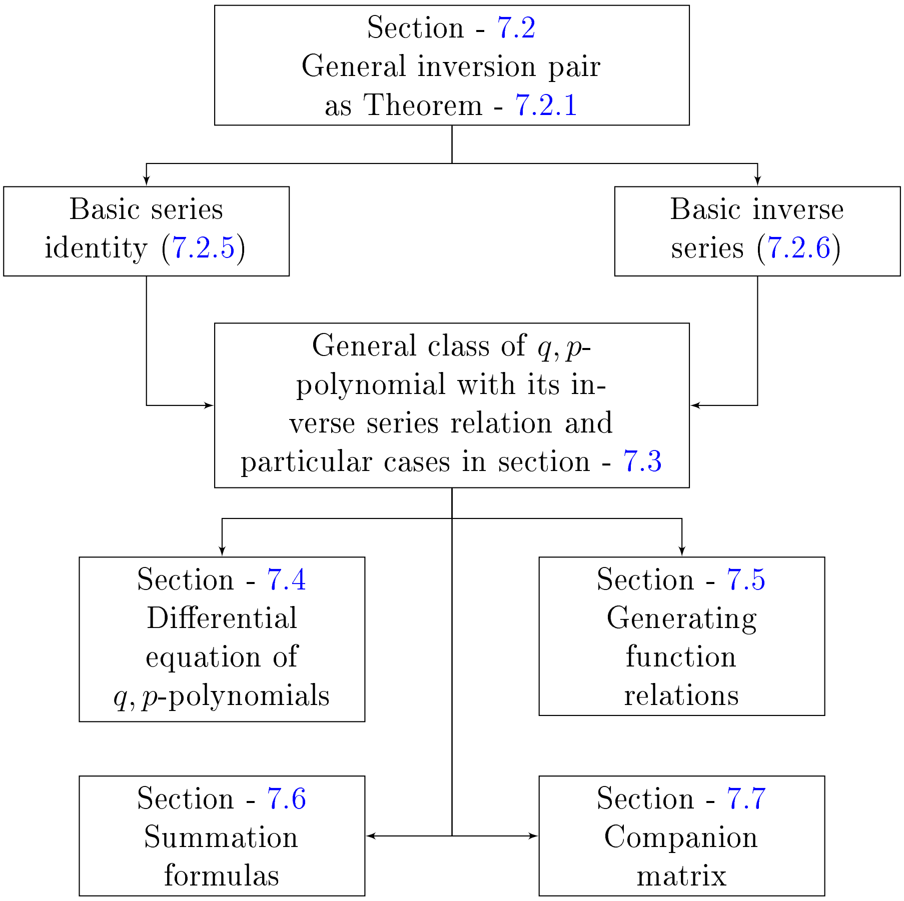
Chapter 7

The p-deformed q-polynomials' system - III



7.1 Introduction

In [57, Eq.1.14, p.228], a general class of p-polynomials (or system of p-deformed polynomials) is introduced in the form:

$$S_{n,p}(l,m,\alpha,\beta:x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk}\ \lambda_k\ x^k}{\Gamma_p(p+\beta-pn\alpha+plk)(n-mk)!}, \tag{7.1.1}$$

in which the floor function $\lfloor r \rfloor = floor\ r$, represents the greatest integer $\leq r$, $0 \leq \alpha \leq 1, \beta \in \mathbb{C},\ m \in \mathbb{N},\ n,\ l = m\alpha \in \{0\} \cup \mathbb{N}$. to provide an extension to the extended Jacobi polynomial [41] (also see [43]), the extended Konhauser polynomial and the extended Laguerre polynomial in p-generalized gamma function: Γ_p and Pochhammer p-symbol: $(x)_{n,p}$.

We define here a q-analogue of the p-polynomial (7.1.1) and obtain its inverse series relation, q,p-difference equation, q,p-differential equation, generating function relations, summation formula and finally, the Companion matrix.
It may be mentioned here that for a parameter $\alpha \in \mathbb{C}$, we shall write α for q^α.
We propose a p-deformed q-extension of the general class of p-polynomial (7.1.1) as follows.

Definition 7.1.1. *For $\beta \in \mathbb{C}$, $r, \alpha \in \mathbb{C}/\{0\}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $|q| < 1$ and $p > 0$,*

$$S_{n,p,r}(l, m, \alpha, \beta : x|q^\alpha) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mkr\alpha(mk-2n+1)/2} \frac{(\beta q^{rlk+p-n\alpha p}; q^\alpha)_{\infty,p}}{(q^{r\alpha}; q^{r\alpha})_{n-mk}} \lambda_k x^k, \tag{7.1.2}$$

where $\lfloor n/m \rfloor = floor\ n/m$, represents the greatest integer $\leq n/m$.

When $q \to 1$ and $r = p$, this coincides with (7.1.1).
Moreover, this general class of q,p-polynomials ((7.1.2) above) extends the general class of p-deformed q-polynomials (5.1.3) of chapter 5 by taking $\alpha = 1$ and $r = \lambda$.
The special case

$$\lambda_n = \frac{(\alpha_1; q)_{n,p} (\alpha_2; q)_{n,p} \cdots (\alpha_c; q)_{n,p}}{(\beta_1; q)_{n,p} (\beta_2; q)_{n,p} \cdots (\beta_d; q)_{n,p} (q^{r\alpha^2}; q^{r\alpha^2})_n}$$

with r is replaced by $r\alpha$, together with property (1.5.10), yields the *tribasic* p-deformed extended q-Jacobi polynomial (cf. [12, Eq.(1.2), p. 77] with $p = 1$):

$$\begin{aligned} &\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha); (\beta) : xq^{r\alpha l}|q^\alpha] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nr\alpha^2}; q^{r\alpha^2})_{mk} (\alpha_1; q)_{k,p} (\alpha_2; q)_{k,p} \cdots (\alpha_c; q)_{k,p}}{(\beta q^{p-pn\alpha}; q^\alpha)_{\frac{rlk}{p},p} (\beta_1; q)_{k,p} (\beta_2; q)_{k,p} \cdots (\beta_d; q)_{k,p} (q^{r\alpha^2}; q^{r\alpha^2})_k} (xq^{r\alpha l})^k. \end{aligned} \tag{7.1.3}$$

Note 7.1.1. *This polynomial approaches to the p-deformed extended Jacobi polynomial[57, Eq.1.15, p.228] as $q \to 1$ and $r = p$ in which the substitution $p = 1$ yields the extended Jacobi polynomial (6.1.1).*

