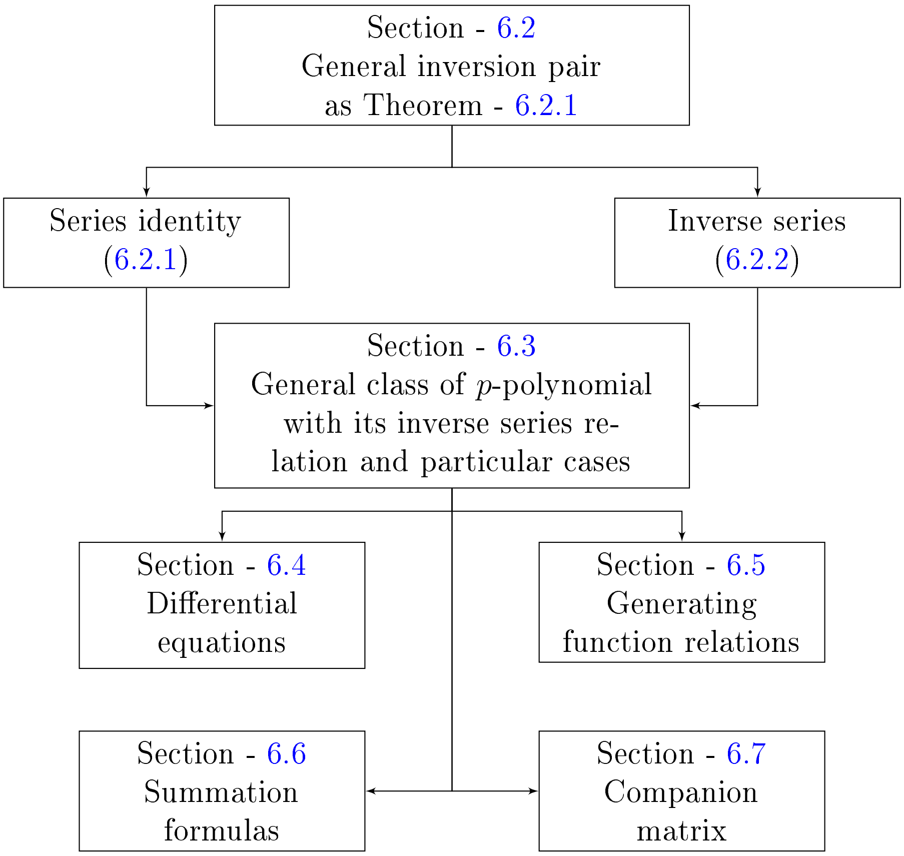
Chapter 6

The p-deformed polynomials' system - III



6.1 Introduction

H. W. Gould[26], J. P. Singhal and Savita kumari[38], Dave and Dalbhide [8] and others proposed various unified of polynomials' systems in the form of a general class of polynomials and derived inverse series relation, generating function relations, differential equation and other properties of it. In this chapter, we provide an extension to the general class of polynomials

$$S_n(l, m, \alpha, \beta : x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk}\ \lambda_k}{\Gamma(1 + \beta - n\alpha + lk)(n - mk)!}\ x^k, \tag{6.1.1}$$

due Dalbhide and Dave[8], in the light of recently proposed one parameter deformation $\Gamma_p(x)$ of the classical gamma function $\Gamma(x)$ such that $\Gamma_p(x)$ reduces to

$\Gamma(x)$ when $p = 1$. Here $\{\lambda_k\}$ is a general sequence; does not involve n. Dave and Dalbhide obtained the inverse series relation of (6.1.1) in the form:

$$\lambda_n x^n = \sum_{k=0}^{nm} \frac{(-1)^k \ \beta \ \Gamma(\beta + nl - k\alpha)}{(mn-k)!} \ S_k(l, m, \alpha, \beta : x).$$

The general class of polynomial (6.1.1) includes the extended Jacobi polynomial:

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(\alpha);(\beta) : x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_k \cdots (\alpha_c)_k}{(\beta+1-n\alpha)_{lk}(\beta_1)_k \cdots (\beta_d)_k \ k!} \ x^k$$

due to H. M. Srivastava and M. A. Pathan[42] whose inverse series relation was obtained in the form [8]:

$$\frac{(-1)^{mn}(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_c)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_d)_n n!} x^n = \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\beta \ \Gamma(\beta+nl-k\alpha)}{\Gamma(1+\beta-k\alpha)(mn-k)!k!} \times \mathcal{H}_{k,l,m}^{(\alpha,\beta)}[(\alpha);(\beta) : x].$$

The deformed version of (6.1.1) besides providing the extension to the extended Jacobi polynomial, also extends the Brafman polynomial (4.1.2), the extended Konhauser polynomial (4.1.3) and the extended Laguerre polynomial (4.1.4) with their inverse series as stated in chapter 4. We define the following p-deformation of polynomial (6.1.1).

Definition 6.1.1. *For* $0 \le \alpha \le 1$, $\beta \in \mathbb{C}$, $m \in \mathbb{N}$, n, $l = m\alpha \in \{0\} \cup \mathbb{N}$ *and* $p > 0$,

$$S_{n,p}(l, m, \alpha, \beta : x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk} \ \lambda_k \ x^k}{\Gamma_p(p+\beta-pn\alpha+plk)(n-mk)!}, \tag{6.1.2}$$

in which the floor function $\lfloor r \rfloor$ = *floor* r, *represents the greatest integer* $\le r$.

The above extended Jacobi polynomial occurs as a special case when $p = 1$ and $\lambda_n = (\alpha_1)_n \cdots (\alpha_c)_n / ((\beta_1)_n \cdots (\beta_d)_n n!)$.
Thus, if

$$\lambda_n = \frac{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}}{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p} n!}$$

then (6.1.2) yields the p-deformed extended Jacobi polynomial (or pEJP):

$$\mathcal{H}_{n,l,m,p}^{(\alpha,\beta)}[(\alpha);(\beta) : x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p} \cdots (\alpha_c)_{k,p}}{(\beta+p-pn\alpha)_{lk,p}(\beta_1)_{k,p} \cdots (\beta_d)_{k,p} k!} x^k. \tag{6.1.3}$$

Here if $l = 0$ then we get the p-deformed Brafman polynomial (pBP):

$$B^m_{n,p}[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\ k!}\ x^k. \tag{6.1.4}$$

