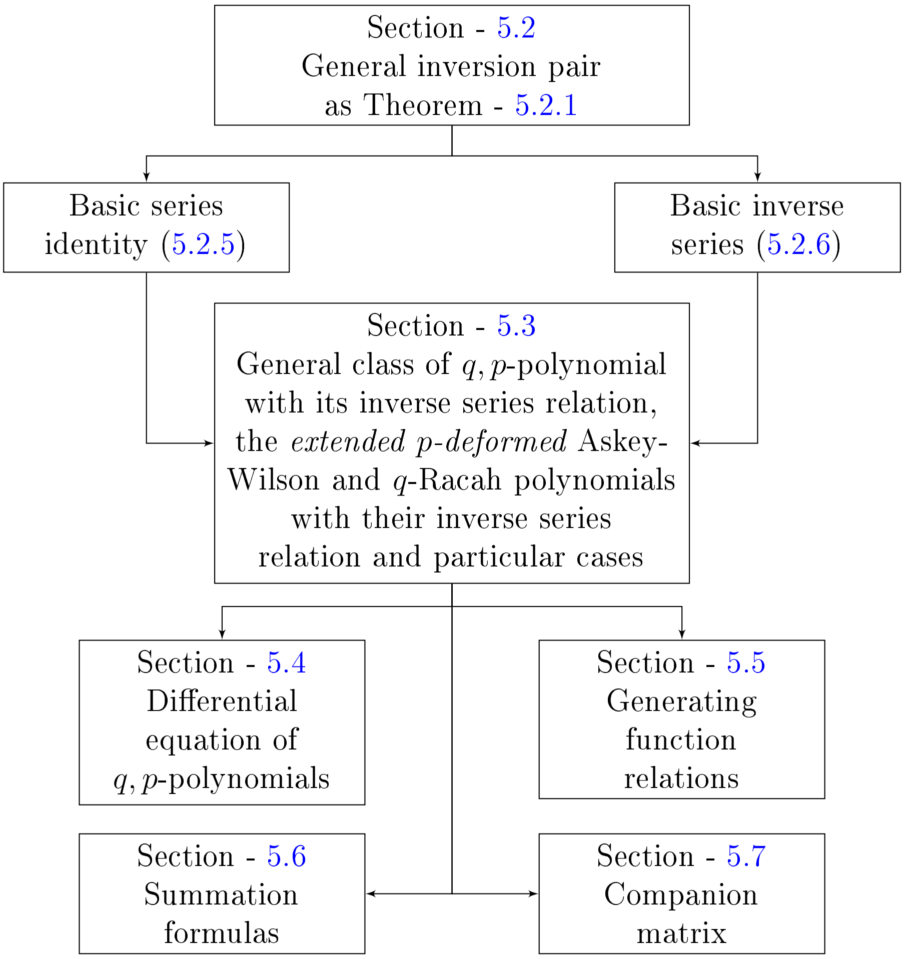
Chapter 5

The p-deformed q-polynomials' system-II



5.1 Introduction

The aim of this chapter is to provide q-analogue of general class of polynomial (4.1.5)

$$\mathcal{B}^a_{n,m,p}(x;l) = \sum_{k=0}^{\lfloor n/m \rfloor} (-n)_{mk}(a+np)_{kl,p}\gamma_k x^k \tag{5.1.1}$$

of Chapter 4 and its inverse series relation(4.3.8)

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+klp/m)}{(a+kp)_{ln+1,p}(mn)!k!}\mathcal{B}^a_{k,m,p}(x;l) \tag{5.1.2}$$

with the help of a general q-inversion pair which will be established in this chapter. We define a q-analogue of (5.1.1) below.

Definition 5.1.1. *For $a \in \mathbb{C},\ m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, 0 < q < 1$ and $p > 0$,*

$$\mathcal{B}^{a}_{n,m,p}(x|q;l) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\ \gamma_k x^k, \tag{5.1.3}$$

in which $l = r - m, r \in \mathbb{C} \setminus \{m\}$, and the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\leq u$.

This general class extends the q-extended Jacobi polynomials [10, Eq. (3.8)] and hence the q-Brafman polynomials and the little q-Jacobi polynomials [35, Eq.(3.12.1), p. 92)] (also [19, Ex. 1.32, p. 27]). As a limiting case, this general class also extends the q-Konhauser polynomials [3, Eq. (3.1), p. 3] and thereby the q-Laguerre polynomials [50]. The main objective of the work is to establish a general inversion pair (GIP) which would invert the aforesaid polynomials; and furthermore, the well known orthogonal polynomials in ${}_4\phi_3$-function forms namely, the Askey-Wilson polynomials [35, Eq.(3.1.1), p. 63] (also [19, Ex. 2.11, p.51]) and the q-Racah polynomials [35, Eq.(3.2.1), p. 66] (also [19, Ex. 2.10, p. 51]). It is interesting to note that the q-analogues of some of the Riordan's classes of inverse series relations [11] also assume the extension by means of this GIP.
The GIP, the main result, will be stated and proved in Section - 5.2 using the auxiliary result (Lemma 5.2.1). Section - 5.3 incorporates several alternative forms of GIP by means of which various particular polynomials will be deduced. The p-deformed q-difference equation and p-deformed q-differential equation of q,p-polynomials are derived in section - 5.4. Next, we emphasis on applicability of both series of GIP; the one for obtaining the generating function relations (GFR) in Section - 5.5 and the other, that is the inverse series, for deducing the summation formulas in section - 5.6. In Section - 5.7, the Companion matrix [48] for the general class (5.1.3) is illustrated. A chart showing the reducibility of the p-deformed Askey-Wilson polynomials and the p-deformed q-Racah polynomials to a number of polynomials, is given at the end. This also includes the inter-connections amongst these particular polynomials.

5.2 Basic inverse series relations

While proving the main result, we shall require the inverse pair which is proved as

Lemma 5.2.1. *For* $0 < q < 1$, $M \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $p > 0$,

$$g(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} \frac{(1 - q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+(M+mj)\lambda-kp-mjp}; q)_{\infty,p}} f(k) \qquad (5.2.1)$$

$$\Leftrightarrow$$

$$f(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-2M+1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} (q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p}; q)_{\infty,p}\ g(k). \qquad (5.2.2)$$

Proof. We first note that the diagonal elements of the coefficient matrix of the first series are

$$(-1)^i q^{i\lambda(i-1)/2}(1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp})/(q^{\alpha+(i+mj)\lambda-ip-mjp}; q)_{\infty,p}$$

and those of the second series are

$$(-1)^i q^{i\lambda(1-i)/2}(q^{\alpha+i\lambda+mj\lambda+p-(i+mj)p}; q)_{\infty,p}.$$

Since these numbers are all non zero; it follows that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We shall show that (5.2.1) implies (5.2.2). For that we denote the right hand side of (5.2.2) by $\Phi(M)$ and substitute for $g(k)$ from (5.2.1) to get

$$\begin{aligned}
\Phi(M) &= \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-2M+1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} \left(q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p}; q\right)_{\infty,p} \\
&\quad \times \sum_{i=0}^{k} (-1)^i q^{i\lambda(i-1)/2} \begin{bmatrix} k \\ i \end{bmatrix}_{q^\lambda} \frac{(1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp})}{(q^{\alpha+(k+mj)\lambda-ip-mjp}; q)_{\infty,p}} f(i) \\
&= \sum_{i=0}^{M} \sum_{k=0}^{M-i} (-1)^k q^{(k+i)\lambda(k+i-2M+1)/2} \begin{bmatrix} M \\ k+i \end{bmatrix}_{q^\lambda} (q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p}; q)_{\infty,p} \\
&\quad \times q^{i\lambda(i-1)/2} \begin{bmatrix} k+i \\ i \end{bmatrix}_{q^\lambda} \frac{(1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp})}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp}; q)_{\infty,p}} f(i) \\
&= \sum_{i=0}^{M} (1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \sum_{k=0}^{M-i} (-1)^k q^{(k+i)\lambda(k+i-2M+1)/2+i\lambda(i-1)/2} \\
&\quad \times \begin{bmatrix} M \\ k+i \end{bmatrix}_{q^\lambda} \begin{bmatrix} k+i \\ i \end{bmatrix}_{q^\lambda} \frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p}; q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp}; q)_{\infty,p}} \\
&= \sum_{i=0}^{M} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2} (1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \sum_{k=0}^{M-i} (-1)^k q^{k\lambda(k+2i-2M+1)/2}
\end{aligned}$$

$$\begin{aligned}
&\times \begin{bmatrix} M-i \\ k \end{bmatrix}_{q^\lambda} \frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}} \\
= &\ f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \sum_{k=0}^{M-i}(-1)^k \\
&\times q^{k\lambda(k+2i-2M+1)/2} \begin{bmatrix} M-i \\ k \end{bmatrix}_{q^\lambda} \frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}}.
\end{aligned}$$

Here, the ratio

$$\frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}} = \sum_{l=0}^{M-i-1} A_l\, q^{\lambda kl},$$

represents a polynomial of degree $M-i-1$ in k, hence we further have

$$\begin{aligned}
\Phi(M) &= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \\
&\quad \times \sum_{k=0}^{M-i}(-1)^k\, q^{k\lambda(k+2i-2M+1)/2} \begin{bmatrix} M-i \\ k \end{bmatrix}_{q^\lambda} \sum_{l=0}^{M-i-1} A_l\, q^{\lambda kl} \\
&= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \\
&\quad \times \sum_{l=0}^{M-i-1} A_l \sum_{k=0}^{M-i}(-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} M-i \\ k \end{bmatrix}_{q^\lambda} q^{\lambda k(l+i-M+1)}. \qquad (5.2.3)
\end{aligned}$$

The inner most series on the right hand side in (5.2.3) may be summed up by means of Lemma - 3.2.1 (based on q-Binomial theorem), to get

$$\begin{aligned}
\Phi(M) &= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \\
&\quad \times \sum_{l=0}^{M-i-1} A_l\ (q^{\lambda(l+i-M+1)};q^\lambda)_{M-i} \\
&= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^\lambda} q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp}) f(i) \\
&\quad \times \sum_{l=0}^{M-i-1} A_l (1-q^{\lambda(l+i-M+1)})(1-q^{\lambda(l+i-M+2)})\cdots \\
&\quad \times (1-q^{\lambda(l-1)})(1-q^{\lambda l}) \\
&= f(M).
\end{aligned}$$

This completes the proof. □

This lemma gives rise to the *q-series orthogonality relation*. In fact, the substitution $\left[\begin{smallmatrix}0\\M\end{smallmatrix}\right]_{q^\lambda}$ for either $f(M)$ or $g(M)$ yields this property. In particular, the following corollary is of our use.