When $\beta \to \infty$ in q^β, this polynomial reduces to the bibasic p-deformed q-Brafman polynomial:

$$\begin{aligned} &B^m_{n,p,r,\alpha}[(\alpha); (\beta) : xq^{r\alpha^2 m}|q^\alpha] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nr\alpha^2}; q^{r\alpha^2})_{mk} (\alpha_1; q)_{k,p} (\alpha_2; q)_{k,p} \cdots (\alpha_c; q)_{k,p}}{(\beta_1; q)_{k,p} (\beta_2; q)_{k,p} \cdots (\beta_d; q)_{k,p} (q^{r\alpha^2}; q^{r\alpha^2})_k} (xq^{r\alpha^2 m})^k. \end{aligned} \tag{7.1.4}$$

Note 7.1.2. *This tends to the p-deformed Brafman polynomial[57, Eq.1.16, p.228] as $q \to 1$ with $r = p$ in which the substitution $p = 1$ gives the Brafman polynomial [62, Eq.(1), p.136].*

Further, replacing x by $x^l q^{nl}$, letting $q^\beta \to 0$, taking $r\alpha = l \in \mathbb{N}$ and

$$\lambda_n = \frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}}$$

in (7.1.2), we obtain the extended p-deformed q-Konhauser polynomial (cf. [3] with $p = 1$ and $m = 1$):

$$Z^{(\alpha)}_{n,m,p}(x;l|q) = \frac{(p\alpha;q)_{nl,p}}{(q^l;q^l)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nl};q^l)_{mk} q^{lk(\alpha+n+1)+lk(lk-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}} x^{kl}. \tag{7.1.5}$$

Note 7.1.3. *The limit of this polynomial as $q \to 1$, is the extended p-deformed Konhauser polynomial[57, Eq.1.17, p.229] wherein the substitutions $p = 1$ and $m = 1$ provide the Konhauser polynomial [3, Eq.(3.1), p.3].*

The obvious specialization $l = 1$ is the extended p-deformed q-Laguerre polynomial:

$$L^{(\alpha)}_{n,m,p}(x|q) = \frac{(p\alpha;q)_{n,p}}{(q;q)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-n};q)_{mk} q^{k(\alpha+n+1)+k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mk}} x^k. \tag{7.1.6}$$

Note 7.1.4. *This polynomial approaches to the extended p-deformed Laguerre polynomial[57] as $q \to 1$ in which the substitution $p = 1$ yields the extended Laguerre polynomial [19].*

We shall abbreviate the polynomials in (7.1.3) by pEqJP, (7.1.4) by pqBP, (7.1.5) by pEqKP and (7.1.6) by pEqLP.
In section - 7.2, we obtain general inversion pair. This will be particularized to get the inverse series relation of (7.1.2) and its further particular cases in section - 7.3. The q,p-difference equation and q,p-differential equation of extended q,p-polynomial are derived in section - 7.4. Certain generating function relations and summation formulas involving q,p-polynomials are derived in section - 7.5 and section - 7.6, respectively. Finally, the Companion matrix of p-deformed monic q-polynomial obtained from (7.1.2) is derived in section - 7.7. While deriving summation formula involving (7.1.2) and its particular cases, the q-Binomial theorem[19, Eq(1.3.2), page 7] (3.5.1) and two q-exponential functions $e_q(x)$[19,

Eq(1.3.15), page 9] (3.5.2), $E_q(x)$[19, Eq(1.3.16), page 9] (3.5.3) of chapter 3 will be required. They are restated below. For $|z| < 1$ and $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \;=\; \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{7.1.7}$$

The q-exponential functions $e_q(z)$ and $E_q(z)$ are immediate consequences of this lemma. In fact, $a \to \infty$, that is $q^a \to 0$ in (7.1.7) gives

$$\sum_{k=0}^{\infty} \frac{(0;q)_k}{(q;q)_k}\, z^k = \frac{1}{(z;q)_\infty} := e_q(z), \quad (|z| < 1,\ |q| < 1) \tag{7.1.8}$$

which we abbreviate here as qeF; whereas replacing z by $-z/a$ and then making limit $a \to \infty$, (7.1.7) leads us to the series

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k}\, z^k = (-z;q)_\infty := E_q(z), \quad (z \in \mathbb{C},\ |q| < 1) \tag{7.1.9}$$

which is abbreviated here as qEF. This qEF will be used later in deriving the generating function relation of (7.1.2) and its particular cases.

7.2 Basic inverse series relation

We shall require a particular inversion pair in deriving the inverse series of the polynomial (7.1.2) which we first prove as

Lemma 7.2.1. *For* $0 < q < 1$, $M \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\beta \in \mathbb{C}$, $r, \alpha \in \mathbb{C} \setminus \{0\}$ *and* $p > 0$,

$$g(M) \;=\; \sum_{k=0}^{M} (-1)^k q^{k r\alpha(k-1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^{r\alpha}} \frac{(1 - q^{(k+mj)r\alpha+\beta-(k+mj+1)\alpha p})}{(q^{(M+mj)r\alpha+\beta-(k+mj+1)\alpha p}; q^{\alpha})_{\infty,p}} f(k) \tag{7.2.1}$$

$$\Leftrightarrow$$

$$f(M) \;=\; \sum_{k=0}^{M} (-1)^k q^{k r\alpha(k-2M+1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^{r\alpha}} (q^{k r\alpha + m j r\alpha+\beta-(M+mj)\alpha p}; q^{\alpha})_{\infty,p}\, g(k). \tag{7.2.2}$$

Proof. We observe that the diagonal elements of the coefficient matrix of the first series are

$$(-1)^i q^{i r\alpha(i-1)/2} \frac{(1 - q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})}{(q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p}; q^{\alpha})_{\infty,p}},$$

and those of the of second series are

$$(-1)^i q^{ir\alpha(1-i)/2}(q^{ir\alpha+mjr\alpha+\beta-(i+mj)\alpha p};q^\alpha)_{\infty,p}.$$