Next, the polynomial (6.1.2) yields the generalization of the Konhauser polynomials $Z^{(\alpha)}_n(x;l)$ when $\beta = 1,\ \alpha = 0,\ \lambda_n = 1/(n!(p+\alpha)_{ln,p}),\ l \in \mathbb{N}$ and x is replaced by x^l; which is given by

$$Z^{(\alpha)}_{n,m,p}(x;l) \quad = \quad \frac{(p+\alpha)_{ln,p}}{n!} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{kl,p}k!} x^{kl}. \tag{6.1.5}$$

We call this polynomial as "the extended p-deformed Konhauser polynomials" or briefly, *EpKP*. The case $l = 1$ yields the "the extended p-deformed Laguerre polynomial", or *EpLP* which we denote by $L^{(\alpha)}_{n,m,p}(x)$ and define by

$$L^{(\alpha)}_{n,m,p}(x) \quad = \quad \frac{(p+\alpha)_{n,p}}{n!} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{k,p}k!} x^k. \tag{6.1.6}$$

It is evident that the polynomial (6.1.2) and its particular cases would reduce to the polynomial (6.1.1) and its particular cases, when $p = 1$.
The flow of the remaining sections are as follows.
In section - 6.2, we derive general inversion pair. Form this, we deduce the inverse series relation of the polynomial (6.1.2) and its further particular cases in section - 6.3. Moreover, We obtained another inversion pair which is associated to (6.1.2) and its particular cases for $m = 1$. The differential equation of the extended p-deformed polynomials are derived in section - 6.4. The generating function relations and the summation formulas involving p-polynomials are derived in section - 6.5 and section - 6.6, respectively. In the last section, section - 6.7, the Companion matrix of the p-deformed monic polynomial obtained from (6.1.2), is derived.

6.2 Inverse series relations

In this section, we derive two general inversion pairs.

Theorem 6.2.1. *Let* $0 \le \alpha \le 1,\ n \in \mathbb{N}\cup\{0\},\ m \in \mathbb{N}$ *such that* αm *is a non negative integer and* $\beta \in \mathbb{C}\setminus\{0\},$ *then*

$$G(n) \quad = \quad \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta + pmk\alpha - pn\alpha + p)(n-mk)!}\ F(k) \tag{6.2.1}$$

$$\Rightarrow$$

$$F(n) = \sum_{k=0}^{nm} \frac{(-1)^{mn-k} \; \beta \; \Gamma_p(\beta + pmn\alpha - pk\alpha)}{(mn-k)!} \; G(k), \tag{6.2.2}$$

and conversely, the series in (6.2.2) implies the series (6.2.1) if for $n \neq mr, \; r \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} \; \beta \; \Gamma_p(\beta + pn\alpha - pk\alpha)}{(n-k)!} \; G(k) = 0. \tag{6.2.3}$$

Proof. We first prove that (6.2.1) $\Rightarrow$ (6.2.2). For that let us denote the right hand side of (6.2.2) by F_n then on substituting $G(k)$ from (6.2.1), we have

$$\begin{aligned} F_n &= \sum_{k=0}^{nm} \frac{(-1)^{mn-k} \beta \; \Gamma_p(\beta + pmn\alpha - pk\alpha)}{(mn-k)!} \; G(k) \\ &= \sum_{k=0}^{nm} (-1)^{mn-k} \frac{\beta \; \Gamma_p(\beta + pmn\alpha - pk\alpha)}{(mn-k)!} \\ &\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} \frac{1}{\Gamma_p(\beta + pmj\alpha - pk\alpha + p)(k - mj)!} \; F(j) \end{aligned} \tag{6.2.4}$$

Here, On making use of the double series relation

$$\sum_{k=0}^{nm} \sum_{j=0}^{\lfloor k/m \rfloor} A(k,j) = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k + mj, j), \tag{6.2.5}$$

we further have from (6.2.4),

$$\begin{aligned} F_n &= \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} \frac{(-1)^{mn-k-mj} \beta \; \Gamma_p(\beta + pmn\alpha - p(k+mj)\alpha)}{(mn-k-mj)! \; \Gamma_p(\beta + pmj\alpha - p(k+mj)\alpha + p) \; k!} \; F(j) \\ &= F(n) + \sum_{j=0}^{n-1} \frac{(-1)^{mn-mj} \beta}{(mn-mj)!} \sum_{k=0}^{mn-mj} (-1)^k \binom{mn-mj}{k} \\ &\quad \times \frac{\Gamma_p(\beta + pmn\alpha - pk\alpha - pmj\alpha)}{\Gamma_p(\beta - pk\alpha + p)} \; F(j) \\ &= F(n) + \sum_{j=0}^{n-1} \frac{(-1)^{mn-mj} \beta}{(mn-mj)!} F(j) \sum_{k=0}^{mn-mj} (-1)^k \binom{mn-mj}{k} \sum_{r=0}^{ln-lj-1} A_r \; k^r, \end{aligned}$$

where $l = \lfloor m\alpha \rfloor$ is an integer $< m$ for $0 \leq \alpha \leq 1$. Now, if $P(a+bk)$ is a polynomial in k of degree less than N then

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} P(a + bk) = 0. \tag{6.2.6}$$

Here, the inner sum in the second term vanishes in view of (6.2.6), giving $F_n = F(n)$.

Next, to show that (6.2.1) $\Rightarrow$ (6.2.3), we take

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!}\ G(k) = G_n.$$

Then substituting from (6.2.1) for $G(k)$, we get

$$\begin{aligned}
G_n &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!} \\
&\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} \frac{1}{\Gamma_p(\beta+pmj\alpha-pk\alpha+p)(k-mj)!}\ F(j) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{\lfloor k/m \rfloor} \frac{(-1)^{n-k}\beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{\Gamma_p(\beta+pmj\alpha-pk\alpha+p)\ (n-k)!\ (k-mj)!}\ F(j) \\
&= \sum_{j=0}^{\lfloor n/m \rfloor} \sum_{k=0}^{n-mj} \frac{(-1)^{n-mj-k}\beta\ \Gamma_p(\beta+pn\alpha-pk\alpha-pmj\alpha)}{(n-mj-k)!\ \Gamma_p(\beta-pk\alpha+p)\ k!}\ F(j) \\
&= \sum_{j=0}^{\lfloor n/m \rfloor} (-1)^{n-mj}\ \frac{\beta\ F(j)}{(n-mj)!} \sum_{k=0}^{n-mj} (-1)^k \binom{n-mj}{k} \sum_{s=0}^{\lfloor n\alpha \rfloor - \lfloor mj\alpha \rfloor - 1} B_s\ k^s.
\end{aligned}$$

Here, in the last expression, the inner sums is the $(n-mj)^{th}$ difference of polynomial of degree less than $n-mj$ for $0 \le \alpha \le 1$, hence it vanishes in view of (6.2.6); thus proving (6.2.1) implies (6.2.3).