Corollary 5.2.1. *For* $0 \leq j \leq n, m \in \mathbb{N}, \lambda \in \mathbb{C}\backslash\{0\}$ *and* $p > 0$,

$$\begin{aligned}\begin{bmatrix}0\\M\end{bmatrix}_{q^\lambda} &= \sum_{k=0}^{M}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}\\ &\quad\times(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}. \qquad (5.2.4)\end{aligned}$$

Proof. In (5.2.1), the substitution $g(k) = \left[\begin{smallmatrix}0\\k\end{smallmatrix}\right]_{q^\lambda}$ gives $f(k) = (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$, and with these $f(k)$ and $g(k)$, (5.2.2) yields the stated series orthogonality relation. □

As a main result, we establish the GIP as

Theorem 5.2.1. $0 < q < 1,\ \lambda \in \mathbb{C}\setminus\{0\},\ \alpha \in \mathbb{C},\ n \in \mathbb{N}\cup\{0\},\ m \in \mathbb{N}$ *and* $p > 0$,

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{mk\lambda(mk-2n+1)/2}\frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-mk}}G(k) \qquad (5.2.5)$$

$$\Rightarrow$$

$$G(n) = \sum_{k=0}^{mn}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}}F(k), \qquad (5.2.6)$$

and conversely, the series in (5.2.6) implies the series (5.2.5) if for $n \neq mr,\ r \in \mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}}F(k) = 0. \qquad (5.2.7)$$

Proof. First we show that (5.2.5)⇒(5.2.6) for that we denote (5.2.6) as $V(n)$ and use (5.2.5) to get

$$\begin{aligned}V(n) &= \sum_{k=0}^{mn}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}}F(k)\\ &= \sum_{k=0}^{mn}(-1)^k q^{k\lambda(k-1)/2}\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}}\\ &\quad\times\sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}q^{mj\lambda(mj-2k+1)/2}\frac{(q^{\alpha+mj\lambda+p-kp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{k-mj}}G(j). \qquad (5.2.8)\end{aligned}$$

Now, on making use of double series relation [62]

$$\sum_{k=0}^{mn}\sum_{j=0}^{\lfloor k/m\rfloor} A(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{mn-mj} A(k+mj,j)$$

in (5.2.8) gives

$$\begin{aligned} V(n) &= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}(-1)^k q^{(k+mj)\lambda(k+mj-1)/2}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^\lambda;q^\lambda)_{mn-mj-k}(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}} \\ &\quad\times q^{mj\lambda(mj-2k-2mj+1)/2}\frac{(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_k}G(j) \\ &= \sum_{j=0}^{n}\frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}}\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^\lambda} \\ &\quad\times\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}} \\ &= G(n)+\sum_{j=0}^{n-1}\frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}}\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^\lambda} \\ &\quad\times\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}. \end{aligned} \tag{5.2.9}$$

In order to prove $\Rightarrow$ part, it is sufficient to show that the second term in (5.2.9) vanishes. In fact, replacing $(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ by $f(k)$ and denoting the inner series by $g(mn-mj)$, one finds

$$g(mn-mj)=\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}f(k), \tag{5.2.10}$$

whose inverse series follows from (5.2.1) and (5.2.2) in the form:

$$\begin{aligned} f(mn-mj) &= \sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-2mn+2mj+1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^\lambda} \\ &\quad\times(q^{\alpha+k\lambda+mj\lambda+p-mnp};q)_{\infty,p}g(k). \end{aligned} \tag{5.2.11}$$

In this last relation, setting $g(k)=\begin{bmatrix} 0 \\ k\end{bmatrix}_{q^\lambda}$ yields $f(k)=(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ and with these $f(k)$ and $g(k)$, (5.2.1) yields the orthogonality relations:

$$\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}$$

$$\times(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p} \;=\; \begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^\lambda}. \quad (5.2.12)$$

On making use of (5.2.12) in (5.2.9), we get

$$\begin{aligned} V(n) &= G(n) + \sum_{j=0}^{n-1} \frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}} \begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^\lambda} \\ &= G(n). \end{aligned}$$

Thus, (5.2.5)⇒(5.2.6). Next to show (5.2.5)⇒(5.2.7), denote (5.2.7) by $R(n)$, that is,

$$R(n) \;=\; \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}} F(k), \quad (5.2.13)$$

and then substitute from (5.2.5) for $F(k)$ in it, then we get

$$\begin{aligned} R(n) &= \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}} \\ &\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj} q^{mj\lambda(mj-2k+1)/2} \frac{(q^{\alpha+mj\lambda+p-kp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{k-mj}} G(j) \\ &= \sum_{j=0}^{\lfloor n/m \rfloor} \sum_{k=0}^{n-mj} (-1)^{k+mj} q^{(k+mj)\lambda(k+mj-1)/2} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^\lambda;q^\lambda)_{n-mj-k}(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} \\ &\quad \times (-1)^{mj} q^{mj\lambda(mj-2k-2mj+1)/2} \frac{(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_k} G(j) \\ &= \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^\lambda;q^\lambda)_{n-mj}} \sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^\lambda} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} \\ &\quad \times (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}. \end{aligned} \quad (5.2.14)$$

Next, following the method employed in obtaining the orthogonality relation (5.2.10), it can be shown that

$$\sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^\lambda} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p} = \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^\lambda},$$

as a result of which (5.2.14) reduces to

$$R(n) \;=\; \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^\lambda;q^\lambda)_{n-mj}} \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^\lambda}.$$

If n/m is not an integer i.e. $n \neq mr, r \in \mathbb{N}$, then the right hand member of the last expression given above vanishes and thus (5.2.5)⇒(5.2.7); which completes the proof of the first part. For the converse part, assume that (5.2.6) and (5.2.7) hold. In view of (5.2.7) and (5.2.13), we have

$$R(n) \;=\; 0, \; n \neq mr, \; r \in \mathbb{N} \tag{5.2.15}$$

and also, by comparing (5.2.6) with (5.2.13), one finds the relation:

$$R(mn) \;=\; G(n). \tag{5.2.16}$$

Since, the inverse pair (5.2.11) and (5.2.10) with $j=0$ and $m=1$ reduces to the result $g(n)=R(n)$ and $f(n)=F_n$, we consequently get

$$\begin{aligned} R(n) \;&=\; \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+n\lambda-kp};q)_{\infty,p}(q^\lambda;q^\lambda)_{n-k}} F_k \\ \Rightarrow \\ F_n \;&=\; \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-2n+1)/2} \frac{(q^{\alpha+k\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-k}} R(k). \end{aligned}$$

It now follows from (5.2.15) and (5.2.16) that

$$\begin{aligned} R(mn) \;&=\; \sum_{k=0}^{mn} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^\lambda;q^\lambda)_{mn-k}} F_k \\ \Rightarrow \\ F_n \;&=\; \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mk\lambda(mk-2n+1)/2} \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-mk}} R(mk), \end{aligned}$$

where $R(mn) = G(n)$. Thus, the relation (5.2.6) with $R(n) = 0, n \neq mr$ for $r \in \mathbb{N}$, implies the relation (5.2.5) which proves the converse part and hence the theorem. □

5.3 Particular cases

In this section, we obtain alternative forms of Theorem - 5.2.1. These forms will be used to deduce the general class of q,p-polynomials (5.1.3) and its particular cases along with their inverse series relations. Besides this, such alternative form will also be used to deduce the *extended p-deformed Askey-Wilson polynomials* and the *extended p-deformed q-Racah polynomials* together with their inverse series. We begin with Theorem - 5.2.1 and apply the formula

$$(q^{-1};q^{-1})_n = (-1)^n q^{-n(n+1)/2}(q;q)_n,$$

in it to get

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^n q^{\lambda(2mk-n(n+1)/2)} \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k) \tag{5.3.1}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^{mn} q^{\lambda(2kmn-mn(mn+1))/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k). \tag{5.3.2}$$

Next, using the formula:

$$(q^{-n};q)_k(q;q)_{n-k} = (-1)^k q^{k(k-2n-1)/2}(q;q)_n,$$

with q is replaced by $q^{-\lambda}$ and k by mk in this pair, this pair transforms to

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{n-mk} q^{-\lambda(n(n+1)-mk(mk-1)+2mnk-2mk)/2} \frac{(q^{n\lambda};q^{-\lambda})_{mk}}{(q^{-\lambda};q^{-\lambda})_n} \\ &\quad \times (q^{\alpha+mk\lambda+p-np};q)_{\infty,p}\, G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn-k} q^{-\lambda(mn(mn+1)-k(k-1))/2} \frac{(q^{mn\lambda};q^{-\lambda})_k(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^{-\lambda};q^{-\lambda})_{mn}} F(k). \end{aligned}$$