They are all non zero; implying that these matrices have the unique inverse. Hence, it suffice to prove that one of these series implies the other. We prefer to show that (7.2.1) implies (7.2.2).
We denote the right hand side of (7.2.2) by $\Phi(M)$ and then substitute for $g(n)$ from (7.2.1) to get

$$\begin{aligned}
\Phi(M) &= \sum_{k=0}^{M}(-1)^k q^{kr\alpha(k-2M+1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^{r\alpha}}(q^{kr\alpha+mjr\alpha+\beta-(M+mj)\alpha p};q^\alpha)_{\infty,p}\\
&\quad\times\sum_{i=0}^{k}(-1)^i q^{ir\alpha(i-1)/2}\begin{bmatrix}k\\i\end{bmatrix}_{q^{r\alpha}}\frac{(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})}{(q^{(k+mj)r\alpha+\beta-(i+mj+1)\alpha p};q^\alpha)_{\infty,p}}f(i)\\
&= \sum_{i=0}^{M}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})f(i)\sum_{k=0}^{M-i}(-1)^k q^{(k+i)r\alpha(k+i-2M+1)/2}\begin{bmatrix}M\\k+i\end{bmatrix}_{q^{r\alpha}}\\
&\quad\times\begin{bmatrix}k+i\\i\end{bmatrix}_{q^{r\alpha}}q^{ir\alpha(i-1)/2}\frac{(q^{(k+i)r\alpha+mjr\alpha+\beta-(M+mj)\alpha p};q^\alpha)_{\infty,p}}{(q^{(k+i+mj)r\alpha+\beta-(i+mj+1)\alpha p};q^\alpha)_{\infty,p}}\\
&= \sum_{i=0}^{M}\begin{bmatrix}M\\i\end{bmatrix}_{q^{r\alpha}}q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})f(i)\\
&\quad\times\sum_{k=0}^{M-i}(-1)^k q^{kr\alpha(k-2M+2i+1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^{r\alpha}}\frac{(q^{(k+i)r\alpha+mjr\alpha+\beta-(M+mj)\alpha p};q^\alpha)_{\infty,p}}{(q^{(k+i+mj)r\alpha+\beta-(i+mj+1)\alpha p};q^\alpha)_{\infty,p}}\\
&= f(M)+\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^{r\alpha}}q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})f(i)\sum_{k=0}^{M-i}(-1)^k\\
&\quad\times q^{kr\alpha(k-2M+2i+1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^{r\alpha}}\frac{(q^{(k+i)r\alpha+mjr\alpha+\beta-(M+mj)\alpha p};q^\alpha)_{\infty,p}}{(q^{(k+i+mj)r\alpha+\beta-(i+mj+1)\alpha p};q^\alpha)_{\infty,p}}. \qquad (7.2.3)
\end{aligned}$$

Here, the ratio of two q,p-gamma functions represents a polynomial of degree $M-i-1$ in k, that is,

$$\frac{(q^{(k+i)r\alpha+mjr\alpha+\beta-(M+mj)\alpha p};q^\alpha)_{\infty,p}}{(q^{(k+i+mj)r\alpha+\beta-(i+mj+1)\alpha p};q^\alpha)_{\infty,p}} = \sum_{l=0}^{M-i-1}A_l\, q^{kr\alpha l},$$

say, hence from (7.2.3), we have

$$\Phi(M) = f(M)+\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^{r\alpha}}q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p})f(i)$$

$$\begin{aligned}
& \times \sum_{k=0}^{M-i}(-1)^k q^{kr\alpha(k-2M+2i+1)/2}\begin{bmatrix} M-i \\ k \end{bmatrix}_{q^{r\alpha}} \sum_{l=0}^{M-i-1} A_l\, q^{kr\alpha l} \\
= \; & f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^{r\alpha}} q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p}) f(i) \\
& \times \sum_{l=0}^{M-i-1} A_l \sum_{k=0}^{M-i}(-1)^k q^{kr\alpha(k-2M+2i+1)/2}\begin{bmatrix} M-i \\ k \end{bmatrix}_{q^{r\alpha}} q^{kr\alpha l} \\
= \; & f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^{r\alpha}} q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p}) f(i) \\
& \times \sum_{l=0}^{M-i-1} A_l \sum_{k=0}^{M-i}(-1)^k q^{kr\alpha(k-2M+2i+1)/2}\begin{bmatrix} M-i \\ k \end{bmatrix}_{q^{r\alpha}} q^{kr\alpha(l+i-M+1)}.
\end{aligned} \tag{7.2.4}$$

On making use of the q-binomial theorem stated as Lemma - 3.2.1 of chapter 3 at the inner most series on the right hand side of (7.2.4), we get

$$\begin{aligned}
\Phi(M) &= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^{r\alpha}} q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p}) f(i) \\
&\quad \times \sum_{l=0}^{M-i-1} A_l\, (q^{r\alpha(l+i-M+1)}; q^{r\alpha})_{M-i} \\
&= f(M) + \sum_{i=0}^{M-1} \begin{bmatrix} M \\ i \end{bmatrix}_{q^{r\alpha}} q^{ir\alpha(i-M)}(1-q^{(i+mj)r\alpha+\beta-(i+mj+1)\alpha p}) f(i) \\
&\quad \times \sum_{l=0}^{M-i-1} A_l \prod_{j=0}^{M-i-1} (1-q^{r\alpha(l+i-M+1+j)}) \\
&= f(M).
\end{aligned}$$

This completes the proof of (7.2.1)⇔(7.2.2). □

Using this inverse relation, we prove a general inversion pair as

Theorem 7.2.1. *If* $r,\ \alpha \in \mathbb{C}\setminus\{0\},\ \beta \in \mathbb{C},\ n \in \mathbb{N}\cup\{0\},\ m \in \mathbb{N},\ p > 0$ *and* $0 < q < 1$, *then*

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mkr\alpha(mk-2n+1)/2} \frac{(q^{mkr\alpha+\beta+p-n\alpha p}; q^{\alpha})_{\infty,p}}{(q^{r\alpha}; q^{r\alpha})_{n-mk}} G(k) \tag{7.2.5}$$

$$\Rightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p}; q^{\alpha})_{\infty,p}(q^{r\alpha}; q^{r\alpha})_{mn-k}} F(k), \tag{7.2.6}$$

and conversely, the series in (7.2.6) implies the series (7.2.5) if for $n \neq mv,\ v \in \mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{n-k}}\ F(k) \ = \ 0. \quad (7.2.7)$$

Proof. Let us denote the series (7.2.6) by $V(n)$ and substitute for $F(k)$ from (7.2.5), then we get

$$\begin{aligned} V(n) &= \sum_{k=0}^{mn}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}}\ F(k) \\ &= \sum_{k=0}^{mn}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}} \\ &\quad\times \sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}q^{mjr\alpha(mj-2k+1)/2}\frac{(q^{mjr\alpha+\beta+p-k\alpha p};q^{\alpha})_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{k-mj}}\ G(j). \end{aligned}$$