We now assume (6.2.2) and (6.2.3) with $n \neq mj,\ j \in \mathbb{N}$, and show that they both imply (6.2.1). For that we first note the inverse series relation:

$$\begin{aligned}
\Omega(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!}\ \Psi(k) \qquad (6.2.7)\\
&\Rightarrow \\
\Psi(n) &= \sum_{k=0}^{n} \frac{1}{\Gamma_p(\beta+pk\alpha-pn\alpha+p)(n-k)!}\ \Omega(k). \qquad (6.2.8)
\end{aligned}$$

Since (6.2.2) and (6.2.3) hold, it follows that if $n \neq mj,\ j \in \mathbb{N}$, then $\Omega(n) = 0$, whereas for $n = mj$, $\Omega(mj) = F(j)$ from (6.2.2). In this case, $\Psi(k) = G(k)$ and with these substitutions, (6.2.8) assumes the form

$$G(n) = \sum_{mk=0}^{n} \frac{1}{\Gamma_p(\beta+pmk\alpha-pn\alpha+p)(n-mk)!}\ \Omega(mk).$$

Thus the inverse pair (6.2.7) and (6.2.8) provide us the series relation:

$$\begin{aligned}
F(n) &= \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\ \beta\ \Gamma_p(\beta + pmn\alpha - pk\alpha)}{(mn-k)!}\ G(k) \\
\Rightarrow & \\
G(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta + pmk\alpha - pn\alpha + p)(n-mk)!}\ F(k)
\end{aligned}$$

whenever (6.2.3) holds. This completes the converse part. □

It is interesting to check whether interchanging of the coefficients in above theorem yields the inverse pair? The attempt made in this direction led us to the inversion pair which is proved as

Theorem 6.2.2. *Let* $0 \le \alpha \le 1$, $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ *such that* αm *is a non negative integer and* $\beta \in \mathbb{C} \setminus \{0\}$, *then*

$$G(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{n-mk}\beta\Gamma_p(\beta + pn\alpha - pmk\alpha)}{(n-mk)!} F(k) \tag{6.2.9}$$

$$\Rightarrow$$

$$F(n) = \sum_{k=0}^{nm} \frac{1}{\Gamma_p(\beta + pk\alpha - pmn\alpha + p)(mn-k)!} G(k), \tag{6.2.10}$$

and conversely, the series in (6.2.10) implies the series (6.2.9) if for $n \neq mr$, $r \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{1}{\Gamma_p(\beta + pk\alpha - pn\alpha + p)(n-k)!} G(k) = 0. \tag{6.2.11}$$

Proof. In order to prove that (6.2.9)⇒(6.2.10), we denote the righthand side of (6.2.10) by $\Theta(n)$ and then substitute for $G(k)$ from (6.2.9) to get

$$\begin{aligned}
\Theta(n) &= \sum_{k=0}^{nm} \frac{1}{\Gamma_p(\beta + pk\alpha - pmn\alpha + p)(mn-k)!} G(k) \\
&= \sum_{k=0}^{nm} \frac{1}{\Gamma_p(\beta + pk\alpha - pmn\alpha + p)(mn-k)!} \\
&\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} \frac{(-1)^{k-mj}\beta\Gamma_p(\beta + pk\alpha - pmj\alpha)}{(k-mj)!} F(j) \\
&= \sum_{k=0}^{nm} \sum_{j=0}^{\lfloor k/m \rfloor} \frac{(-1)^{k-mj}\beta\Gamma_p(\beta + pk\alpha - pmj\alpha)}{\Gamma_p(\beta + pk\alpha - pmn\alpha + p)(mn-k)!(k-mj)!} F(j).
\end{aligned}$$

This, in view of the double series relation (6.2.5), further simplifies to

$$\begin{aligned}
\Theta(n) &= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}\frac{(-1)^{k+mj-mj}\beta\ \Gamma_p(\beta+p(k+mj)\alpha-pmj\alpha)}{(mn-k-mj)!\ (k-mj+mj)!}\\
&\quad\times\frac{1}{\Gamma_p(\beta+p(k+mj)\alpha-pmn\alpha+p)}F(j)\\
&= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}\frac{(-1)^{k}\beta\ \Gamma_p(\beta+pk\alpha)}{(mn-mj-k)!k!\Gamma_p(\beta+p(k+mj)\alpha-pmn\alpha+p)}\ F(j)\\
&= F(n)+\sum_{j=0}^{n-1}\sum_{k=0}^{mn-mj}\frac{(-1)^{k}\beta\ \Gamma_p(\beta+pk\alpha)}{(mn-mj-k)!\ k!}\\
&\quad\times\frac{1}{\Gamma_p(\beta+p(k+mj)\alpha-pmn\alpha+p)}F(j)\\
&= F(n)+\sum_{j=0}^{n-1}\frac{\beta}{(mn-mj)!}\sum_{k=0}^{mn-mj}\binom{mn-mj}{k}(-1)^{k}F(j)\sum_{r=0}^{ln-lj-1}A_r k^r,
\end{aligned}$$

where $l=\lfloor m\alpha\rfloor$ is a non negative integer $<m$ for $0\le\alpha\le 1$. Here, the inner sum in the second term vanishes in view of (6.2.6), giving $\Theta(n)=F(n)$. Thus, (6.2.9)$\Rightarrow$(6.2.10). Next, in order to show that (6.2.9)$\Rightarrow$(6.2.11) for $n\neq mr,\ r\in\mathbb{N}$, we denote right hand side of (6.2.11) by $\Phi(n)$ and then substitute for $G(k)$ from (6.2.9) as follows.