Here, if $F(n)$ is replaced by $q^{-\lambda n(n+1)/2}F(n)$ and $G(n)$ is replaced by

$$q^{-\lambda(mn(mn+1)/2)}G(n)/(q^{\alpha+mn\lambda+p};q)_{\infty,p}G(n),$$

then we obtain after little simplification, the pair:

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda mnk} \frac{(q^{n\lambda};q^{-\lambda})_{mk}}{(q^{\alpha+mk\lambda+p};q)_{-n,p}(q^{-\lambda};q^{-\lambda})_n} G(k)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} q^{-k\lambda} \frac{(q^{mn\lambda};q^{-\lambda})_k(1-q^{\alpha+k\lambda-kp})(q^{\alpha+mn\lambda+p};q)_{-k,p}}{(q^{-\lambda};q^{-\lambda})_{mn}(1-q^{\alpha+mn\lambda-kp})} F(k).$$

But since

$$\begin{aligned}\frac{q^{-\lambda mnk}}{(q^{\alpha+mk\lambda+p};q)_{-n,p}} &= \frac{q^{-\lambda mnk}(q^{p-\alpha-mk\lambda-p};q)_{n,p}}{(-1)^n q^{pn(n+1)/2-(\alpha+mk\lambda+p)n}} \\ &= (-1)^n q^{-pn(n+1)/2+(\alpha+p)n}(q^{-\alpha-mk\lambda};q)_{n,p},\end{aligned}$$

and

$$\begin{aligned}q^{-k\lambda}(q^{\alpha+mn\lambda+p};q)_{-k,p} &= \frac{(-1)^k q^{-k\lambda} q^{pk(k+1)/2-(\alpha+mn\lambda+p)k}}{(q^{p-\alpha-mn\lambda-p};q)_{k,p}} \\ &= \frac{(-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k}}{(q^{-\alpha-mn\lambda};q)_{k,p}},\end{aligned}$$

the above pair changes to

$$\begin{aligned}F(n) &= \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^n q^{-pn(n+1)/2+(\alpha+p)n}\frac{(q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}}{(q^{-\lambda};q^{-\lambda})_n}\;G(k) \\ \Leftrightarrow& \\ G(n) &= \sum_{k=0}^{mn}(-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k}\frac{(q^{mn\lambda};q^{-\lambda})_k(1-q^{\alpha+k\lambda-kp})}{(q^{-\lambda};q^{-\lambda})_{mn}(1-q^{\alpha+mn\lambda-kp})} \\ &\quad\times\frac{F(k)}{(q^{-\alpha-mn\lambda};q)_{k,p}}.\end{aligned}$$

Further, Replacing $F(n)$ by $(-1)^n q^{-pn(n+1)/2+(\alpha+p)n}F(n)/(q^{-\lambda};q^{-\lambda})_n$, and noticing that

$$\begin{aligned}\frac{q^{-k\lambda}(1-q^{\alpha+k\lambda-kp})}{(1-q^{\alpha+mn\lambda-kp})(q^{-\alpha-mn\lambda};q)_{k,p}} &= \frac{q^{-k\lambda}q^{\alpha+k\lambda-kp}(1-q^{-\alpha-k\lambda+kp})}{q^{\alpha+mn\lambda-kp}(1-q^{-\alpha-mn\lambda+kp})(q^{-\alpha-mn\lambda};q)_{k,p}} \\ &= \frac{(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{k+1,p}},\end{aligned}$$

the above pair assumes the form:

$$\begin{aligned}F(n) &= \sum_{k=0}^{\lfloor n/m\rfloor}(q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}\;G(k) \\ \Leftrightarrow& \\ G(n) &= \sum_{k=0}^{mn}\frac{q^{-mnk\lambda}(q^{mn\lambda};q^{-\lambda})_k(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\lambda};q^{-\lambda})_{mn}(q^{-\alpha-mn\lambda};q)_{k+1,p}(q^{-\lambda};q^{-\lambda})_k}\;F(k).\end{aligned}$$

Finally, replacing $G(n)$ by $G(n)/(q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{\infty,p})$, $F(n)$ by $F(n)/(q^{a+np};q)_{\infty,p}$ and substituting $\alpha=-a,\ m\lambda=-l$, where $l=r-m,\ r\neq m$, in this last pair, we obtain

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m}; q^{l/m})_{mk}(q^{a+np}; q)_{\frac{kl}{p},p}\, G(k) \tag{5.3.3}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-nl}; q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp}; q)_{\frac{ln}{p}+1,p}(q^{l/m}; q^{l/m})_{mn}(q^{l/m}; q^{l/m})_k}\, F(k). \tag{5.3.4}$$

We shall use the adjective *extended p-deformed* or *p-deformed* whichever suitable, for the polynomials belonging to the general class (5.1.3).
We first deduce the inverse series of the polynomials (5.1.3). In fact, the choice $G(n) = \gamma_n x^n$ in (5.3.3) yields the polynomials (5.1.3); whereas the same substitution in (5.3.4) yields the inverse series:

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)}; q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp}; q)_{\frac{ln}{p}+1,p}(q^{l/m}; q^{l/m})_{mn}(q^{l/m}; q^{l/m})_k}\, \mathcal{B}^a_{k,m,p}(x|q;l). \tag{5.3.5}$$

Next, regarding $l \in \mathbb{C} \setminus \{0\}$, and putting $a = e$ and

$$\gamma_n = (q^{\alpha_1}; q)_{n,p} \cdots (q^{\alpha_c}; q)_{n,p}/((q^{\beta_1}; q)_{n,p} \cdots (q^{\beta_d}; q)_{n,p}(q^{l/m}; q^{l/m})_n)$$

in (5.1.3) and (5.3.5), then the basic analogue of the p-deformed extended Jacobi polynomials $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$ and its inverse series occur in the forms:

$$\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q] = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)}; q^{l/m})_{mk}(q^{e+np}; q)_{\frac{kl}{p},p}(q^{\alpha_1}; q)_{k,p} \cdots (q^{\alpha_c}; q)_{k,p}}{(q^{\beta_1}; q)_{k,p} \cdots (q^{\beta_d}; q)_{k,p}(q^{l/m}; q^{l/m})_k} x^k \tag{5.3.6}$$

$$\Leftrightarrow$$

$$\frac{(q^{\alpha_1}; q)_{n,p} \cdots (q^{\alpha_c}; q)_{n,p}}{(q^{\beta_1}; q)_{n,p} \cdots (q^{\beta_d}; q)_{n,p}(q^{l/m}; q^{l/m})_n} x^n = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)}; q^{l/m})_k(1-q^{e+Lk+kp})}{(q^{l/m}; q^{l/m})_{mn}(q^{e+kp}; q)_{\frac{ln}{p}+1,p}(q^{l/m}; q^{l/m})_k} \mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q], \tag{5.3.7}$$

where (α) indicates the array of c parameters $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_c$ and (β) indicates the array of d parameters $\beta_1,\ \beta_2,\ \ldots,\ \beta_d$. Here the limit $q^e \to 0$ leads us to the bibasic p-deformed q-Brafman polynomials and its inverse series as follows.

$$B^m_{n,p}[(\alpha);(\beta):xq^l|q] = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)}; q^{l/m})_{mk}(q^{\alpha_1}; q)_{n,p} \cdots (q^{\alpha_c}; q)_{n,p}}{(q^{\beta_1}; q)_{k,p} \cdots (q^{\beta_d}; q)_{k,p}(q^{l/m}; q^{l/m})_k}\, x^k$$

$$\Leftrightarrow$$

$$\frac{(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n}\,x^n=\sum_{k=0}^{mn}q^{nkl}\frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\times B^m_{k,p}[(\alpha);(\beta):q^lx|q].$$

The extended p-deformed little q-Jacobi polynomials (cf.[19, p.27] with $m=l=p=1$) and its inverse series may be deduced from (5.1.3) and (5.3.5) by replacing a by $a+b+p$ and taking $\gamma_n=1/((aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$ which are stated below.

$$p_{n,m,p,l}(x;a,b;q)=\sum_{k=0}^{\lfloor n/m\rfloor}q^{kl}\frac{(q^{-n(l/m)};q^{l/m})_{mk}(abq^{np+p};q)_{\frac{kl}{p},p}}{(aq^p;q)_{k,p}(q^{l/m};q^{l/m})_k}\,x^k$$

$\Leftrightarrow$

$$\frac{x^n}{(aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}=\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{k(l/m)+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\times p_{k,m,p,l}(x;a,b;q).$$

Next, in (5.1.3) and (5.3.5), making the limit $q^a\to 0$, putting

$$\gamma_n=q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/((p\alpha;q)_{nl,p}(q^l;q^l)_{mn}),$$

and replacing l and x by lm and $(xq^n)^l$ respectively, lead us to the inverse pair of the extended p-deformed q-Konhauser polynomial (cf. [3] with $p=1$ and $m=1$) and its inverse series:

$$Z^{(\alpha)}_{n,m,p}(x;l|q)=\frac{(p\alpha;q)_{nl,p}}{(q^l;q^l)_n}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{kl(\alpha+n+1)+kl(kl-1)/2}(q^{-nl};q^l)_{mk}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}}x^{kl}\quad(5.3.8)$$

$\Leftrightarrow$

$$\frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}}\,x^{ln}=\sum_{k=0}^{mn}\frac{(-1)^kq^{kl(kl-1)/2}}{(\alpha q^p;q)_{kl,p}(q^l;q^l)_{mn-k}}Z^{(\alpha)}_{k,m,p}(x;l|q).\quad(5.3.9)$$

The instance $l=1$ is the pair of inverse series relations involving the extended p-deformed q-Laguerre polynomials (cf. [50] with $m=p=1$):

$$L^{(\alpha)}_{n,m,p}(x|q)=\frac{(p\alpha;q)_{n,p}}{(q;q)_n}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(q^{-n};q)_{mk}\,q^{k(\alpha+n+1)+k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mk}}\,x^k\quad(5.3.10)$$