In view of the double series relation [62]:

$$\sum_{k=0}^{mn}\sum_{j=0}^{\lfloor k/m\rfloor} A(k,j) \ = \ \sum_{j=0}^{n}\sum_{k=0}^{mn-mj} A(k+mj,j),$$

this further simplifies to

$$\begin{aligned} V(n) &= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}(-1)^{k+mj}q^{(k+mj)r\alpha(k+mj-1)/2}\ \frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+mj+1)\alpha p};q^{\alpha})_{\infty,p}} \\ &\quad\times(-1)^{mj}q^{mjr\alpha(mj-2k-2mj+1)/2}\ \frac{(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{mn-k-mj}(q^{r\alpha};q^{r\alpha})_{k+mj-mj}}\ G(j) \\ &= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj}(-1)^{k}q^{(k+mj)r\alpha(k+mj-1)/2}\ \frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+mj+1)\alpha p};q^{\alpha})_{\infty,p}} \\ &\quad\times\frac{(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{mn-mj-k}(q^{r\alpha};q^{r\alpha})_{k}}\ q^{mjr\alpha(mj-2k-2mj+1)/2}\ G(j) \\ &= \sum_{j=0}^{n}\frac{G(j)}{(q^{r\alpha};q^{r\alpha})_{mn-mj}}\ \sum_{k=0}^{mn-mj}(-1)^k q^{kr\alpha(k-1)/2}\begin{bmatrix} mn-mj \\ k\end{bmatrix}_{q^{r\alpha}} \\ &\quad\times\frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+mj+1)\alpha p};q^{\alpha})_{\infty,p}}(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}. \qquad (7.2.8) \end{aligned}$$

In order to prove $'\Rightarrow'$ part, it is sufficient to show that inner series of (7.2.8) vanishes. For hat, we replace $(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}$ by $f(k)$, put $mn-mj=$

N and denote the inner series by $g(mn-mj) = g(N)$ to get

$$g(N) = \sum_{k=0}^{N} (-1)^k q^{kr\alpha(k-1)/2} \begin{bmatrix} N \\ k \end{bmatrix}_{q^{r\alpha}} \frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+mj+1)\alpha p};q^\alpha)_{\infty,p}} f(k), \quad (7.2.9)$$

whose inverse series follows from (7.2.1) and (7.2.2) with $M = mn - mj$ and β is replaced by $\beta + p$. This is given by

$$\begin{aligned} f(mn-mj) &= \sum_{k=0}^{mn-mj} (-1)^k q^{kr\alpha(k-2mn+2mj+1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^{r\alpha}} \\ &\quad \times (q^{kr\alpha+mjr\alpha+\beta+p-mn\alpha p};q^\alpha)_{\infty,p}\, g(k). \end{aligned}$$

In this last series, setting $g(k) = \left[\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right]_{q^{r\alpha}}$ we find $f(k) = (q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^\alpha)_{\infty,p}$ and with these $f(k)$ and $g(k)$, (7.2.1) yields the series orthogonality relation:

$$\begin{aligned} &\sum_{k=0}^{mn-mj} (-1)^k q^{kr\alpha(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^{r\alpha}} \frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+mj+1)\alpha p};q^\alpha)_{\infty,p}} \\ &\qquad \times (q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^\alpha)_{\infty,p} = \begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^{r\alpha}}. \quad (7.2.10) \end{aligned}$$

On making use of (7.2.10) in (7.2.8), we arrive at

$$\begin{aligned} V(n) &= G(n) + \sum_{j=0}^{n-1} \frac{G(j)}{(q^{r\alpha};q^{r\alpha})_{mn-mj}} \begin{bmatrix} 0 \\ mn-mj \end{bmatrix}_{q^{r\alpha}} \\ &= G(n). \end{aligned}$$

Thus, (7.2.5)⇒(7.2.6). Next to show (7.2.5)⇒(7.2.7), denote (7.2.7) by $R(n)$, that is,

$$R(n) = \sum_{k=0}^{n} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{n-k}} F(k), \quad (7.2.11)$$

and then use (7.2.5) for $F(k)$ in it to get

$$\begin{aligned} R(n) &= \sum_{k=0}^{n} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{n-k}} \\ &\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj} q^{mjr\alpha(mj-2k+1)/2} \frac{(q^{mjr\alpha+\beta+p-k\alpha p};q^\alpha)_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{k-mj}} G(j) \\ &= \sum_{j=0}^{\lfloor n/m \rfloor} \sum_{k=0}^{n-mj} (-1)^{k+mj} q^{(k+mj)r\alpha(k+mj-1)/2} \frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+mj+1)\alpha p};q^\alpha)_{\infty,p}} \end{aligned}$$

$$\times(-1)^{mj}q^{mjr\alpha(mj-2k-2mj+1)/2}\frac{(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{n-mj-k}(q^{r\alpha};q^{r\alpha})_{k+mj-mj}}G(j)$$

$$= \sum_{j=0}^{\lfloor n/m\rfloor}\frac{G(j)}{(q^{r\alpha};q^{r\alpha})_{n-mj}}\sum_{k=0}^{n-mj}(-1)^k q^{kr\alpha(k-1)/2}\begin{bmatrix}n-mj\\ k\end{bmatrix}_{q^{r\alpha}}$$

$$\times\frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+mj+1)\alpha p};q^{\alpha})_{\infty,p}}(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p}. \quad (7.2.12)$$

Next, following the method employed in obtaining the orthogonality relation (7.2.9), it can be shown that

$$\sum_{k=0}^{n-mj}(-1)^k q^{kr\alpha(k-1)/2}\begin{bmatrix}n-mj\\ k\end{bmatrix}_{q^{r\alpha}}\frac{(1-q^{(k+mj)r\alpha+\beta+p-(k+mj+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+mj+1)\alpha p};q^{\alpha})_{\infty,p}}$$

$$\times(q^{mjr\alpha+\beta+p-(k+mj)\alpha p};q^{\alpha})_{\infty,p} = \begin{bmatrix}0\\ n-mj\end{bmatrix}_{q^{r\alpha}},$$

as a result of which (7.2.12) gets reduced to

$$R(n) = \sum_{j=0}^{\lfloor n/m\rfloor}\frac{G(j)}{(q^{r\alpha};q^{r\alpha})_{n-mj}}\begin{bmatrix}0\\ n-mj\end{bmatrix}_{q^{r\alpha}}.$$

If n/m is not an integer i.e. $n\neq mr, r\in\mathbb{N}$, then the right hand member of the last expression given above vanishes and thus (7.2.5)⇒(7.2.7); which completes the proof of the first part.