$$\begin{aligned}
\Phi(n) &= \sum_{k=0}^{n}\frac{1}{\Gamma_p(\beta+pk\alpha-pn\alpha+p)(n-k)!}G(k)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor k/m\rfloor}\frac{(-1)^{k-mj}\beta\ \Gamma_p(\beta+pk\alpha-pmk\alpha)}{\Gamma_p(\beta+pk\alpha-pn\alpha+p)(n-k)!(k-mj)!}F(j)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor k/m\rfloor}\frac{(-1)^{k-mj}\beta\Gamma_p(\beta+pk\alpha-pmj\alpha)}{\Gamma_p(\beta+pk\alpha-pn\alpha+p)(n-k)!(k-mj)!}F(j)\\
&= \sum_{j=0}^{\lfloor n/m\rfloor}\sum_{k=0}^{n-mj}\frac{(-1)^{k+mj-mj}\beta\ \Gamma_p(\beta+p(k+mj)\alpha-mj\alpha)}{(n-k-mj)!\ k!\ \Gamma_p(\beta+p(k+mj)\alpha-pn\alpha+p)}\ F(j)\\
&= \sum_{j=0}^{\lfloor n/m\rfloor}\sum_{k=0}^{n-mj}\frac{(-1)^{n-k-mj}\beta}{(n-k-mj)!\ k!}\ F(j)\sum_{s=0}^{\lfloor n\alpha\rfloor-\lfloor mj\alpha\rfloor-1}B_s k^s.
\end{aligned}$$

Now since $0\le\alpha\le 1$ and $m\alpha$ is a non negative integer, the inner sum is $(n-mj)^{th}$ difference of the polynomial in k of degree less than $\lfloor n\alpha\rfloor-\lfloor mj\alpha\rfloor-1$, hence is zero (from (6.2.6)). Thus $\Phi(n)=0$ giving (6.2.9)$\Rightarrow$(6.2.11). We now assume that (6.2.10) and (6.2.11) with $n\neq mj, j\in\mathbb{N}$, hold true and show that they together imply (6.2.9). For that we first denote the inverse series relation $G(n)$ and $F(n)$ by $\Lambda(n)$ and $\Upsilon(n)$ for $m = 1$ as follows.

$$\begin{aligned}\Lambda(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\beta\Gamma_p(\beta + pn\alpha - pk\alpha)}{(n-k)!}\Upsilon(k) &(6.2.12)\\ \Rightarrow & \\ \Upsilon(n) &= \sum_{k=0}^{n} \frac{1}{\Gamma_p(\beta + pk\alpha - pn\alpha + p)(n-k)!}\Lambda(k). &(6.2.13)\end{aligned}$$

If $n \neq mr$, $r \in \mathbb{N}$, then $\Lambda(n) = 0$, whereas for $n = mr$, $\Lambda(mr) = F(r)$ from (6.2.10). In this case, $\Upsilon(k) = G(k)$ and with these substitutions, (6.2.13) assumes the form

$$G(n) = \sum_{mk=0}^{n} \frac{(-1)^{n-mk}\beta\Gamma_p(\beta + pn\alpha - pmk\alpha)}{(n-mk)!}\Lambda(mk).$$

Thus the inverse pair (6.2.12) and (6.2.13) provide us the series relation:

$$\begin{aligned}F(n) &= \sum_{k=0}^{nm} \frac{1}{\Gamma_p(\beta + pk\alpha - pmn\alpha + p)(mn-k)!} G(k)\\ \Rightarrow & \\ G(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{n-mk}\beta\Gamma_p(\beta + pn\alpha - pmk\alpha)}{(n-mk)!} F(k),\end{aligned}$$

whenever (6.2.11) holds. This completes the converse part ad hence the theorem. □

6.3 Particular cases

In this section, we obtain inverse series relation of (6.1.2) and its particular cases with the help of Theorem - 6.2.1. In order to obtain the inverse series relation of (6.1.2), we substitute $F(n) = (-1)^{mn}\lambda_n x^n$ and $m\alpha = l$ in this theorem. We then have $G(n) = S_{n,p}(l, m, \alpha, \beta : x)$ and consequently from (6.2.2), we obtain the inverse series:

$$\lambda_n x^n = \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta + pnl - pk\alpha)}{(mn-k)!}\ S_{k,p}(l, m, \alpha, \beta : x). \tag{6.3.1}$$

The extended Jacobi polynomial (6.1.3) and its inverse series may be deduced by taking

$$G(n) = \frac{\mathscr{H}^{(\alpha,\beta)}_{n,l,m,p}[(\alpha); (\beta) : x]}{\Gamma_p(p + \beta - pn\alpha)\ n!}$$

and

$$F(n) = \frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}...(\beta_d)_{n,p}n!}x^n,$$

and it is given by

$$\begin{aligned}\frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}\cdots(\beta_d)_{n,p}n!}x^n &= \sum_{k=0}^{nm}\frac{(-1)^{mn-k}\beta\Gamma_p(\beta+pnl-pk\alpha)}{\Gamma_p(p+\beta-pk\alpha)(mn-k)!k!}\\ &\quad\times\mathscr{H}^{(\alpha,\beta)}_{k,l,m,p}[(\alpha);(\beta):x]. \end{aligned}\tag{6.3.2}$$

From this, the inverse series of (6.1.4) occurs in straightforward manner by taking $\alpha = 0$ in (6.3.2), given by

$$\frac{(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}\cdots(\beta_d)_{n,p}n!}x^n = \sum_{k=0}^{nm}\frac{(-1)^k}{(mn-k)!k!}\ B^m_{k,p}[(\alpha);(\beta):x].$$

The polynomial EpKP possesses the inverse series:

$$x^{ln} = \sum_{k=0}^{nm}\frac{(-1)^k n!(p+\alpha)_{ln,p}}{(p+\alpha)_{kl,p}(mn-k)!}\ Z^{(\alpha)}_{k,m,p}(x;l),\tag{6.3.3}$$

and for $l = 1$, it furnishes the inverse of EpLP (cf. [53, Eq.(2), p. 207] with p=1):

$$x^n = \sum_{k=0}^{nm}\frac{(-1)^k n!(p+\alpha)_{n,p}}{(p+\alpha)_{k,p}(mn-k)!}\ L^{(\alpha)}_{k,m,p}(x).\tag{6.3.4}$$

6.4 Differential equations

In this section, we derive the differential equation of the polynomial (6.1.2) by specializing the sequence $\{\lambda_n\}$ as $\{\frac{1}{k!}\}$. The equation will be obtained with the help of the differential equation (1.3.11) of the function (1.3.10).
The particular polynomial thus obtained is denoted by $R^m_{n,p}(x;l)$ which is given by

$$\begin{aligned} R^m_{n,p}(x;l) &= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-np)_{mk,p}\ x^k\ p^{-mk}}{(p+\beta-pn\alpha)_{lk,p}\ k!}\\ &= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{1}{k!}\left\{\prod_{i=1}^{m}\left(\frac{-np+ip-p}{m}\right)_{k,p}\right\}\left\{\prod_{j=1}^{l}\left(\frac{\beta-pn\alpha+jp}{l}\right)_{k,p}\right\}^{-1}\\ &\quad\times\left(\frac{m^m xp^{-m}}{l^l}\right)^k\\ &= {}_mF_l\left(\triangle_p(m;-np),p,\triangle_p(l;p+\beta-pn\alpha),p\right)\left(\frac{m^m xp^{-m}}{l^l}\right). \end{aligned}$$