$\Leftrightarrow$

$$\frac{q^{n(\alpha+1)-mn+n(n-1)/2}}{(p\alpha;q)_{n,p}(q;q)_{mn}}\,x^n=\sum_{k=0}^{mn}\frac{(-1)^kq^{k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mn-k}}\,L^{(\alpha)}_{k,m,p}(x|q).\quad(5.3.11)$$

It is interesting to see that the inverse pair (5.3.3) and (5.3.4) provides the extension to the Askey-Wilson polynomials and q-Racah polynomials to which we call the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials; and denote them by $p_{n,l,m,p}(cos\theta; a, b, c, d|q)$ and $R_{n,m,p,l}(q^{-x} + cdq^{x+1}; a, b, c, d|q)$, respectively. These polynomials may be deduced from (5.3.3) and (5.3.4) as follows. First replacing a by $a+b+c+d-p$ and then choosing

$$G(n) = (ae^{i\theta}; q)_{n,p}(ae^{-i\theta}; q)_{n,p}/((ab; q)_{n,p}(ac; q)_{n,p}(ad; q)_{n,p}(q^{l/m}; q^{l/m})_n),$$

then $F(n) = p_{n,l,m,p}(cos\theta; a, b, c, d|q)a^n/((ab; q)_{n,p}(ac; q)_{n,p}(ad; q)_{n,p})$ yields the pair:

$$\frac{p_{n,l,m,p}(cos\theta; a, b, c, d|q)a^n}{(ab; q)_{n,p}(ac; q)_{n,p}(ad; q)_{n,p}} = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)}; q^{l/m})_{mk}(abcdq^{np-p}; q)_{kl/p,p}\ (ae^{i\theta}; q)_{k,p}\ (ae^{-i\theta}; q)_{k,p}}{(ab; q)_{k,p}(ac; q)_{k,p}(ad; q)_{k,p}(q^{l/m}; q^{l/m})_k} \tag{5.3.12}$$

$$\Leftrightarrow$$

$$\frac{(ae^{i\theta}; q)_{n,p}(ae^{-i\theta}; q)_{n,p}}{(ab; q)_{n,p}(ac; q)_{n,p}(ad; q)_{n,p}(q^{l/m}; q^{l/m})_n} = \sum_{k=0}^{mn} q^{nkl}\ \frac{(q^{-mn(l/m)}; q^{l/m})_k}{(q^{l/m}; q^{l/m})_k}$$
$$\times \frac{(1 - abcdq^{kL+kp-p})\ a^k\ p_{k,l,m,p}(cos\theta; a, b, c, d|q)}{(abcdq^{kp-p}; q)_{\frac{ln}{p}+1,p}(ab; q)_{k,p}(ac; q)_{k,p}(ad; q)_{k,p}(q^{l/m}; q^{l/m})_{mn}}. \tag{5.3.13}$$

Likewise, in the inverse pair (5.3.3) and (5.3.4) if a is replaced by $a+b+p$ and $G(n)$ is chosen as

$$(q^{-x}; q)_{n,p}(cdq^{x+p}; q)_{n,p}/((aq^p; q)_{n,p}(bdq^p; q)_{n,p}(cq^p; q)_{n,p}(q^{l/m}; q^{l/m})_n),$$

then $F(n) = R_{n,m,p,l}(q^{-x} + cdq^{x+1}; a, b, c, d|q)$ yields the inverse pair:

$$R_{n,m,p,l}(q^{-x} + cdq^{x+1}; a, b, c, d|q) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}\ \frac{(q^{-n(l/m)}; q^{l/m})_{mk}}{(q^{l/m}; q^{l/m})_k}$$
$$\times \frac{(abq^{np+p}; q)_{\frac{kl}{p},p}(q^{-x}; q)_{k,p}(cdq^{x+p}; q)_{k,p}}{(aq^p; q)_{k,p}(bdq^p; q)_{k,p}(cq^p; q)_{k,p}} \tag{5.3.14}$$

$$\Leftrightarrow$$

$$\frac{(q^{-x}; q)_{n,p}(cdq^{x+p}; q)_{n,p}}{(aq^p; q)_{n,p}(bdq^p; q)_{n,p}(cq^p; q)_{n,p}(q^{l/m}; q^{l/m})_n} = \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)}; q^{l/m})_k}{(q^{l/m}; q^{l/m})_k}$$
$$\times \frac{(1 - abq^{kL+kp+p})}{(abq^{kp+p}; q)_{ln/p+1,p}(q^{l/m}; q^{l/m})_{mn}}\ R_{k,m,p,l}(q^{-x} + cdq^{x+1}; a, b, c, d|q), \tag{5.3.15}$$

These p-deformed q-polynomials provide p-extension to a number of particular q-polynomials (see [35, p. 61, 62] for complete reducibility chart and [35, Ch. 3]). They include among several polynomials the q-Hahn, dual q-Hahn, continuous q-Hahn, continuous dual q-Hahn, q-Meixner-Pollaczek, Meixner, q-Krawtchouk and q-Charlier polynomials together with their inverse series relations in p-deformed sense.

5.4 q,p-Difference equations and q,p-Differential equations

In deriving the differential equation for the polynomial (5.1.3), we use the theory that the function

$$w = {}_A\phi_B \left[\begin{matrix} a_1, & a_2, & \dots, & a_A; & q & x \\ b_1, & b_2, & \dots, & b_B; & & \end{matrix} \right]$$

with $A \leq B$ and $|x| < \infty$, or $A = B+1$ and $|x| < 1$, $|q| < 1$, satisfies the q-difference equation [13, Eq.(25), p.329]:

$$\left[\left\{ \theta \prod_{i=1}^{B} (\theta + q^{1-b_i} - 1) \right\} - x q^{B+\sum a_i - \sum b_j} \left\{ \prod_{j=1}^{A} (\theta + q^{-a_j} - 1) \right\} \right] w = 0, \quad (5.4.1)$$

where $\theta f(x) = f(x) - f(xq)$.

Now for $p > 0$, let us consider the operator $\theta_{q,p} f(x) = f(x) - f(xq^p)$, then we find that

$$\frac{\theta_{q,p} f(x)}{(1-q)x} = D_{q,p} f(x), \quad (|q| < 1) \qquad (5.4.2)$$

which may be regarded as p-deformed q-derivative of $f(x)$ in which $p = 1$ yields θ-form q-derivative of $f(x)$(cf. [19, Ex.1.12, p.22] with $p = 1$). We shall obtain the q,p-difference equation and q,p-differential equation of the p-deformed extended q-Jacobi polynomial by replacing l by lp (in which case, $q^{pl/m} = r$) and using the properties (1.5.12) and (1.5.13) stated in Chapter 1 with $p = 1$. Then we get the following simplified form:

$$(r^{-n}; r)_{mk} = \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k ;$$

and using same properties for $p > 0$, we get

$$(q^{e+np};q)_{kl,p} = \prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}.$$

In view of this, the explicit representation of the p-deformed extended q-Jacobi polynomial (5.3.6) takes the form:

$$\begin{aligned}
&\mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x|q] \\
&= \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(r^{-n};r)_{mk}(q^{e+np};q)_{kl,p}(q^{\alpha_1};q)_{k,p}\cdots(q^{\alpha_c};q)_{k,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(r;r)_k} q^{klp}x^k \\
&= \sum_{k=0}^{\lfloor n/m\rfloor} \left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\} \\
&\quad\times\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}(r;r)_k\right\}^{-1}(xq^{pl})^k. \qquad (5.4.3)
\end{aligned}$$

Here w is m^{th} root of unity and ν is l^{th} root of unity. The q,p-difference equation satisfies by (5.4.3) is as given below.

Corollary 5.4.1.

$$\begin{aligned}
&\Bigg[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\} - xq^{pl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{l}\prod_{t=1}^{l}\prod_{u=1}^{c}\prod_{v=1}^{d}\Big\{q^{\alpha_u-\beta_v+p+(e+p(n+s-1))/l} \\
&\times r^{(i-1-n)/m}\ w^{j-1}\nu^{t-1}(\theta_r+r^{-((i-1)-n)/m}w^{1-j}-1)(\theta_{q,p}+q^{-(e+p(n+s-1))/l}\nu^{1-t}-1) \\
&\times(\theta_{q,p}+q^{-\alpha_u}-1)\Big\}\Bigg]\ \mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x|q]=0.(5.4.4)
\end{aligned}$$

Proof. We begin with

$$\begin{aligned}
&\left[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]\mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x|q] \\
&= \left[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]\sum_{k=0}^{\lfloor n/m\rfloor}\frac{1}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\} \\
&\quad\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}(xq^{pl})^k \\
&= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{klp}}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}
\end{aligned}$$

$$\times \left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]x^k$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{klp}}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}$$

$$\times \left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}(x^k)+q^{p-\beta_v}x^k-x^k)\right\}\right]$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{klp}}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}$$

$$\times \left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\{\theta_r(x^k)\}\left\{\prod_{v=1}^{d}(-q^{pk}+q^{p-\beta_v})\right\}$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{klp}(1-r^k)}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}$$

$$\times \left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{v=1}^{d}q^{p-\beta_v}(1-q^{pk-p+\beta_v})\right\}(xq^{pl})^k$$

$$= \left\{\prod_{v=1}^{d}q^{p-\beta_v}\right\}\sum_{k=1}^{\lfloor n/m\rfloor}\frac{1}{(r;r)_{k-1}}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}$$

$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k-1,p}\right\}^{-1}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}(xq^{pl})^k$$

$$= \left\{\prod_{v=1}^{d}q^{p-\beta_v}\right\}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{1}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_{k+1}\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k+1,p}\right\}$$

$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k+1,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}(xq^{pl})^{k+1}.$$

