further reduces to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+b+p)_{n,p}}{n!}\ R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)\ t^n \\
&= (1-tp)^{\frac{-a-b-p}{p}}\ {}_{m+l+2}F_3\left(\left(\triangle_p(m+l, a+b+p), -x, x+c+d+p\right), p,\right. \\
&\qquad\qquad\qquad \left.\left(a+p, b+d+p, c+p\right), p\right)\left(\frac{(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}}\right),
\end{aligned}
$$

in which $m+l=1,2$ for convergence.

Next section deals with the summation formulas obtained with the help of inverse series relation of earlier section.

4.6 Summation formulas

In this section, We use the series identity (4.3.7) to derive certain summation formulas involving the polynomials $\mathcal{B}^{a}_{n,m,p}(x;l)$, the *extended p-deformed* Wilson as well as the *extended p-deformed* Racah polynomials. We shall require the p-deformed Gauss summation formula (Lemma - 2.7.1)

If $p > 0,\ c \neq -p,\ -2p,\ \ldots$ and $\Re(c-a-b) > 0$ then

$$ {}_2F_1((a,b),p,(c),p)(1/p) = \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-a)\Gamma_p(c-b)} \tag{4.6.1} $$

and the p-Chu-Vandermonde identity (cf. [53, Ex. 4, p.69] with $p=1$)

Corollary 4.6.1.

$$ {}_2F_1((-np,b),p,(c),p)(1/p) = \frac{(c-b)_{n,p}}{(c)_{n,p}}. \tag{4.6.2} $$

We now obtain the summation formulas involving the above considered polynomials. For that we begin with the inverse series (4.3.8) assuming $\gamma_n \neq 0\ \forall n \in \mathbb{N}$, in the form:

$$ \frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\ k!}\ \mathcal{B}^a_{k,m,p}(x;l) = x^n, \tag{4.6.3} $$

and multiply the both sides by $1/n!$ and take the summation n from 0 to ∞, then we find

$$ \sum_{n=0}^{\infty}\frac{1}{n!\ \gamma_n}\sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\ k!}\ \mathcal{B}^a_{k,m,p}(x;l) = e^x. \tag{4.6.4} $$

Here x may be assigned particular values to get a number of particular sums; for instance, $x=0$ in (4.6.4) gives

$$ \sum_{n=0}^{\infty}\frac{\gamma_0}{n!\ \gamma_n}\sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\ k!} = 1. $$

Next, by taking $\gamma_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}/((\beta_1)_{n,p}\cdots\ (\beta_d)_{n,p}\ n!)$ in (4.6.4), we get a summation formula involving the p-deformed extended Jacobi polynomial:

$$ \sum_{n=0}^{\infty}\frac{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p}}{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}}\sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!k!}\ \mathcal{F}^{(a)}_{k,l,m,p}[(\alpha);(\beta):x] = e^x. $$

The p-Brafman polynomial case follows immediately when $l=0$.
Now, the summation formula involving the extended p-deformed Konhauser polynomial can be obtained from (4.6.3) by taking $l=0, \gamma_n = 1/((p+\alpha)_{sn,p}n!)$ and replacing x by x^s. With this, $\mathcal{B}^a_{n,m,p}(x;0) = Z^{(\alpha)}_{n,m,p}(x;s)/(p+\alpha)_{sn,p}$ yields the sum:

$$ \sum_{n=0}^{\infty}(p+\alpha)_{sn,p}\sum_{k=0}^{mn}\frac{(-mn)_k}{(p+\alpha)_{sk,p}(mn)!}Z^{(\alpha)}_{k,m,p}(x;s) = e^{x^s}. $$

When $s = 1$, this readily yields the summation formula involving the extended p-deformed Laguerre polynomial.

Next assuming $|x| < 1$, and taking summation n from 0 to ∞ in (4.6.3), we find

$$\sum_{n=0}^{\infty} \frac{1}{\gamma_n} \sum_{k=0}^{mn} \frac{(-mn)_k (a + kp + lkp/m)}{(a + kp)_{ln+1,p} (mn)!\ k!} \mathcal{B}^a_{k,m,p}(x; l) = \frac{1}{1 - x}.$$

By assigning different values to x from $(-1, 1)$, a number of particular summation formulas can be derived. For example, $x = 1/2$ gives the following one.

$$\sum_{n=0}^{\infty} \frac{1}{\gamma_n} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a + kp + lkp/m)}{(a + kp)_{ln+1,p}\ (mn)!\ k!}\ \mathcal{B}^a_{k,m,p}\left(\frac{1}{2}; l\right) = 2.$$

Here, if $\gamma_n = (\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}/((\beta_1)_{n,p} \cdots\ (\beta_d)_{n,p} n!)$, then we get

$$\sum_{n=0}^{\infty} \frac{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p} n!}{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}} \sum_{k=0}^{mn} \frac{(-mn)_k (a + kp + lkp/m)}{(a + kp)_{ln+1,p} (mn)!\ k!}\ \mathcal{F}^{(a)}_{k,l,m,p}[(\alpha); (\beta) : x] = \frac{1}{1 - x}$$

involving the p-extended Jacobi polynomial.