Now comparing this with (1.3.11), we obtain in straight forward manner, the differential equation:

$$\begin{aligned}&[D\,(lpD+\beta-pn\alpha+p-lp)\,(lpD+\beta-pn\alpha+2p-lp)\cdots(lpD+\beta-pn\alpha)\\&\qquad -x\,(mD-n)\,(mD-n+1)\cdots(mD-n+m-1)]\,R^m_{n,p}(x;l)=0.\end{aligned}$$

This may be further reduced to the differential equations satisfied by pEJP, pBP, EpKP and EpLP by specializing the parameters appropriately.

Explicit representation of (6.1.3) for $l\in\mathbb{N}$ with the help of (1.3.7) is given by

$$\begin{aligned}&\mathcal{H}^{(\alpha,\beta)}_{n,l,m,p}[\alpha_1,\ldots,\alpha_c;\beta_1,\ldots,\beta_d:x]\\&=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\;k!}\left\{\prod_{i=1}^{m}\left(\frac{-np+ip-p}{m}\right)_{k,p}\right\}\left\{\prod_{j=1}^{l}\left(\frac{\beta-pn\alpha+jp}{l}\right)_{k,p}\right\}^{-1}\\&\times\left(\frac{m^m xp^{-m}}{l^l}\right)^k\\&={}_{m+c}F_{l+d}\left((\triangle_p(m;-np),\{\alpha_r\}_{r=1}^{c}),p,(\triangle_p(l;p+\beta-pn\alpha),\{\beta_s\}_{i=1}^{d},p)\left(\frac{m^m xp^{-m}}{l^l}\right)\right.\end{aligned}$$

Now comparing this with (1.3.11), we get the differential equation of pEJP:

$$\begin{aligned}&\left[D\left\{\prod_{j=1}^{l}\prod_{s=1}^{d}(lpD+\beta-pn\alpha+jp-lp)\,(pD+\beta_s-p)\right\}\right.\\&\qquad\left.-x\left\{\prod_{i=1}^{m}\prod_{r=1}^{c}(mD-n+i-1)\,(pD+\alpha_r)\right\}\right]\mathcal{H}^{(\alpha,\beta)}_{n,l,m,p}[(\alpha);(\beta):x]=0,\end{aligned}$$

where $D=x\dfrac{d}{dx}$. The p-deformed differential equation of the deformed Brafman polynomial (6.1.4), the extended p-deformed Konhauser polynomials (6.1.5) and the extended p-deformed Laguerre polynomial (6.1.6) are derived in section - 4.4 of chapter 4.

6.5 Generating function relations

In this section, we derive the generating function relations (GFR) of the polynomials pEJP, EpKP and their particular cases pBP and EpLP. For that we use the series (6.2.1) of Theorem - 6.2.1 given by

$$G(n)=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{1}{\Gamma_p(\beta+pmk\alpha-pn\alpha+p)(n-mk)!}\,F(k).$$

Noticing that the Bessel function occurs in the generating function relation of the Laguerre polynomial (see [53, Eq. (2), p. 201]), we use here the p-deformed Bessel function (2.3.12) of chapter 2 and define the p-deformed generalized Bessel function as well as the p-deformed modified Bessel function respectively, as follows (cf. [53, 62] with $p = 1$).

Definition 6.5.1. *For $p > 0$ and $n, \mu, \nu \in \mathbb{C}$,*

$$J_{n,p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p + np + kp)k!} \left(\frac{x}{2}\right)^{n+2k}, \tag{6.5.1}$$

$$J^{\mu}_{\nu,p}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma_p(p + \nu p + kp\mu)k!}(-x)^k, \tag{6.5.2}$$

and

$$I_{n,p}(x) = i^{-n} J_{n,p}(ix). \tag{6.5.3}$$

Now from (6.2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} G(n)\, t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta + pkm\alpha - pn\alpha + p)(n - mk)!}\, F(k) t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\Gamma_p(\beta + pkm\alpha - pn\alpha - pkm\alpha + p)(n + mk - mk)!} \\ &\quad \times F(k) t^{n+mk} \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{\Gamma_p(\beta - pn\alpha + p)\; n!}\, t^n \right) \left(\sum_{k=0}^{\infty} F(k)\, t^{mk} \right).\end{aligned} \tag{6.5.4}$$

Next consider

$$\begin{aligned}\sum_{n=0}^{\infty} (\gamma)_{n,p}\, G(n)\, t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(\gamma)_{n,p}}{\Gamma_p(\beta + pkm\alpha - pn\alpha + p)(n - mk)!}\, F(k)\, t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_{n+mk,p}}{\Gamma_p(\beta - pn\alpha + p)n!}\, F(k)\, t^{n+mk} \\ &= \sum_{k=0}^{\infty} (\gamma)_{mk,p} \left(\sum_{n=0}^{\infty} \frac{(\gamma + mkp)_{n,p}}{\Gamma_p(\beta - pn\alpha + p)n!} t^n \right) F(k)\, t^{mk} \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p} F(k)}{\Gamma_p(\beta + p)} t^{mk} \sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(\gamma + mkp)_{n,p}(-\beta)_{n\alpha,p}}{n!}\, t^n.\end{aligned} \tag{6.5.5}$$

By specializing α, β and $F(k)$ appropriately, we obtain the generating function relations of the polynomials pEJP and EpKP.
The substitution

$$F(n) = \frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}\cdots(\beta_d)_{n,p}\; n!}\; x^n$$

in (6.5.4) implies

$$G(n) = \frac{\mathcal{H}^{(\alpha,\beta)}_{n,m\alpha,m,p}[(\alpha);(\beta):x]}{\Gamma_p(p+\beta-pn\alpha)\; n!},$$

and consequently, with the help of (6.5.4) and (1.3.10), we are led to the generating function relation:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\; \mathcal{H}^{(\alpha,\beta)}_{n,m\alpha,m,p}[(\alpha);(\beta):x]\; t^n \\
=\;& \left(\sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\; t^n\right)\left(\sum_{k=0}^{\infty} \frac{(\alpha_1)_{k,p}(\alpha_2)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p}\cdots(\beta_d)_{k,p}k!}\; x^k\; (-t)^{mk}\right) \\
=\;& {}_cF_d\left((\alpha),p,(\beta),p\right)\left(x(-t)^m\right)\sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\; t^n.
\end{aligned}$$