Thus we get,

$$\left[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]\mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x|q]$$

$$= xq^{pl}\left\{\prod_{v=1}^{d}q^{p-\beta_v}\right\}\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_{k+1}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}$$

$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k+1,p}\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k+1,p}\right\}\frac{(xq^{pl})^k}{(r;r)_k}. \quad (5.4.5)$$

On the other side we have

$$
\begin{aligned}
& \left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r + r^{-(i-1-n)/m}w^{1-j} - 1)\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q,p} + q^{-(e+p(n+s-1))/l}\nu^{1-t} - 1)\right\}\right.\\
& \left.\times\left\{\prod_{u=1}^{c}(\theta_{q,p} + q^{-\alpha_u} - 1)\right\}\right] \mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta):x|q]\\
={}& \left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r + r^{-(i-1-n)/m}w^{1-j} - 1)\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q,p} + q^{-(e+p(n+s-1))/l}\nu^{1-t} - 1)\right\}\right.\\
& \left.\times\left\{\prod_{u=1}^{c}(\theta_{q,p} + q^{-\alpha_u} - 1)\right\}\right] \sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\\
& \times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}(r;r)_k\right\}^{-1}(xq^{pl})^k\\
={}& \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(q^{pl})^k}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}\\
& \times\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r + r^{-(i-1-n)/m}w^{1-j} - 1)\right\}\right.\\
& \left.\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q,p} + q^{-(e+p(n+s-1))/l}\nu^{1-t} - 1)\right\}\right]\left[\prod_{u=1}^{c}(\theta_{q,p} + q^{-\alpha_u} - 1)\right]x^k\\
={}& \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(q^{pl})^k}{(r;r)_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k,p}\right\}\\
& \times\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}(-r^k + r^{-(i-1-n)/m}w^{1-j})\right\}\right.\\
& \left.\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(-q^{pk} + q^{-(e+p(n+s-1))/l}\nu^{1-t})\right\}\left\{\prod_{u=1}^{c}(-q^{pk} + q^{-\alpha_u})\right\}\right]x^k\\
={}& \left\{\prod_{i=1}^{m}\prod_{j=1}^{m}r^{-(i-1-n)/m}w^{1-j}\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}q^{-(e+p(n+s-1))/l}\nu^{1-t}\right\}\left\{\prod_{u=1}^{c}q^{-\alpha_u}\right\}\\
& \times\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(p(i-1)-np)/m}w^{j-1};r\right)_{k+1}\right\}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(e+p(n+s-1))/l}\nu^{t-1};q\right)_{k+1,p}\right\}\\
& \times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k+1,p}\right\}\frac{(xq^{pl})^k}{(r;r)_k}.
\end{aligned}
\tag{5.4.6}
$$

Finally, we get the bibasic p-deformed q-difference equation (5.4.4) satisfied by (5.4.3) from (5.4.5) and (5.4.6). □

Corollary 5.4.2. *The function $y = \mathcal{F}^{(e)}_{n,l,m,p}[(\alpha);(\beta) : x|q]$ satisfies the bibasic q,p-differential equation*

$$\Bigg[(1-r)xD_r\left\{\prod_{v=1}^{d}((1-q)xD_{q,p}+q^{p-\beta_v}-1)\right\} - xq^{pl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{l}\prod_{t=1}^{l}\prod_{u=1}^{c}\prod_{v=1}^{d}$$
$$\times\Bigg\{r^{(i-1-n)/m}q^{\alpha_u-\beta_v+p+(e+p(n+s-1))/l}w^{j-1}\nu^{t-1}((1-r)xD_r+r^{-(i-1-n)/m}w^{1-j}-1)$$
$$\times(x(1-q)D_{q,p}+q^{-(e+p(n+s-1))/l}\nu^{1-t}-1)((1-q)xD_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]y=0. \tag{5.4.7}$$

Proof. This follows when (5.4.2) is used in (5.4.4). □

The following corollaries are the consequence of the limiting case $q^e \to 0$ of (5.4.4) and (5.4.7).

Corollary 5.4.3. *The function $y = B^m_{n,p}[(\alpha);(\beta) : xq^{pl}|q]$ satisfies the q,p-difference equation*

$$\Bigg[\theta_r\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\} - xq^{pl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{r^{(i-1-n)/m}$$
$$\times q^{\alpha_u-\beta_v+p}w^{j-1}(\theta_r+r^{-(i-1-n)/m}w^{1-j}-1)(\theta_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]y=0$$

and

Corollary 5.4.4. *The function $y = B^m_{n,p}[(\alpha);(\beta) : xq^{pl}|q]$ satisfies the bibasic p-deformed q-differential equation:*

$$\Bigg[(1-r)xD_r\left\{\prod_{v=1}^{d}((1-q)xD_{q,p}+q^{p-\beta_v}-1)\right\} - xq^{pl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{r^{((i-1)p-np)/m}$$
$$\times q^{\alpha_u-\beta_v+p}w^{j-1}((1-r)xD_r+r^{-(i-1-n)/m}w^{1-j}-1)((1-q)xD_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]y=0,$$

where w is m^{th} root of unity.

In order to obtain the bibasic p-deformed q-difference as well as q-differential equation of the extended p-deformed q-Konhauser polynomial (5.3.8), we first write (5.3.8) with $q^l = r$, that is

$$Z^{(\alpha)}_{n,m,p}(x;l|q) = \frac{(p\alpha;q)_{nl,p}}{(r;r)_n}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{r^{k(\alpha+n+1)+k(lk-1)/2}(r^{-n};r)_{mk}x^{lk}}{(p\alpha;q)_{kl,p}(r;r)_{mk}}. \tag{5.4.8}$$

In view of the properties (1.5.12) and (1.5.13), we have

$$(p\alpha;q)_{kl,p} = \prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}$$

and

$$\begin{aligned}(r;r)_{mk} &= (r,r^2,r^3,\cdots,r^m;r^m)_{k,p}\\ &= \prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k.\end{aligned}$$

Using these in (5.4.8), we get

$$\begin{aligned}Z^{(\alpha)}_{n,m,p}(x;l|r) &= \frac{(p\alpha;q)_{nl,p}}{(r;r)_n}\sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\\ &\times\left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1}x^{lk}, \quad (5.4.9)\end{aligned}$$

where w and ν are m^{th} root of unity. For this polynomial, we have the

Corollary 5.4.5. *The polynomial (5.4.8) satisfies the q,p-difference equation:*

$$\begin{aligned}&\left[\prod_{t=1}^{l}\prod_{u=1}^{m}\prod_{v=1}^{m}\left\{(\theta_{r,p}+r^{p-(\alpha+pt)/l}-1)(\theta_r+r^{1-(u/m)}\nu^{1-v}-1)\right\}\right]Z^{(\alpha)}_{n,m,p}(x;l|r)\\ &-x^l r^{(\alpha+n+1)+(l-1)/2}\left[\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{t=1}^{l}\prod_{u=1}^{m}\prod_{v=1}^{m}\left\{r^{p-((\alpha+pt)/l)+1-(u/m)+(i-1-n)/m}\right.\right.\\ &\left.\left.\times\nu^{1-v}w^{j-1}(\theta_r+r^{-(i-1-n)/m}w^{1-j}-1)\right\}\right]Z^{(\alpha)}_{n,m,p}(xr;l|r)=0. \quad (5.4.10)\end{aligned}$$

Proof. We start with

$$\begin{aligned}&\left[\left\{\prod_{t=1}^{l}(\theta_{r,p}+r^{p-(\alpha+pt)/l}-1)\right\}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}(\theta_r+r^{1-(u/m)}\nu^{1-v}-1)\right\}\right]\frac{(r;r)_n Z^{(\alpha)}_{n,m,p}(x;l|r)}{(p\alpha;q)_{nl,p}}\\ &=\sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}\\ &\times\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1}\left\{\prod_{t=1}^{l}(\theta_{r,p}+r^{p-(\alpha+pt)/l}-1)\right\}\\ &\times\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}(\theta_r+r^{1-(u/m)}\nu^{1-v}-1)\right\}x^{lk}\end{aligned}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} \left(r^{u/m} \nu^{v-1}; r \right)_k \right\}^{-1} \left\{ \prod_{t=1}^{l} (\theta_{r,p} + r^{p-(\alpha+pt)/l} - 1) \right\}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} (\theta_r(x^{lk}) + r^{1-(u/m)} \nu^{1-v} x^{lk} - x^{lk}) \right\}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} \left(r^{u/m} \nu^{v-1}; r \right)_k \right\}^{-1} \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} (-r^k + r^{1-(u/m)} \nu^{1-v}) \right\}$$

$$\times \left\{ \prod_{t=1}^{l} (\theta_{r,p}(x^{lk}) + r^{p-(\alpha+pt)/l} x^{lk} - x^{lk}) \right\}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} \left(r^{u/m} \nu^{v-1}; r \right)_k \right\}^{-1} \left\{ \prod_{t=1}^{l} (-r^{pk} + r^{p-(\alpha+pt)/l}) \right\}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} (-r^k + r^{1-(u/m)} \nu^{1-v}) \right\} x^{lk}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} \left(r^{u/m} \nu^{v-1}; r \right)_k \right\}^{-1} \left\{ \prod_{t=1}^{l} r^{p-(\alpha+pt)/l} \right\} \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} r^{1-(u/m)} \nu^{1-v} \right\} x^{lk}$$

$$= \left\{ \prod_{t=1}^{l} r^{p-(\alpha+pt)/l} \right\} \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} r^{1-(u/m)} \nu^{1-v} \right\}$$

$$\times \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$

$$\times \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} \left(r^{u/m} \nu^{v-1}; r \right)_k \right\}^{-1} x^{lk}$$

$$= \left\{ \prod_{t=1}^{l} r^{p-(\alpha+pt)/l} \right\} \left\{ \prod_{u=1}^{m} \prod_{v=1}^{m} r^{1-(u/m)} \nu^{1-v} \right\}$$

$$\times \sum_{k=0}^{\lfloor n/m \rfloor} r^{(k+1)(\alpha+n+1)+(k+1)(lk+l-1)/2} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r \right)_{k+1} \right\}$$

$$\times \left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1} x^{lk+l}$$

$$= \left\{\prod_{t=1}^{l} r^{p-(\alpha+pt)/l}\right\}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m} r^{1-(u/m)}\nu^{1-v}\right\} x^l r^{(\alpha+n+1)+(l-1)/2}$$

$$\times \sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_{k+1}\right\}$$

$$\times \left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1}(xr)^{lk}.$$