For the converse part, assume that (7.2.6) and (7.2.7) hold. In view of (7.2.7) and (7.2.11), one can say that

$$R(n) = 0,\ n\neq mr,\ r\in\mathbb{N}, \quad (7.2.13)$$

and also, by comparing (7.2.6) with (7.2.11), one finds that

$$R(mn) = G(n). \quad (7.2.14)$$

Since, the inverse pair (7.2.9) and (7.2.10) with $j = 0$ and $m = 1$ reduces to $g(n) = R(n)$ and $f(n) = F_n$ given by

$$R(n) = \sum_{k=0}^{n}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{n-k}}F_k \quad (7.2.15)$$

$\Rightarrow$

$$F_n = \sum_{k=0}^{n}(-1)^k q^{kr\alpha(k-2n+1)/2}\frac{(q^{kr\alpha+\beta+p-n\alpha p};q^\alpha)_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{n-k}}R(k), \tag{7.2.16}$$

it follows from (7.2.13) and (7.2.14) that

$$R(mn) = \sum_{k=0}^{mn}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}}F_k$$

$$\Rightarrow$$

$$F_n = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk} q^{mkr\alpha(mk-2n+1)/2}\frac{(q^{mkr\alpha+\beta+p-n\alpha p};q^\alpha)_{\infty,p}}{(q^{r\alpha};q^{r\alpha})_{n-mk}}R(mk).$$

We note the relations $R(mn)=G(n)$ and $F_n=F(n)$. Thus, the series (7.2.6), together with the condition $R(n)=0, n\neq mr$ for $r\in\mathbb{N}$, implies the series (7.2.5). This proves the converse part and hence the theorem. □

7.3 Particular cases

The inverse series relation of (7.1.2) is obtained from (7.2.6) by putting $m\alpha=l,\ \alpha\in\mathbb{C}\setminus\{0\}$ and $G(n)=\lambda_n\ x^n$ which is given below.

$$\lambda_n\ x^n = \sum_{k=0}^{mn}(-1)^k q^{kr\alpha(k-1)/2}\frac{(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})\ S_{k,p,r}(l,m,\alpha,\beta:x|q^\alpha)}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}. \tag{7.3.1}$$

The inverse series of pEqJP is obtained by making substitution

$$\lambda_n = \frac{(\alpha_1;q)_{n,p}(\alpha_2;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}{(\beta_1;q)_{n,p}(\beta_2;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{r\alpha^2};q^{r\alpha^2})_n}$$

and replacing r by $r\alpha$ in (7.3.1) which is given below.

$$\frac{(\alpha_1;q)_{n,p}(\alpha_2;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}{(\beta_1;q)_{n,p}(\beta_2;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{r\alpha^2};q^{r\alpha^2})_n}\ x^n = \sum_{k=0}^{mn}(-1)^k q^{kr\alpha^2(k-1)/2}$$
$$\times\frac{(\beta q^{p-pn\alpha};q^\alpha)_{\infty,p}(1-q^{kr\alpha^2+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha^2+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha^2};q^{r\alpha^2})_k(q^{r\alpha^2};q^{r\alpha^2})_{mn-k}}\ \mathscr{H}^{(\alpha,\beta)}_{k,m,l,p,r}[(\alpha);(\beta):xq^{r\alpha l}|q^\alpha].$$

The consequence $\beta\equiv q^\beta\to 0$ of this inverse series is the series

$$\frac{(\alpha_1;q)_{n,p}(\alpha_2;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}{(\beta_1;q)_{n,p}(\beta_2;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{r\alpha^2};q^{r\alpha^2})_n}\ x^n$$
$$= \sum_{k=0}^{mn}(-1)^k q^{kr\alpha^2(k-1)/2}\frac{B^m_{k,p,r,\alpha}[(\alpha);(\beta):xq^{r\alpha^2 m}|q^\alpha]}{(q^{r\alpha^2};q^{r\alpha^2})_{mn-k}(q^{r\alpha^2};q^{r\alpha^2})_k},$$

the inverse series of pqBP. Similarly, with specializations $r\alpha = l \in \mathbb{N}$, x is replaced by $(xq^n)^l$, $q^\beta \to 0$ and

$$\lambda_n = \frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}}$$

in (7.3.1) yield the series

$$\frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}} = \sum_{k=0}^{mn} \frac{(-1)^k q^{lk(k-1)/2}}{(p\alpha;q)_{lk,p}(q^l;q^l)_{mn-k}} Z^{(\alpha)}_{k,m,p}(x;l|q),$$

as the inverse of pEqKP (7.1.5). Here taking $l = 1$, we get

$$\frac{q^{n(\alpha+1)-mn+n(n-1)/2}}{(p\alpha;q)_{n,p}(q;q)_{mn}} = \sum_{k=0}^{mn} \frac{(-1)^k q^{k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mn-k}} L^{(\alpha)}_{k,m,p}(x|q),$$

the inverse series of pEqLP (7.1.6).
It can be seen that general class of q-polynomials (7.1.2) and its particular cases along with the inverse series relations coincide to the general class of p-polynomials (7.1.1) and its particular cases together with the inverse series relation as $q \to 1$ (see for instance [57]).

7.4 q,p-Difference equations and q,p-Differential equations

In deriving the p-deformed q-difference and q-differential equations for the particular cases of the polynomial (7.1.2), we follow the procedure adopted in chapter 5, section 5.4, wherein we considered the operator $\theta_{q,p}f(x) = f(x) - f(xq^p)$ for $p > 0$, (Eq.(5.4.2)) and the p-deformed q-derivative (cf. [19, Ex.1.12, p.22] with $p = 1$) related by the identity:

$$\frac{\theta_{q,p}f(x)}{(1-q)x} = D_{q,p}f(x), \quad (|q| < 1). \tag{7.4.1}$$

We first obtain the ${}_A\phi_B[x]$ representation of the polynomial (7.1.2) by choosing $\lambda_i = 1/(q^{r\alpha};q^{r\alpha})_i$. We denote the particular polynomial thus obtained by $V_{n,p,r}(l,m,\alpha,\beta : x|q^\alpha)$, then we have

$$V_{n,p,r}(l,m,\alpha,\beta : x|q^\alpha) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nr\alpha};q^{r\alpha})_{mk}}{(\beta q^{p-pn\alpha};q^\alpha)_{lk,p}\,(q^{r\alpha};q^{r\alpha})_k} (q^{rl}x)^k. \tag{7.4.2}$$