Here for $\alpha = 0$, the first series on the right hand side converges for $|t| < \infty$ and for $\alpha = 1$, it converges for $|t| < 1/p$. For $\alpha = 1$, the above GFR takes more elegant form:

$$\sum_{n=0}^{\infty} \mathcal{H}^{(1,\beta)}_{n,m,m,p}[(\alpha);(\beta):x]\;(-\beta)_{n,p}\frac{(-t)^n}{n!} \;=\; (1+tp)^{\frac{\beta}{p}}\; {}_cF_d\left((\alpha),p,(\beta),p\right)\left(x(-t)^m\right).$$

On the other hand, the choice $\alpha = 0$ (hence $l = 0$) leads us to the GFR of pBP which is given by

$$\sum_{n=0}^{\infty} \mathcal{B}^{m}_{n,p}[(\alpha);(\beta):x]\;\frac{t^n}{n!} = e^t\; {}_cF_d\left((\alpha),p,(\beta),p\right)\;\left(x(-t)^m\right).$$

The special case $p = 1$ yields the generating function relation occurring in [62, Ex. 67, p.199]. Next, taking $\alpha = 0, \beta = 0$, and

$$F(n) = \frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p},\cdots,(\beta_d)_{n,p}n!}x^n$$

in (6.5.5), we get

$$G(n) = \frac{1}{n!}\; B^{m}_{n,p}[(\alpha);(\beta):x].$$

Hence in view of (1.3.12), we find yet another relation with the help of (6.5.5):

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\gamma)_{n,p}}{n!}B^m_{n,p}[(\alpha);(\beta):x]t^n\\
=&\sum_{k=0}^{\infty}\frac{(-1)^{mk}(\gamma)_{mk,p}(\alpha_1)_{k,p}(\alpha_2)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p}\cdots(\beta_d)_{k,p}k!}\left(\sum_{n=0}^{\infty}\frac{(\gamma+mkp)_{n,p}}{k!}x^kt^n\right)t^{mk}\\
=&\sum_{k=0}^{\infty}\frac{(\gamma)_{mk,p}(\alpha_1)_{k,p}(\alpha_2)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p}\cdots(\beta_d)_{k,p}k!}(1-pt)^{\left(\frac{-\gamma-mkp}{p}\right)}x^k(-t)^{mk}\\
=&(1-pt)^{-\gamma/p}\sum_{k=0}^{\infty}\frac{(\gamma)_{mk,p}(\alpha_1)_{k,p}(\alpha_2)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p}\cdots(\beta_d)_{k,p}k!}(1-pt)^{-mk}x^k(-t)^{mk}.
\end{aligned} \tag{6.5.6}$$

By making use of (1.3.12) together with properties (1.3.7) and (1.3.9) in (6.5.6), we get (cf. [62, Eq.(2), p.136] with p=1)

$$\begin{aligned}
&\sum_{n=0}^{\infty}B^m_{n,p}[(\alpha);(\beta):x]\ \frac{(\gamma)_{n,p}\ t^n}{n!}=(1-pt)^{-\gamma/p}\\
&\times_{m+c}F_d\left((\triangle_p(m;\gamma),(\alpha),p),((\beta),p)\right)\left(\frac{(-mt)^mx}{(1-pt)^m}\right).
\end{aligned}$$

For $m+c\leq d+1$, this yields convergent generating function relations.
For obtaining the GFR of EpKP and EpLP, we take $\alpha=0$, $\beta=1$ and replace $F(n)$ by $(-1)^{mn}x^{ln}/(\Gamma_p(p+\alpha+pnl)n!)$ in (6.5.4). Then $G(n)=Z^{\alpha}_{n,m,p}(x;l)/((p+\alpha)_{nl,p}\Gamma_p(p+\alpha))$ the GFR:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{Z^{\alpha}_{n,m,p}(x;l)}{(p+\alpha)_{nl,p}\Gamma_p(p+\alpha)}t^n&=\left(\sum_{n=0}^{\infty}\frac{1}{\Gamma_p(1+p)n!}t^n\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^{mk}x^{lk}}{\Gamma_p(p+\alpha+pkl)k!}t^{mk}\right)\\
&=\frac{e^t}{\Gamma_p(1+p)}\sum_{k=0}^{\infty}\frac{1}{\Gamma_p(p+\alpha+pkl)k!}\left((-1)^mx^lt^{(m)}\right)^k,
\end{aligned}$$

that is

$$\sum_{n=0}^{\infty}\frac{Z^{\alpha}_{n,m,p}(x;l)}{(p+\alpha)_{nl,p}}t^n=\frac{e^t\Gamma_p(p+\alpha)}{\Gamma_p(1+p)}J^l_{\frac{\alpha}{p},p}\left((-1)^{m+1}x^lt^m\right), \tag{6.5.7}$$

involving the deformed generalized Bessel function (6.5.2). The particular cases $p=1$, $m=1$ provide us the generating function relation obtained in [62, Ex. 65, p. 198]. Yet another generating function relation is obtainable by making the substitutions $\alpha=0$, $\beta=1$, and $F(n)=(-1)^{mn}\ x^{ln}/(\Gamma_p(p+\alpha+pnl)n!)$ in

(6.5.5), yields $G(n) = Z^{\alpha}_{n,m,p}(x;l)/((p+\alpha)_{nl,p}\Gamma_p(p+\alpha))$, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} Z^{\alpha}_{n,m,p}(d,x;l)}{(p+\alpha)_{nl,p}\Gamma_p(p+\alpha)} t^n \\
&= \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}(-1)^{mk} x^{lk} t^{mk}}{\Gamma_p(p+\alpha+pkl)k!\Gamma_p(1+p)} \left(\sum_{n=0}^{\infty} \frac{(\gamma+mkp)_{n,p}}{n!} t^n\right) \\
&= \frac{1}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}\ x^{lk}}{(p+\alpha)_{lk,p}k!} \left(\sum_{n=0}^{\infty} \frac{(\gamma+mkp)_{n,p}}{n!} t^n\right) (-t)^{mk} \\
&= \frac{1}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}}{(p+\alpha)_{lk,p}k!} (1-pt)^{\frac{-\gamma-mkp}{p}} x^{lk}(-t)^{mk} \\
&= \frac{(1-pt)^{\frac{-\gamma}{p}}}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}}{(p+\alpha)_{lk,p}k!} (1-pt)^{-mk} x^{lk}(-t)^{mk}.
\end{aligned}$$

In the notations of (1.3.7) and (1.3.9), this takes the form:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p}}{(p+\alpha)_{nl,p}}\ Z^{(\alpha)}_{n,m,p}(x;l)\ t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)} \\
&\times {}_mF_l\left((\triangle_p(m;\gamma),p),(\triangle_p(l;\alpha+p),p)\right)\left(\left(\frac{x}{l}\right)^l\left(\frac{-mt}{1-pt}\right)^m\right). \qquad (6.5.8)
\end{aligned}$$

Here $m \leq l+1$ for convergence; the divergent generating function relations occur for $m > l+1$. This reduces to the generating function relation appearing in [62, Ex. 66, p. 198] with $p = 1$ and $m = 1$.