Thus,

$$\left[\left\{\prod_{t=1}^{l}(\theta_{r,p}+r^{p-(\alpha+pt)/l}-1)\right\}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}(\theta_r+r^{1-(u/m)}\nu^{1-v}-1)\right\}\right]$$

$$\times\frac{(r;r)_n Z^{(\alpha)}_{n,m,p}(x;l|r)}{(p\alpha;q)_{nl,p}}$$

$$= \left[\left\{\prod_{t=1}^{l} r^{p-(\alpha+pt)/l}\right\}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m} r^{1-(u/m)}\nu^{1-v}\right\}\right] x^l r^{(\alpha+n+1)+(l-1)/2}$$

$$\times \sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_{k+1}\right\}$$

$$\times \left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}(xr)^{lk}. \qquad (5.4.11)$$

Next on the other side we have

$$\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r+r^{-(i-1-n)/m}w^{1-j}-1)\right]\frac{(r;r)_n}{(p\alpha;q)_{nl,p}} Z^{(\alpha)}_{n,m,p}(xr;l|r)$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}$$

$$\times \left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1} r^{lk}\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r+r^{-(i-1-n)/m}w^{1-j}-1)\right] x^{lk}$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor} r^{k(\alpha+n+1)+k(lk-1)/2}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(r^{(i-1-n)/m}w^{j-1};r\right)_k\right\}\left\{\prod_{t=1}^{l}(r^{(\alpha+pt)/l};r)_{k,p}\right\}^{-1}$$

$$\times \left\{\prod_{u=1}^{m}\prod_{v=1}^{m}\left(r^{u/m}\nu^{v-1};r\right)_k\right\}^{-1} r^{lk}\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_r(x^{lk})+r^{-(i-1-n)/m}w^{1-j}x^{lk}-x^{lk})\right]$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m}\prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r\right)_k \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{u=1}^{m}\prod_{v=1}^{m} \left(r^{u/m}\nu^{v-1}; r\right)_k \right\}^{-1} (xr^l)^k \left[\prod_{i=1}^{m}\prod_{j=1}^{m} (-r^k + r^{-(i-1-n)/m} w^{1-j}) \right]$$
$$= \left[\prod_{i=1}^{m}\prod_{j=1}^{m} (r^{-(i-1-n)/m} w^{1-j}) \right] \sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{i=1}^{m}\prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r\right)_{k+1} \right\} \left\{ \prod_{u=1}^{m}\prod_{v=1}^{m} \left(r^{u/m}\nu^{v-1}; r\right)_k \right\}^{-1} (xr^l)^k.$$

Thus,

$$\sum_{k=0}^{\lfloor n/m \rfloor} r^{k(\alpha+n+1)+k(lk-1)/2} \left\{ \prod_{i=1}^{m}\prod_{j=1}^{m} \left(r^{(i-1-n)/m} w^{j-1}; r\right)_{k+1} \right\} \left\{ \prod_{t=1}^{l} (r^{(\alpha+pt)/l}; r)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{u=1}^{m}\prod_{v=1}^{m} \left(r^{u/m}\nu^{v-1}; r\right)_k \right\}^{-1} (xr^l)^k$$
$$= \left[\prod_{i=1}^{m}\prod_{j=1}^{m} (r^{(i-1-n)/m} w^{j-1})(\theta_r + r^{-(i-1-n)/m} w^{1-j} - 1) \right] \frac{Z^{(\alpha)}_{n,m,p}(xr; l|r)}{(r;r)_n^{-1} (p\alpha; q)_{nl,p}}. \quad (5.4.12)$$

The p-deformed q-difference equation of (5.4.9) now follows from (5.4.11) and (5.4.12). □

The corresponding q, p-differential equation of (5.4.9) is obtained by making use of (5.4.2) in (5.4.10) which is given by

Corollary 5.4.6. *The polynomial* $y = Z^{(\alpha)}_{n,m,p}(x; l|r)$ *satisfies the* q, p*-differential equation:*

$$\left[\prod_{t=1}^{l}\prod_{u=1}^{m}\prod_{v=1}^{m} \left\{ ((1-r)xD_{r,p} + r^{p-(\alpha+pt)/l} - 1)((1-r)xD_r + r^{1-(u/m)}\nu^{1-v} - 1) \right\} \right]$$
$$\times Z^{(\alpha)}_{n,m,p}(x; l|r) - x^l r^{(\alpha+n+1)+(l-1)/2} \left[\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{t=1}^{l}\prod_{u=1}^{m}\prod_{v=1}^{m} \left\{ r^{p-((\alpha+pt)/l)+1-(u/m)+(i-1-n)/m} \right.\right.$$
$$\left.\left. \times \nu^{1-v} w^{j-1} ((1-r)xD_r + r^{-(i-1-n)/m} w^{1-j} - 1) \right\} \right] Z^{(\alpha)}_{n,m,p}(xr; l|r) = 0. \quad (5.4.13)$$

Since the extended p-deformed q-Konhauser polynomial reduces to the extended p-deformed q-Laguerre polynomial (5.3.10) when $l = 1$ in which case $r = q$, we obtain the p-deformed q-difference equation for the extended p-deformed q-Laguerre polynomial by making use of $l = 1$ in (5.4.10). We thus have the

Corollary 5.4.7. *The extended p-deformed q-Laguerre polynomial satisfies the q, p-difference equation:*

$$\left[(\theta_{q,p}+q^{-\alpha}-1)\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}(\theta_q+q^{1-(u/m)}\nu^{1-v}-1)\right\}\right]L^{(\alpha)}_{n,m,p}(x|q)-xq^{n+1}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{m}\prod_{v=1}^{m}\left\{q^{1-(u/m)+(i-1-n)/m}\nu^{1-v}w^{j-1}\right.\right.$$
$$\left.\left.\times(\theta_q+q^{-(i-1-n)/m}w^{1-j}-1)\right\}\right]L^{(\alpha)}_{n,m,p}(xq|q)=0. \quad (5.4.14)$$

The differential equation then follows from (5.4.13) with $l = 1$ which is stated as

Corollary 5.4.8. *The extended p-deformed q-Laguerre polynomial satisfies the q, p-differential equation:*

$$\left[((1-q)xD_{q,p}+q^{-\alpha}-1)\left\{\prod_{u=1}^{m}\prod_{v=1}^{m}((1-q)xD_q+q^{1-(u/m)}\nu^{1-v}-1)\right\}\right]L^{(\alpha)}_{n,m,p}(x|q)$$
$$-xq^{n+1}\left[\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{m}\prod_{v=1}^{m}\left\{q^{1-(u/m)+(i-1-np)/m}\nu^{1-v}w^{j-1}\right.\right.$$
$$\left.\left.\times((1-q)xD_q+q^{-(i-1-np)/m}w^{1-j}-1)\right\}\right]L^{(\alpha)}_{n,m,p}(xq|q)=0.(5.4.15)$$

The choices $m = 1$ and $p = 1$ in (5.4.15) lead us to

$$\left[(1-q)xD_q+q^{-\alpha}-1\right](1-q)xD_qL^{(\alpha)}_n(x|q)$$
$$-xq\left[((1-q)xD_q+q^n-1)\right]L^{(\alpha)}_n(xq|q)=0,$$

that is,

$$(1-q)xD_q\left[(1-q)xD_qL^{(\alpha)}_n(x|q)\right]+(q^{-\alpha}-1)(1-q)xD_qL^{(\alpha)}_n(x|q)$$
$$-xq\left[((1-q)xD_q+q^n-1)\right]L^{(\alpha)}_n(xq|q)=0.$$

It can be shown that this equation approaches to the ordinary differential equation of $L^{\alpha}_n(x)$ as $q\to1^-$ from within the interval $(0,1)$. For that we replace x by $x(1-q)$ to get

$$(1-q)xD_q\left[(1-q)xD_qL^{(\alpha)}_n(x(1-q)|q)\right]+(q^{-\alpha}-1)(1-q)xD_qL^{(\alpha)}_n(x(1-q)|q)$$
$$-xq(1-q)\left[x(1-q)D_q+q^n-1\right]L^{(\alpha)}_n(xq(1-q)|q)=0.$$

Dividing by $x(1-q)$, it becomes

$$D_q\left[(1-q)xD_qL_n^{(\alpha)}(x(1-q)|q)\right]+(q^{-\alpha}-1)D_qL_n^{(\alpha)}(x(1-q)|q)$$
$$-q\left[x(1-q)D_q+q^n-1\right]L_n^{(\alpha)}(xq(1-q)|q)=0.$$

Now with $f(x)=(1-q)x$ and $g(x)=D_qL_n^{(\alpha)}(x|q)$, the q-product rule:

$$D_q\left[f(x)g(x)\right]=f(xq)D_qg(x)+g(x)D_qf(x)$$

transforms this equation to the form:

$$(1-q)xqD_q^2L_n^{(\alpha)}(x(1-q)|q)+(1-q)D_qL_n^{(\alpha)}(x(1-q)|q)$$
$$+(q^{-\alpha}-1)D_qL_n^{(\alpha)}(x(1-q)|q)-q\left[x(1-q)D_q+q^n-1\right]L_n^{(\alpha)}(xq(1-q)|q)=0,$$

that is,

$$xqD_q^2L_n^{(\alpha)}(x(1-q)|q)+D_qL_n^{(\alpha)}(x(1-q)|q)+\left(\frac{q^{-\alpha}-1}{1-q}\right)D_qL_n^{(\alpha)}(x(1-q)|q)$$
$$-q\left[xD_q+\frac{q^n-1}{1-q}\right]L_n^{(\alpha)}(xq(1-q)|q)=0.$$

Applying now $q\to 1^-$, we find

$$x\left[\frac{d^2L_n^{(\alpha)}(x)}{dx^2}\right]+(\alpha+1-x)\frac{dL_n^{(\alpha)}(x)}{dx}+nL_n^{(\alpha)}(x)=0,$$

which is the differential equation of Laguerre polynomial $L_n^{(\alpha)}(x)$ [53, Eq.(1), p. 204].