Now by using the properties (1.5.12) and (1.5.13) with $p = 1$, we get

$$(q^{-nr\alpha};q^{r\alpha})_{mk} = (q^{-nr\alpha}, q^{-nr\alpha+r\alpha}, \cdots, q^{-nr\alpha+r\alpha(m-1)}; q^{mr\alpha})_k$$

$$\begin{aligned}
&= \prod_{i=1}^{m}(q^{(i-1)r\alpha-nr\alpha};q^{mr\alpha})_k \\
&= \prod_{i=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m},q^{((i-1)r\alpha-nr\alpha)/m}w,\cdots,\right. \\
&\quad \left. q^{((i-1)r\alpha-nr\alpha)/m}w^{m-1};q^{r\alpha}\right)_k \\
&= \prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k,
\end{aligned}$$

and

$$\begin{aligned}
(\beta q^{p-pn\alpha};q^{\alpha})_{lk,p} &= (\beta q^{p-pn\alpha},\beta q^{p-pn\alpha+p\alpha},\beta q^{p-pn\alpha+2\alpha},\cdots,\beta q^{p-pn\alpha+(l-1)\alpha};q^{l\alpha})_{k,p} \\
&= \prod_{s=1}^{l}\left(\beta q^{p-pn\alpha+p\alpha(s-1)};q^{l\alpha}\right)_{k,p} \\
&= \prod_{s=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\cdots,q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{l-1};q^{\alpha}\right)_{k,p} \\
&= \prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p},
\end{aligned}$$

where w is m^{th} root of unity and μ is l^{th} root of unity. Thus the polynomial (7.4.2) assumes the desired form:

$$\begin{aligned}
V_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha}) = \sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\} \\
\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(q^{rl}x)^k}{(q^{r\alpha};q^{r\alpha})_k}. \quad (7.4.3)
\end{aligned}$$

We derive the bibasic p-deformed q-difference equation for the polynomial (7.4.3) in

Theorem 7.4.1. *The function $y = V_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})$ satisfies the equation*

$$\begin{aligned}
&\left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}-1)\right\}\right. \\
&-xq^{rl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{l}\prod_{t=1}^{l}\left\{q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l+((i-1)r\alpha-nr\alpha)/m}\nu^{1-t}w^{j-1}\right. \\
&\left.\left.\times(\theta_{q^{r\alpha}}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}-1)\right\}\right]V_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha}) \;=\; 0, \quad (7.4.4)
\end{aligned}$$

where $\theta_{q,p}(f(x)) = f(x) - f(xq^p)$ *for any* $p > 0$.

Proof. We begin with

$$\left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q^{\alpha},p} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t} - 1)\right\}\right] V_{n,p,r}(l,m,\alpha,\beta : x|q^{\alpha})$$

$$= \left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q^{\alpha},p} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t} - 1)\right\}\right]$$
$$\times \sum_{k=0}^{\lfloor n/m \rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1}; q^{r\alpha}\right)_k\right\}$$
$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1} \frac{(q^{rl}x)^k}{(q^{r\alpha};q^{r\alpha})_k}$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1}; q^{r\alpha}\right)_k\right\}$$
$$\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times \left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q^{\alpha},p} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t} - 1)\right\}\right] x^k$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1}; q^{r\alpha}\right)_k\right\}$$
$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times \left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(\theta_{q^{\alpha},p}x^k + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}x^k - x^k)\right\}\right]$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1}; q^{r\alpha}\right)_k\right\}$$
$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times \left[\theta_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}(x^k - x^k q^{p\alpha k} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}x^k - x^k)\right\}\right]$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1}; q^{r\alpha}\right)_k q^{rlk}\right\}$$
$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1}$$

$$\times \left[\theta_{q^{r\alpha}}(x^k)\prod_{s=1}^{l}\prod_{t=1}^{l}(-q^{p\alpha k}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t})\right]$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}(1-q^{r\alpha k})}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k(xq^{rl})^k\right\}$$

$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}$$

$$\times\left[\prod_{s=1}^{l}\prod_{t=1}^{l}q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}(1-q^{(\beta+p-pn\alpha+p\alpha(s-1))/l+p\alpha k-p\alpha})\nu^{t-1}\right]$$

$$= \sum_{k=1}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_{k-1}}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k(xq^{rl})^k\right\}$$

$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k-1,p}\right\}^{-1}$$

$$\times\left[\prod_{s=1}^{l}\prod_{t=1}^{l}q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}\right]$$

$$= xq^{rl}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t}\right\}$$

$$\times\sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_{k+1}\right\}$$

$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}x^k. \tag{7.4.5}$$

On the other hand,

$$\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha}}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}-1)\right]V_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})$$

$$= \left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha}}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}-1)\right]$$

$$\times\sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$

$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}x^k$$

$$= \sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$

$$\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha}}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}-1)\right]x^k$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$
$$\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha}}x^k+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}x^k-x^k)\right]$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{q^{rlk}}{(q^{r\alpha};q^{r\alpha})_k}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$
$$\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(x^k-x^kq^{r\alpha k}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}x^k-x^k)\right]$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$
$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{rl})^k}{(q^{r\alpha};q^{r\alpha})_k}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(-q^{r\alpha k}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j})\right]$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m}w^{j-1};q^{r\alpha}\right)_k\right\}$$
$$\times\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{rl})^k}{(q^{r\alpha};q^{r\alpha})_k}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}(1-q^{((i-1)r\alpha-nr\alpha)/m+r\alpha k}w^{j-1})\right].$$

From this, we have

$$\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha}}+q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j}-1)\right]V_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})$$

$$= \left\{\prod_{i=1}^{m}\prod_{j=1}^{m} q^{-((i-1)r\alpha-nr\alpha)/m} w^{1-j}\right\} \sum_{k=0}^{\lfloor n/m \rfloor} \left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha-nr\alpha)/m} w^{j-1}; q^{r\alpha}\right)_{k+1}\right\}$$
$$\times \left\{\prod_{s=1}^{l}\prod_{t=1}^{l}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/l} \nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1} \frac{(xq^{rl})^k}{(q^{r\alpha}; q^{r\alpha})_k}. \qquad (7.4.6)$$

The bi-basic p-deformed q-difference equation (7.4.4) now follows from (7.4.5) and (7.4.6). □

The bibasic p-deformed q-differential equation satisfied by (7.4.3) is obtained by making use of (7.4.1) for $p > 0$ (and $p = 1$ in (7.4.4)). It is stated as

Corollary 7.4.1.