The summation formula corresponding to the extended p-deformed Konhauser polynomial can be obtained from (4.6.3) by taking $l = 0$, $\gamma_n = 1/((p + \alpha)_{sn,p} n!)$ and replacing x by x^s. Then with $\mathcal{B}^a_{n,m,p}(x; 0) = Z^{(\alpha)}_{n,m,p}(x; s)/(p + \alpha)_{sn,p}$, it gives

$$\sum_{n=0}^{\infty} n!(p + \alpha)_{sn,p} \sum_{k=0}^{mn} \frac{(-mn)_k}{(p + \alpha)_{sk,p} (mn)!} Z^{(\alpha)}_{k,m,p}(x; s) = \frac{1}{1 - x^s}.$$

We now derive certain summation formulas involving the *extended p-deformed* Wilson polynomials. For that we first multiply both sides of (4.3.20) by p^{-n} and then take summation from $n = 0$ to ∞, to get

$$\sum_{n=0}^{\infty} \frac{(a + c)_{n,p} (a + d)_{n,p}}{p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a + b + c + d - p + kp + lkp/m)}{(a + b + c + d + kp - p)_{ln+1,p}} \times \frac{W_{k,l,m,p}(x^2; a, b, c, d).}{(a + b)_{k,p} (a + c)_{k,p} (a + d)_{k,p}\ k!} = \sum_{n=0}^{\infty} \frac{(a + ix)_{n,p} (a - ix)_{n,p}}{(a + b)_{n,p}\ p^n\ n!}.$$

This with the aid of the p-Gauss sum (4.6.1), gets simplified to the form:

$$\sum_{n=0}^{\infty} \frac{(a + c)_{n,p} (a + d)_{n,p}}{p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a + b + c + d - p + kp + lkp/m)}{(a + b + c + d + kp - p)_{ln+1,p}}$$

$$\times \frac{W_{k,l,m,p}(x^2; a, b, c, d).}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!} = \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{\Gamma_p(b-ix)\Gamma_p(b+ix)}.$$

Since for $x = 0$,

$$\begin{aligned}
&\frac{W_{k,l,m,p}(0; a, b, c, d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}} \\
= \ &{}_{m+l+2}F_3((\Delta_p(m; -k), \Delta_p(l; a+b+c+d+kp-p+lkp/m), a, a), p, \\
&(a+b, a+c, a+d), p)(m^m l^l),
\end{aligned}$$

hence, we find the summation formula:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a+b+c+d-p+kp+lkp/m)}{(a+b+c+d+kp-p)_{ln+1,p}\ k!} \\
&\times_{m+l+2}F_3((\Delta_p(m; -k), \Delta_p(l; a+b+c+d+kp-p+lkp/m), \\
&a, a), p, (a+b, a+c, a+d), p)(m^m l^l) = \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{[\Gamma_p(b)]^2}.
\end{aligned}$$

Now, if both sides of (4.3.20) is multiplied by $(-jp)_{n,p}\, p^{-n}$ and then the summation from $n = 0$ to j is taken, then we obtain

$$\begin{aligned}
&\sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}\ p^n (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a+b+c+d-p+kp+lkp/m)}{(a+b+c+d+kp-p)_{ln+1,p}} \\
&\qquad \times \frac{W_{k,l,m,p}(x^2; a, b, c, d).}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!} = \sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+ix)_{n,p}}{(a+b)_{n,p} p^n n!}.
\end{aligned}$$

Here the left hand series when summed up by using the terminating p-Gauss sum or the p-Chu-Vandermonde identity (4.6.2), then we find

$$\begin{aligned}
&\sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}\ p^n (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (a+b+c+d-p+kp+lkp/m)}{(a+b+c+d+kp-p)_{ln+1,p}} \\
&\qquad \times \frac{W_{k,l,m,p}(x^2; a, b, c, d).}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!} = \frac{(b-ix)_{j,p}}{(a+b)_{j,p}}.
\end{aligned}$$

In this, the choice $x = 0$ gives rise to the sum

$$\begin{aligned}
&\frac{(b)_{j,p}}{(a+b)_{j,p}} = \sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}\ p^n (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k\ (h-p+kp+lkp/m)}{(a+b+c+d+kp-p)_{ln+1,p}\ k!} \\
&\times_{r+2}F_3((\Delta_p(m; -kp), \Delta_p(l; h+kp-p+lkp/m), a, a), p, (a+b, a+c, a+d), p) \\
&\qquad (m^m l^l p^{-m}).
\end{aligned}$$