5.5 Generating function relations

The generating function relations (GFR) for the general class of q,p-polynomials (5.1.3), the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials will be derived with the help of the alternative form (5.3.3) given by

$$\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\,t^n$$
$$=\sum_{n=0}^{\infty}q^{ln(n-1)/2m}\frac{(a;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\sum_{k=0}^{\lfloor n/m\rfloor}q^{kl}(q^{-ln/m};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}G(k)\,t^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}q^{ln(n-1)/2m}\frac{(q^{-ln/m};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_n}\,(a;q)_{n+\frac{kl}{p},p}\,q^{klp}G(k)\,t^n.$$

Now, in formula (1.5.10), replacing k by mk and taking $p=1$, it becomes

$$(q^{l/m};q^{l/m})_{n-mk}=(-1)^{mk}q^{lk(mk+1)/2-lk-lnk}\ \frac{(q^{l/m};q^{l/m})_n}{(q^{-ln/m};q^{l/m})_{mk}},$$

consequently, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{ln(n-1)/2m+mk(mk+1)/2-lk-lnk}\frac{(a;q)_{n+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_{n-mk}}\ q^{kl}G(k)\ t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{ln(n-1)/2m+lk(mk-1)/2-lnk}\frac{(a;q)_{n+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_{n-mk}}\ q^{kl}G(k)\ t^n.\end{aligned}$$

Here the double sum may be replaced by means of the identity:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk),$$

to get

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{l(n+mk)(n+mk-1)/2m+lk(mk-1)/2-(n+mk)lk}\frac{(a;q)_{n+mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_n}\ q^{kl}\\ &\times G(k)\ t^{n+mk}\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{ln(n-1)/2m}\frac{(a;q)_{n+mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_n}\ G(k)\ t^{n+mk}.\end{aligned}\tag{5.5.1}$$

From this, we deduce the GFR of the q-polynomials as follows.

∗ **GFR of** $\mathcal{B}^a_{n,m,p}(x|q;l)$.

In (5.5.1), the choice $G(n)=\gamma_n x^n$ implies $F(n)=\mathcal{B}^a_{n,m,p}(x|q;l)$ which yields a general generating function relation:

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}\frac{(a;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\mathcal{B}^a_{n,m,p}(x|q;l)\ t^n\\ =\ &\sum_{k=0}^{\infty}(-1)^{mk}(a;q)_{mk+\frac{kl}{p},p}\ {}_1\phi_1\left(aq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right)\ \gamma_k\ \left((-t)^m x\right)^k,\end{aligned}\tag{5.5.2}$$

in which we have used (1.5.8). We shall also assume that $|t| < 1$ whenever required. This relation when further specialized appropriately, provides us the GFR of the particular polynomials which are illustrated below.

$*$ **GFR of** $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$.

Here choosing $l \in \mathbb{C}\setminus\{0\}$, $a = e$ and $\gamma_n = \dfrac{(\alpha_1;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}{(\beta_1;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{l/m};q^{l/m})_n}$ in (5.5.2), we immediately obtain the GFR

$$\sum_{n=0}^{\infty} q^{(l/m)n(n-1)/2}\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]\ \frac{(e;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\ t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}\frac{q^{ln(n-1)/2m}(e;q)_{n+mk+\frac{kl}{p},p}(\alpha_1;q)_{k,p}\cdots(\alpha_c;q)_{k,p}}{(\beta_1;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{l/m};q^{l/m})_k(q^{l/m};q^{l/m})_n}x^k\ t^{n+mk}$$
$$= \sum_{k=0}^{\infty}\ {}_1\phi_1\left(eq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right)\frac{(\alpha_1;q)_{k,p}\cdots(\alpha_c;q)_{k,p}\,((-t)^m x)^k}{(\beta_1;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{l/m};q^{l/m})_n}.$$

$*$ **GFR of** $B^m_{n,p}[(\alpha);(\beta):xq^l|q]$.

The limiting case $e \equiv q^e \to 0$ of the above GFR yields

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\ B^m_{n,p}[(\alpha);(\beta):xq^l|q]\ \frac{t^n}{(q^{l/m};q^{l/m})_n}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}\frac{q^{ln(n-1)/2m-lk}(\alpha_1;q)_{k,p}\cdots(\alpha_c;q)_{k,p}}{(\beta_1;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{l/m};q^{l/m})_k(q^{l/m};q^{l/m})_n}x^k\ t^{n+mk}$$
$$= E_{q^{l/m}}(t)\sum_{k=0}^{\infty}\frac{(q^{\alpha_1};q)_{k,p}\cdots(q^{\alpha_c};q)_{k,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(q^{l/m};q^{l/m})_k}(x(-t)^m q^{-l})^k$$
$$= E_{q^{l/m}}(t)\ {}_c\phi_d((\alpha);\ (\beta)\ ;p)\left(x(-t)^m q^{-l}x\middle|q,q^{l/m}\right),$$

wherein we have used the q-exponential function (3.5.3) and we assume for convergence that $c < d+1$, and $|t| < 1$ if $c = d+1$.

$*$ **GFR of** $p_{n,m,p,l}(x;a,b;q)$.

In (5.5.2), we replace a by $a+b+p$ and substitute $\gamma_n = 1/((ap;q)_{n,p}(q^{l/m};q^{l/m})_n)$ to get

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n}p_{n,m,p,l}(x;a,b;q)\ t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{mk}q^{ln(n-1)/2m}(abq^p;q)_{n+mk+\frac{kl}{p},p}}{(ap;q)_{n,p}(q^{l/m};q^{l/m})_k(q^{l/m};q^{l/m})_n}\ x^k\ t^{n+mk}$$
$$= \sum_{k=0}^{\infty}\frac{(abq^p;q)_{mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_k}\ {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m}).$$

$*$ **GFR of** $Z^{(\alpha)}_{n,m,p}(x;l|q)$.

In (5.5.2), taking limit $q^a \to 0$, replacing l and x by lm and $(xq^n)^l, l \in \mathbb{N}$, and putting $\gamma_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/((p\alpha;q)_{nl,p}(q^l;q^l)_{mn})$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{ln(n-1)/2}\frac{Z^{(\alpha)}_{n,m,p}(x;l|q)}{(p\alpha;q)_{nl,p}}\ t^n\\
=&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{nl(n-1)/2-klm}\frac{q^{kl(\alpha+n+1)+kl(kl-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}(q^l;q^l)_n}x^{kl}\ t^{n+mk}.\\
=&\sum_{k=0}^{\infty}(-1)^{mk}q^{-klm}\left\{\sum_{n=0}^{\infty}q^{ln(n-1)/2}\frac{(tq^{lk})^n}{(q^l;q^l)_n}\right\}\frac{q^{kl(\alpha+1)+kl(kl-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}}(x^lt^m)^k.\\
=&\sum_{k=0}^{\infty}(-1)^{mk}q^{-klm}(-tq^{lk};q^l)_\infty\frac{q^{kl(\alpha+1)+kl(kl-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}}(x^lt^m)^k\\
=&(-t;q^l)_\infty\sum_{k=0}^{\infty}q^{kl(kl-1)/2}\frac{w^k}{(p\alpha;q)_{lk,p}(-t;q^l)_k(q^l;q^l)_{mk}},
\end{aligned}$$

which is a bibasic convergent hypergeometric series with $w=(-t)^m x^l q^{(\alpha-m+1)l}$.

$*$ **GFR of** $L^{(\alpha)}_{n,m,p}(x|q)$.

It is an instance $l=1$ the last GFR of $Z^{(\alpha)}_{n,m,p}(x;l|q)$.

We now deduce the GFRs of the polynomials (5.3.12) and (5.3.14) from the general GFR (5.5.1). Here also wherever necessary, we assume that $|t|<1$.

Now, if a is replaced by $a+b+c+d-p$ and

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n),$$

then

$$F(n) = p_{n,l,m,p}(cos\theta;a,b,c,d|q)a^n/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p})$$

leads us to the GFR

$$\begin{aligned}
&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(abcdq^{-p};q)_{n,p}\frac{p_{n,l,m,p}(cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}\ t^n\\
=&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}\frac{q^{ln(n-1)/2m}(abcdq^{-p};q)_{n+mk+\frac{kl}{p},p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_k(q^{l/m};q^{l/m})_n}t^{n+mk}\\
=&\sum_{k=0}^{\infty}\frac{(abcdq^{-p};q)_{mk+\frac{kl}{p},p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_k}\\
&\times{}_1\phi_1\left(abcdq^{mkp+kl-p};0;p\right)(t|q,q^{l/m})(-t)^{mk}.
\end{aligned}$$

Likewise, replacing a by $a+b+p$

$$G(n) = (q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (5.5.1), we obtain with $F(n) = R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)$, the following GFR.