$$\Bigg[(1-q^{r\alpha})xD_{q^{r\alpha}}\left\{\prod_{s=1}^{l}\prod_{t=1}^{l}((1-q^{\alpha})xD_{q^{\alpha},p} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l}\nu^{1-t} - 1)\right\}$$
$$-xq^{rl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{l}\prod_{t=1}^{l}\Bigg\{q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/l+((i-1)r\alpha-nr\alpha)/m}\nu^{1-t}w^{j-1}$$
$$\times((1-q^{r\alpha})xD_{q^{r\alpha}} + q^{-((i-1)r\alpha-nr\alpha)/m}w^{1-j} - 1)\Bigg\}\Bigg] V_{n,p,r}(l, m, \alpha, \beta : x|q^{\alpha}) = 0.$$

Next, The p-deformed q-difference equation and q-differential equation of pEqJP (7.1.3) are derived with $rl/p = h \in \mathbb{N}$. Alternative form of pEqJP with the help of derived simplified forms is given by

$$\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha);(\beta) : xq^{r\alpha l}|q^{\alpha}]$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nr\alpha^2}; q^{r\alpha^2})_{mk}}{(\beta q^{p-pn\alpha}; q^{\alpha})_{hk,p}} \left\{\prod_{u=1}^{c}(\alpha_u; q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v; q)_{k,p}\right\}^{-1} \frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1}; q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u; q)_{k,p}\right\}$$
$$\times\left\{\prod_{v=1}^{d}(\beta_v; q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1}; q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}, \qquad (7.4.7)$$

where w is m^{th} root of unity and ν is h^{th} root of unity. The tribasic p-deformed q-difference equation satisfied by (7.4.7) is given in

Corollary 7.4.2. *The polynomial* $y = \mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha);(\beta) : xq^{r\alpha l}|q^{\alpha}]$ *satisfies the* q,p*-difference equation with the usual notations and restrictions:*

$$\Bigg[\theta_{q^{r\alpha^2}}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{v=1}^{d}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)(\theta_{q,p}+q^{p-\beta_v}-1)\right\}$$
$$-xq^{r\alpha l}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h+((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1}\nu^{1-t}$$
$$\times q^{p-\beta_v+\alpha_u}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)(\theta_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]y=0. \quad (7.4.8)$$

Proof. We have

$$\left[\theta_{q^{r\alpha^2}}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{v=1}^{d}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]y$$
$$=\left[\theta_{q^{r\alpha^2}}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)\right\}\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]$$
$$\times\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}$$
$$\times\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}$$
$$\times\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(q^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}$$
$$\times\left[\theta_{q^{r\alpha^2}}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)\right\}\left\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]x^k$$
$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}$$
$$\times\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(q^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}$$
$$\times\Bigg[\theta_{q^{r\alpha^2}}\Bigg\{\prod_{s=1}^{h}\prod_{t=1}^{h}(\theta_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)$$
$$\times\prod_{v=1}^{d}(\theta_{q,p}x^k+q^{p-\beta_v}x^k-x^k)\Bigg\}\Bigg]$$

$$= \sum_{k=0}^{\lfloor n/m \rfloor} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(q^{((i-1)r\alpha^2 - nr\alpha^2)/m} w^{j-1}; q^{r\alpha^2} \right)_k \right\} \left\{ \prod_{u=1}^{c} (\alpha_u; q)_{k,p} \right\} \left\{ \prod_{v=1}^{d} (\beta_v; q)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} \left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{t-1}; q^{\alpha} \right)_{k,p} \right\}^{-1} \frac{(q^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}$$
$$\times \left[\theta_{q^{r\alpha^2}} \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} (\theta_{q^{\alpha},p} x^k + q^{p\alpha - (\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{1-t} x^k - x^k) \right.\right.$$
$$\left.\left. \times \prod_{v=1}^{d} q^{p-\beta_v} (1 - q^{\beta_v + pk - p}) \right\} \right]$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(q^{((i-1)r\alpha^2 - nr\alpha^2)/m} w^{j-1}; q^{r\alpha^2} \right)_k \right\} \left\{ \prod_{u=1}^{c} (\alpha_u; q)_{k,p} \right\} \left\{ \prod_{v=1}^{d} (\beta_v; q)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} \left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{t-1}; q^{\alpha} \right)_{k,p} \right\}^{-1} \frac{(q^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}$$
$$\times \left[\theta_{q^{r\alpha^2}} x^k \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} q^{p\alpha - (\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{1-t} \right.\right.$$
$$\left.\left. \times (1 - q^{(\beta+p-pn\alpha+p\alpha(s-1))/h + kp\alpha - p\alpha} \nu^{t-1}) \prod_{v=1}^{d} q^{p-\beta_v} (1 - q^{\beta_v + pk - p}) \right\} \right]$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(q^{((i-1)r\alpha^2 - nr\alpha^2)/m} w^{j-1}; q^{r\alpha^2} \right)_k \right\} \left\{ \prod_{u=1}^{c} (\alpha_u; q)_{k,p} \right\} \left\{ \prod_{v=1}^{d} (\beta_v; q)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} \left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{t-1}; q^{\alpha} \right)_{k,p} \right\}^{-1} \frac{(q^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}$$
$$\times \left[(1 - q^{r\alpha^2 k}) \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} q^{p\alpha - (\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{1-t} \right.\right.$$
$$\left.\left. \times (1 - q^{(\beta+p-pn\alpha+p\alpha(s-1))/h + kp\alpha - p\alpha} \nu^{t-1}) \prod_{v=1}^{d} q^{p-\beta_v} (1 - q^{\beta_v + pk - p}) \right\} \right] x^k$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor} \left\{ \prod_{i=1}^{m} \prod_{j=1}^{m} \left(q^{((i-1)r\alpha^2 - nr\alpha^2)/m} w^{j-1}; q^{r\alpha^2} \right)_k \right\} \left\{ \prod_{u=1}^{c} (\alpha_u; q)_{k,p} \right\} \left\{ \prod_{v=1}^{d} (\beta_v; q)_{k,p} \right\}^{-1}$$
$$\times \left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} \left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h} \nu^{t-1}; q^{\alpha} \right)_{k,p} \right\}^{-1} \frac{(q^{r\alpha l})^k}{(q^{r\alpha^2}; q^{r\alpha^2})_k}$$
$$\times \left[\left\{ \prod_{s=1}^{h} \prod_{t=1}^{h} q^{p\alpha - (\beta+p-p6kn\alpha+p\alpha(s-1))/h} \nu^{1-t} \prod_{v=1}^{d} q^{p-\beta_v} \right\} \right] x$$