As noted in Section -6.1, the case $l = 1$ of (6.1.5) is the extended p-deformed Laguerre polynomial $L^{(\alpha)}_{n,m,p}(x)$. Hence the GFRs (6.5.7) and (6.5.8) when $l = 1$, will reduce to those corresponding to the EpLP. They are however also directly deducible from (6.5.4) with $\alpha = 0$, $\beta = 1$, and $F(n) = (-1)^{mn}x^n/((p+\alpha)_{n,p}\ n!)$ and thereby $G(n) = L^{(\alpha)}_{n,m,p}(x)/(p+\alpha)_{n,p}$. In either case, we have

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}\Gamma_p(p+\alpha)} t^n = \left(\sum_{n=0}^{\infty} \frac{1}{\Gamma_p(1+p)n!} t^n\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^{mk}x^k}{\Gamma_p(p+\alpha+kp)k!} t^{mk}\right),$$

that is,

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{e^t\Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{(-1)^{mk}}{\Gamma_p(p+\alpha+kp)k!}(xt^m)^k.$$

Here the cases $m = 2r$ and $m = 2r+1, r \in \mathbb{N} \cup \{0\}$, are worth mentioning. If we take $m = 2r+1$, then we have the following GFR involving (6.5.1).

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r+1,p}(x)}{(p+\alpha)_{n,p}} t^n &= \frac{e^t \Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \left(\sum_{k=0}^{\infty} \frac{(-1)^{(2r+1)k}}{\Gamma_p(p+\alpha+kp)k!} (\sqrt{x(t)^{2r+1}})^{2k} \right) \\
&= \frac{e^t \Gamma_p(p+\alpha)(\sqrt{x(t)^{2r+1}})^{-\frac{\alpha}{p}}}{\Gamma_p(1+p)} \\
&\quad \times \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p+\alpha+kp)k!} ((\sqrt{x(t)^{2r+1}}))^{\frac{\alpha}{p}+2k} \right)
\end{aligned}$$

On making use of (6.5.1), this becomes

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r+1,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{e^t \ \Gamma_p(p+\alpha) \ (t^r\sqrt{xt})^{-\alpha/p}}{\Gamma_p(1+p)} \ J_{\frac{\alpha}{p},p}(2t^r\sqrt{xt}).$$

When $p = 1$ and $r = 0$, this further reduces to the GFR as obtained in [53, Eq.(2), p.201]. On the other hand, for $m = 2r$, the following GFR occurs which involves p-deformed Bessel function (6.5.1) and p-deformed modified Bessel function(6.5.3) respectively.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r,p}(x)}{(p+\alpha)_{n,p}} t^n &= e^t \frac{\Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{(-1)^{(2r+1)k}}{\Gamma_p(p+\alpha+kp)k!} \left(t^r\sqrt{-x}\right)^{2k} \\
&= e^t \frac{\Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \left(it^r\sqrt{x}\right)^{-\frac{\alpha}{p}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p+\alpha+kp)k!} \left(it^r\sqrt{x}\right)^{\frac{\alpha}{p}+2k} \\
&= e^t \frac{\Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \left(it^r\sqrt{x}\right)^{-\frac{\alpha}{p}} J_{\frac{\alpha}{p},p}\left(2it^r\sqrt{x}\right).
\end{aligned}$$

In the notation of (6.5.3), this becomes

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r,p}(x)}{(p+\alpha)_{n,p}} t^n = e^t \ \frac{\Gamma_p(p+\alpha)}{\Gamma_p(1+p)} \ \left(t^r\sqrt{x}\right)^{-\alpha/p} \ I_{\frac{\alpha}{p},p}\left(2t^r\sqrt{x}\right).$$

Similarly, if $\alpha = 0$, $\beta = 1$, and $F(n) = (-1)^{mn}x^n/(\Gamma_p(p+\alpha+np)\ n!)$, then with $G(n) = L^{(\alpha)}_{n,m,p}(x)/((p+\alpha)_{n,p}\Gamma_p(p+\alpha))$ and (6.5.5) we find the relation:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}\Gamma_p(p+\alpha)} t^n &= \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p} \ (-1)^{mk} x^k t^{mk}}{\Gamma_p(1+p)\Gamma_p(p+\alpha+kp)k!} \\
&\quad \times \left(\sum_{n=0}^{\infty} \frac{(\gamma+mkp)_{n,p}}{n!} t^n \right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}}{\Gamma_p(1+p)\Gamma_p(p+\alpha+kp)k!} \times (1-pt)^{\frac{-\gamma-mkp}{p}}(-t)^{mk}x^k.$$

Thus we get,

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}}\, t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)} \sum_{k=0}^{\infty} \frac{\left\{\prod_{j=1}^{m}\left(\frac{\gamma+jp-p}{m}\right)_{k,p}\right\}}{(p+\alpha)_{k,p}\ k!}\, x^k \times \left(\frac{-mt}{1-pt}\right)^{mk}.$$

Here the series on the right hand side converges for $m = 1, 2$. These cases are illustrated below. For $m = 1$, we have