In a similar manner, we may find the summation formulas involving the *extended p-deformed* Racah polynomials. From the inverse series (4.3.22) of the *extended p-deformed* Racah polynomials, we obtain the following formula by multiplying both sides by p^{-n} and then taking the summation from $n = 0$ to ∞.

$$\sum_{n=0}^{\infty} \frac{(b+d+p)_{n,p}(c+p)_{n,p}}{p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!}$$
$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \sum_{n=0}^{\infty} \frac{(x+c+d+p)_{n,p}(-x)_{n,p}}{(a+p)_{n,p}\ p^n\ n!}.$$

In view of the p-Gauss sum (4.6.1), the right hand side gets summed up as follows.

$$\sum_{n=0}^{\infty} \frac{(b+d+p)_{n,p}(c+p)_{n,p}}{p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!}$$
$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \frac{\Gamma_p(p+a)\Gamma_p(a-c-d)}{\Gamma_p(a-x-c-d)\Gamma_p(p+a+x)}.$$

If we multiply both sides of (4.3.22) by $(-jp)_{n,p}\ p^{-n}$ and then take the summation from $n = 0$ to j, then we have

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!}$$
$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \sum_{n=0}^{j} \frac{(-jp)_{n,p}(-x)_{n,p}}{(a+p)_{n,p}p^n n!}.$$

Applying the p-Chu-Vandermonde identity (4.6.2) on the left hand side, gives

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!}$$
$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \frac{(x+a+p)_{j,p}}{(a+p)_{j,p}}.$$

A worth mentioning sum occurs when $x = 0$. In this case, $R_{k,p}\,(0(c+d+p);a,b,c,d) = 1$ hence, this summation formula reduces to

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!} = 1.$$

4.7 Companion Matrix

Taking $\lfloor n/m \rfloor = N$ in (4.1.5) and converting it to the monic form $\widetilde{\mathcal{B}}^{a}_{n,m,p}(x;l)$, we get

$$\widetilde{\mathcal{B}}^{a}_{n,m,p}(x;l) \;=\; \sum_{k=0}^{N} \lambda_k \, x^k, \tag{4.7.1}$$

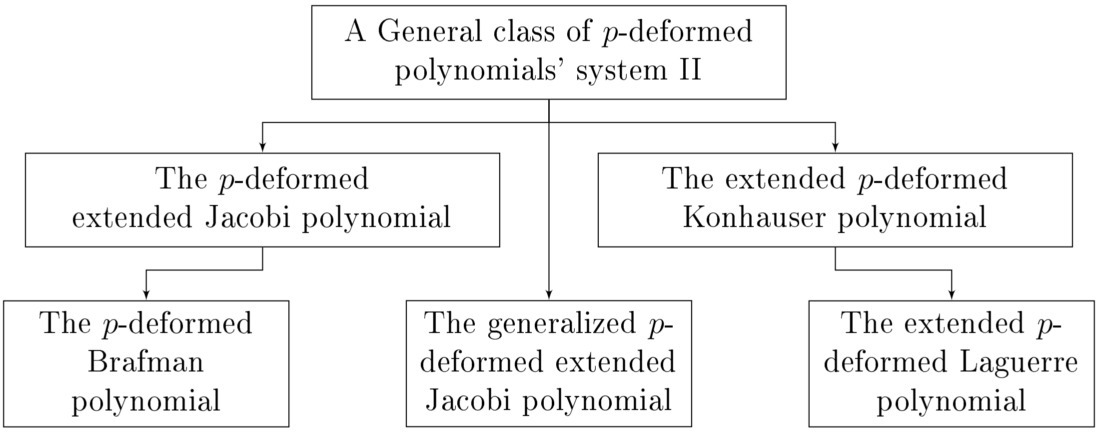
where

$$\lambda_k \;=\; \frac{(-n)_{mk}(a+np)_{lk,p}\gamma_k}{(-n)_{mN}(a+np)_{lN,p}\gamma_N},$$

assumes the form as stated in definition 1.3.1. The eigen values of this matrix will be then precisely the zeros of $\widetilde{\mathcal{B}}^{a}_{n,m,p}(x;l)$ (see [48, p. 39]).

Besides yielding the extended p-polynomials, Theorem - 4.2.1 and its alternative forms also provide an effective tool for carrying out the extension of certain inverse series relations belonging to the Riordan's classification[55] involving the p-gamma function and the Pochhammer p-symbol. They are derived in Chapter 8.

POLYNOMIALS' REDUCIBILITY



SCHEME OF p-DEFORMED EXTENDED CLASSICAL POLYNOMIALS