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\; R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)\; t^n \\
=\;& \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{mk}q^{ln(n-1)/2m}(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}(abq^p;q)_{n+mk+\frac{kl}{p},p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}(q^{l/m};q^{l/m})_k(q^{l/m};q^{l/m})_n}\; q^{kl}\; t^{n+mk} \\
=\;& \sum_{k=0}^{\infty} \frac{(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}(abq^p;q)_{mk+\frac{kl}{p},p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}(q^{l/m};q^{l/m})_k} \\
&\times {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)\left(t|q,q^{l/m}\right)(-t)^{mk}.
\end{aligned}$$

5.6 Summation formulas

In this section, we illustrate the use of the inverse series of the GIP and in particular the inverse series of the general class (5.1.3), to deduce certain summation formulas. The inverse series (5.2.6) of Theorem - 5.2.1 provides the sums involving the p-Askey-Wilson polynomials (5.3.12) and q,p-Racah polynomials (5.3.14); whereas the inverse series (5.3.5) takes care of the sums involving the other particular polynomials.

We Begin with the inverse series (5.3.5) with the assumption that $\gamma_n \neq 0, \forall n = 0,1,2,\ldots$, then we have

$$\frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) \;=\; x^n. \quad (5.6.1)$$

In this, multiplying both sides by $(a;q)_n/(q;q)_n$ and taking the summation from $n=0$ to ∞ and then using the q-Binomial theorem (3.5.1) with $|x|<1$, we get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) \;=\; \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

If $x=0$, then $\mathcal{B}^a_{k,m,p}(0|q;l) = \gamma_0$ simplifies this sum to the form:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n\gamma_0}{(q;q)_n\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = 1.$$

Next, multiplying both sides by $q^{n(n-1)/2}/(q;q)_n$ and taking summation from $n=0$ to ∞, in (5.6.1), we find in view of (3.5.3), the sum

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{\gamma_n(q;q)_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l)=E_q(x).$$

Similarly, we obtain

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n(q;q)_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l)=e_q(x),$$

using (3.5.2) with $|x|<1$. Taking summation $n=0$ to ∞ and assuming $|x|<1$ in (5.6.1) yields

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = \frac{1}{1-x}. \tag{5.6.2}$$

By assigning different values to x from $(-1,1)$, a number of particular summation formulas can be derived. For example, $x=1/2$ in this formula gives the following one.

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}\left(\frac{1}{2}\Big|q;l\right) = 2.$$

Now consider the summation formula of ${}_1\phi_1$ given by [19, Eq.(II.5), p.236]

$${}_1\phi_1(a;c;q,c/a)=\frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{5.6.3}$$

The sum of ${}_1\phi_1[*]$ in (5.6.3) enables us to obtain one more summation formula by multiplying

$$\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{(c;q)_n(q;q)_n}$$

to both sides of (5.6.1), replacing x by c/a and then summing-up from $n=0$ to ∞. We then obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{\gamma_n(c;q)_n(q;q)_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}\left(\frac{c}{a}\Big|q;l\right)$$
$$=\frac{(c/a;q)_\infty}{(c;q)_\infty}.$$

The reducibility to all these summation formulas to the particular polynomials may be obtained by making the substitutions as specified in Section 5.3. For instance, taking $\gamma_n = (q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}/((q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_{n,p})$ and $a = e$ in (5.6.2) yields the summation formula involving the p-deformed extended q-Jacobi polynomials as follows.

$$\sum_{n=0}^{\infty}\frac{(\beta_1;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{l/m};q^{l/m})_{n,p}}{(\alpha_1;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}\sum_{k=0}^{mn}q^{kln}\frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}}$$
$$\times\frac{(1-q^{e+Lk+kp})}{(q^{e+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_k}\mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q]=\frac{1}{1-x}.$$

We now obtain summation formulas involving the extended p-deformed Askey-Wilson polynomials and p-deformed extended q-Racah polynomials with the help of the sum (3.5.13):

$${}_2\phi_1\left(a,b;c;q^p,\frac{c}{ab}\right) = \frac{(c/a;q)_{\infty,p}\,(c/b;q)_{\infty,p}}{(c;q)_{\infty,p}\,(c/ab;q)_{\infty,p}}. \tag{5.6.4}$$

and the sum (3.5.14):

$${}_2\phi_1\left(q^{-np},b;c;q^p,\frac{cq^{np}}{b}\right) = \frac{(c/b;q)_{n,p}}{(c;q)_{n,p}}. \tag{5.6.5}$$

We rewrite the inverse series (5.3.13) by introducing $(q^p;q)_{n,p}$ to get

$$\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}}$$
$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}=\frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}. \tag{5.6.6}$$

We intend to use (5.6.4), and for that we multiply both sides of (5.6.6) by

$$q^{n(a+b-(a+cos\theta+isin\theta+a+cos\theta-isin\theta))}=q^{n(b-a-2cos\theta)}$$

and then take sum from $n=0$ to ∞, then after little simplification, we find

$$\sum_{n=0}^{\infty}\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}q^{n(b-a-2cos\theta)}=\frac{\left(be^{-i\theta};q\right)_{\infty,p}\left(be^{i\theta};q\right)_{\infty,p}}{(ab;q)_{\infty,p}\left(q^{b-a-2cos\theta};q\right)_{\infty,p}}.$$

In (5.6.6), we transfer $(ae^{-i\theta};q)_{n,p}$ to the other side to get

$$\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}=\frac{(ae^{i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}.$$

In this, multiplying both sides by $(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n$ and then taking the sum from $n=0$ to j, then using (5.6.5) on the right hand side, we obtain

$$\sum_{n=0}^{j}\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n=\frac{\left(be^{-i\theta};q\right)_{j,p}}{(ab;q)_{j,p}}.$$

We proceed in a similar manner to derive summation formulas from the inverse series (5.3.15). We rewrite it by introducing the factor $(q^p;q)_{n,p}$ and transfer the factors $(cdq^{x+p};q)_{n,p}$, $(bdq^p;q)_{n,p}$, $(cq^p;q)_{n,p}$ and $(q^{l/m};q^{l/m})_n$ to the other side to get

$$\frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}$$
$$\times R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)=\frac{(q^{-x};q)_{n,p}}{(aq^p;q)_{n,p}(q^p;q)_{n,p}}.$$

Now multiplying both sides by $(q^{-jp};q)_{n,p}(axq^{jp+p})^n$ and then taking the summation from $n=0$ to j, we obtain

$$\sum_{n=0}^{j}\frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}}$$
$$\times\frac{(q^{-jp};q)_{n,p}(axq^{jp+p})^n}{(q^{l/m};q^{l/m})_k}R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)=\frac{\left(aq^{x+p};q\right)_{j,p}}{(aq^p;q)_{j,p}}.$$

5.7 Companion matrix

Taking $\lfloor n/m\rfloor = N$ in (5.1.3) and converting it to the monic form $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$, we get

$$\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l) = \sum_{k=0}^{N}\delta_k\, x^k,$$

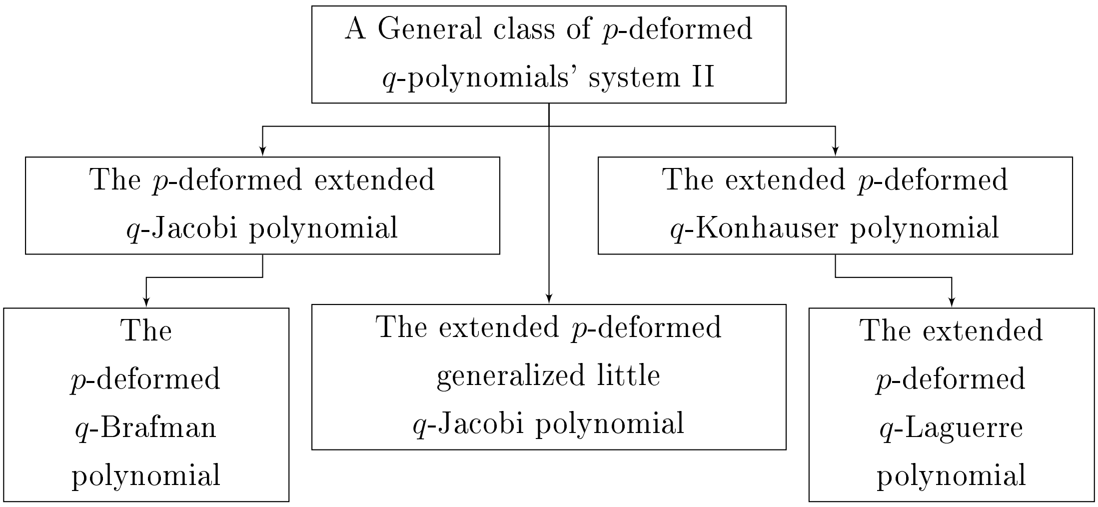
where

$$\delta_k \;=\; \frac{q^{lk}(q^{-n(l/m)};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\gamma_k}{q^{lN}(q^{-(l/m)n};q^{(l/m)})_N(q^{a+np};q)_{\frac{Nl}{p},p}\gamma_N}.$$

Thus, $\widetilde{B}^a_{n,m,p}(x|q;l)$ is of the form as stated in Definition 1.3.1. The eigen values of the Companion matrix will be then precisely the zeros of the polynomial $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$.

Apart from yielding the extended p-deformed q-polynomials, Theorem - 5.2.1 and its alternative forms also provide an effective tool for carrying out the extension of certain inverse series relations belonging to q-Riordan's classification [11, Table 2,5,7, p. 17-20] in the sense of p-deformation (also see [55] for *ordinary* forms). They are derived in chapter 9.

BASIC POLYNOMIALS' REDUCIBILITY



THE INTERCONNECTIONS OF THE p-DEFORMED EXTENDED BASIC HYPERGEOMETRIC POLYNOMIALS