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L^{(\alpha)}_{n,1,p}(x)}{(p+\alpha)_{n,p}}\, t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)}\, {}_1F_1((\gamma,p),(\alpha+p,p))\left(\frac{-xt}{1-pt}\right), \tag{6.5.9}$$

whereas for $m = 2$, we find

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L^{(\alpha)}_{n,2,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)}\, {}_2F_1\left((\triangle_p\,(2;\gamma)\,,p),((\alpha+p),p)\right)\left(\frac{-4xt^2}{(1-pt)^2}\right), \tag{6.5.10}$$

with $\left|\frac{4xt^2}{(1-pt)^2}\right| < 1$. If $p = 1$, then (6.5.9) reduces to the GFR given in [53, Eq.(3), p.202]. Further, for $\gamma = p + \alpha$, (6.5.9) reduces to the GFR:

$$\sum_{n=0}^{\infty} L^{(\alpha)}_{n,1,p}(x) t^n = \frac{(1-pt)^{-1-\frac{\alpha}{p}}}{\Gamma_p(1+p)}\, exp\left(\frac{-xt}{1-pt}\right),$$

whose particular case $p = 1$ appears in [53, Eq.(4), p.202].
In (6.5.10), the substitution $\gamma = 2\alpha + p$ gives an elegant form:

$$\sum_{n=0}^{\infty} \frac{(2\alpha+p)_{n,p} L^{(\alpha)}_{n,2,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{(1-pt)^{-1-\frac{2\alpha}{p}}}{\Gamma_p(1+p)} \left(1 - \frac{4pxt^2}{(pt-1)^2}\right)^{-\frac{1}{2}-\frac{\alpha}{p}}.$$

6.6 Summation formulas

It is interesting to observe that the inverse series relation obtained from Theorem - 6.2.1 of section - 6.2 leads us to derive certain summation formulas. In fact, From (6.3.1) we have

$$\frac{1}{\lambda_n}\sum_{k=0}^{nm}\frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta:x)\ =\ x^n, \tag{6.6.1}$$

assuming $\lambda_n \neq 0, \forall n \in \mathbb{N}$.
Now multiplying both sides by $1/n!$ and taking summation from $n=0$ to ∞, we obtain

$$\sum_{n=0}^{\infty}\frac{1}{n!\lambda_n}\sum_{k=0}^{nm}\frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta:x)\ =\ \sum_{n=0}^{\infty}\frac{x^n}{n!}\ =\ e^x, \tag{6.6.2}$$

for all x. The choice $\lambda_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}/((\beta_1)_{n,p}\cdots(\beta_d)_{n,p}n!)$ here, provides the summation formula of pEJP as follows.

$$\sum_{n=0}^{\infty}\frac{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p}}{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}}\sum_{k=0}^{nm}\frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\mathcal{H}^{(\alpha,\beta)}_{k,l,m,p}[(\alpha);(\beta):x]=e^x. \tag{6.6.3}$$

Further, $l=m\alpha=0$ in (6.6.3) simplifies to the summation formula of pBP(6.1.4) in the form:

$$\sum_{n=0}^{\infty}\frac{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p}}{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}}\sum_{k=0}^{nm}\frac{(-1)^k\ \beta\ \Gamma_p(\beta)}{(mn-k)!}\ B^m_{k,p}[(\alpha);(\beta):x]=e^x.$$

Next taking $\beta=1,\ \alpha=0,\ \lambda_n=1/(n!(p+\alpha)_{ln,p})$ and x is replaced by x^l; in (6.6.2), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{nm}\frac{(-1)^k(p+\alpha)_{ln,p}}{(p+\alpha)_{kl,p}(mn-k)!}\ Z^{(\alpha)}_{k,m,p}(x;l)=e^{x^l},$$

the summation formula of EpKP. The immediate consequence of this, is the summation formula involving EpLP:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{nm}\frac{(-1)^k(p+\alpha)_{n,p}}{(p+\alpha)_{k,p}(mn-k)!}\ L^{(\alpha)}_{k,m,p}(x)=e^x,$$

when $l=1$.
The number e is the value if $x=1$ in above formulas.

Further on taking sum from $n = 0$ to ∞ (6.6.1) with $|x| < 1, \lambda_n \neq 0,\ \forall n$, we get

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta + pnl - pk\alpha)}{(mn-k)!}\ S_{k,p}(l, m, \alpha, \beta : x) \;=\; \sum_{n=0}^{\infty} x^n \;=\; \frac{1}{1-x}.$$

By assigning different values to x from $(-1, 1)$, a number of particular summation formulas can be derived. For example, $x = 1/2$ in this formula gives the following one:

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta + pnl - pk\alpha)}{(mn-k)!}\ S_{k,p}\left(l, m, \alpha, \beta : \frac{1}{2}\right) \;=\; 2.$$

The reducibility of this summation formula corresponding to the particular cases of $S_{n,p}(l, m, \alpha, \beta : x)$ may be obtained by the substitutions as stated above. We consider the sum

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p}\ n!}{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}} \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta + pnl - pk\alpha)}{(mn-k)!} & \\
\times \mathcal{H}^{(\alpha,\beta)}_{k,l,m,p}\left[(\alpha); (\beta) : \frac{1}{2}\right] &= 2, \\
\sum_{n=0}^{\infty} \frac{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p}\ n!}{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}} \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta)}{(mn-k)!} B^m_{k,p}\left[(\alpha); (\beta) : \frac{1}{2}\right] &= 2, \\
\sum_{n=0}^{\infty} \sum_{k=0}^{nm} \frac{(-1)^k n! (p+\alpha)_{ln,p}}{(p+\alpha)_{kl,p} (mn-k)!}\ Z^{(\alpha)}_{k,m,p}\left(\frac{1}{2}; l\right) &= \frac{2^l}{2^l - 1}.
\end{aligned}$$

Similarly one can derive other summation formulas from (6.6.1).

6.7 Companion matrix

Taking $\lfloor n/m \rfloor = N$ in (6.1.2) and converting it to the monic form $\widetilde{S}_{n,p}(l, m, \alpha, \beta : x)$, we get

$$\widetilde{S}_{n,p}(l, m, \alpha, \beta : x) \;=\; \sum_{k=0}^{N} \delta_k\ x^k,$$

where

$$\delta_k \;=\; \frac{(-1)^{(k-N)m}\ \Gamma_p(p + \beta - pn\alpha + plN)\lambda_k\ (n - mN)!\ x^k}{\Gamma_p(p + \beta - pn\alpha + plk)\lambda_N (n - mk)!}.$$

With this δ_k, $C\left(\widetilde{S}_{n,p}(l, m, \alpha, \beta : x)\right)$ assumes the form as stated in Definition 1.3.1. The eigen values of this matrix will be then precisely the zeros of $\widetilde{S}_{n,p}(l, m, \alpha, \beta : x)$

(see [48, p. 39]).
Next chapter, Chapter-7, provides q-extension to the polynomial (6.1.2) and derive analogous properties.

POLYNOMIALS' REDUCIBILITY