Thus we get,

$$\begin{aligned}
&\left[\theta_{q^{r\alpha^2}}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{v=1}^{d}(\theta_{q^\alpha,p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)(\theta_{q,p}+q^{p-\beta_v}-1)\right\}\right]\\
&\quad\times\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[\alpha_1,\alpha_2,\ldots,\alpha_c;\beta_1,\beta_2,\ldots,\beta_d:xq^{r\alpha l}|q^\alpha]\\
&= xq^{r\alpha l}\left[\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}\prod_{v=1}^{d}q^{p-\beta_v}\right\}\right]\\
&\quad\times\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\\
&\quad\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^\alpha\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}.
\end{aligned} \tag{7.4.9}$$

On other side we have,

$$\begin{aligned}
&\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)\prod_{u=1}^{c}(\theta_{q,p}+q^{-\alpha_u}-1)\right]\\
&\quad\times\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[\alpha_1,\alpha_2,\ldots,\alpha_c;\beta_1,\beta_2,\ldots,\beta_d:xq^{r\alpha l}|q^\alpha]\\
&= \left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)\prod_{u=1}^{c}(\theta_{q,p}+q^{-\alpha_u}-1)\right]\\
&\quad\times\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\\
&\quad\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^\alpha\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}\\
&= \sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\\
&\quad\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^\alpha\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}\\
&\quad\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)\prod_{u=1}^{c}(\theta_{q,p}+q^{-\alpha_u}-1)\right]x^k\\
&= \sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}\\
&\quad\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^\alpha\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}
\end{aligned}$$

$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)\prod_{u=1}^{c}(\theta_{q,p}x^k+q^{-\alpha_u}x^k-x^k)\right]$$

$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}$$
$$\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}$$
$$\times\left[\prod_{i=1}^{m}\prod_{j=1}^{m}(\theta_{q^{r\alpha^2}}x^k+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}x^k-x^k)\prod_{u=1}^{c}q^{-\alpha_u}(1-q^{\alpha_u+pk})\right]$$

$$=\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}$$
$$\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}$$
$$\times\left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}(1-q^{((i-1)r\alpha^2-nr\alpha^2)/m+r\alpha^2k}w^{j-1})\right\}\right.$$
$$\left.\times\left\{\prod_{u=1}^{c}q^{-\alpha_u}(1-q^{\alpha_u+pk})\right\}\right]$$

Finally we get,

$$\left[\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)(\theta_{q,p}+q^{-\alpha_u}-1)\right\}\right]y$$
$$=\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}q^{-((i-1)r\alpha^2-nr\alpha^2)/m-\alpha_u}w^{1-j}\right\}$$
$$\times\sum_{k=0}^{\lfloor n/m\rfloor}\left\{\prod_{i=1}^{m}\prod_{j=1}^{m}\left(q^{((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1};q^{r\alpha^2}\right)_k\right\}\left\{\prod_{u=1}^{c}(\alpha_u;q)_{k,p}\right\}$$
$$\times\left\{\prod_{v=1}^{d}(\beta_v;q)_{k,p}\right\}^{-1}\left\{\prod_{s=1}^{h}\prod_{t=1}^{h}\left(q^{(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{t-1};q^{\alpha}\right)_{k,p}\right\}^{-1}\frac{(xq^{r\alpha l})^k}{(q^{r\alpha^2};q^{r\alpha^2})_k}.$$
(7.4.10)

We get bibasic p-deformed q-difference equation (7.4.8) satisfied by (7.4.7) from (7.4.9) and (7.4.10). □

Corollary 7.4.3. *There holds the tribasic p-deformed q-differential equation satisfied by the polynomial* $y=\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha);(\beta):xq^{r\alpha l}|q^{\alpha}]$ *given by (7.4.7) which* *follows from (7.4.8) and (7.4.1), is*

$$\Bigg[(1-q^{r\alpha^2})xD_{q^{r\alpha^2}}\Bigg\{\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{v=1}^{d}((1-q^{\alpha})xD_{q^{\alpha},p}+q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t}-1)$$
$$\times((1-q)xD_{q,p}+q^{p-\beta_v}-1)\Bigg\}-xq^{r\alpha l}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}$$
$$\times\, q^{p+((i-1)r\alpha^2-nr\alpha^2)/m-\beta_v+\alpha_u}w^{j-1}\nu^{1-t}((1-q)xD_{q,p}+q^{-\alpha_u}-1)$$
$$\times((1-q^{r\alpha^2})xD_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)\Bigg\}\Bigg]y=0. \quad (7.4.11)$$

Next on making $q^{\beta}\to 0$ on (7.4.8) and (7.4.11), we get p-deformed q-difference equation and p-deformed q-differential equation for the p-deformed q-Brafman polynomial given respectively as follows.

Corollary 7.4.4. *The polynomial* $u = B^{m}_{n,p,r,\alpha}[(\alpha);(\beta) : xq^{r\alpha^2 m}|q^{\alpha}$ *satisfies the equation:*

$$\Bigg[\theta_{q^{r\alpha^2}}\Bigg\{\prod_{v=1}^{d}(\theta_{q,p}+q^{p-\beta_v}-1)\Bigg\}-xq^{r\alpha^2 m}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{q^{p-\beta_v+\alpha_u+((i-1)r\alpha^2-nr\alpha^2)/m}$$
$$\times\, w^{j-1}(\theta_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)(\theta_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]u=0,$$

and

Corollary 7.4.5. *The same polynomial satisfies the equation:*

$$\Bigg[(1-q^{r\alpha^2})xD_{q^{r\alpha^2}}\Bigg\{\prod_{v=1}^{d}((1-q)xD_{q,p}+q^{p-\beta_v}-1)\Bigg\}-xq^{r\alpha^2 m}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{u=1}^{c}\prod_{v=1}^{d}$$
$$\times\Bigg\{q^{p-\beta_v+\alpha_u+((i-1)r\alpha^2-nr\alpha^2)/m}w^{j-1}((1-q^{r\alpha^2})xD_{q^{r\alpha^2}}+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j}-1)$$
$$\times((1-q)xD_{q,p}+q^{-\alpha_u}-1)\Bigg\}\Bigg]u=0,$$

in which w is m^{th} root of unity. The bibasic p-deformed q-difference and q-differential equation satisfy by the extended p-deformed q-Konhauser polynomial and the extended p-deformed q-Laguerre polynomial remain same as derived in chapter 5.

7.5 Generating function relations

The generating function relation for the polynomial (7.1.2) is first considered in the following form.

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{r\alpha n(n-1)/2}\, \frac{S_{n,p,r}(l,m,\alpha,\beta : x|q^{\alpha})}{(\beta q^{p-pn\alpha})_{\infty,p}}\, t^n \\
&= \sum_{n=0}^{\infty} q^{r\alpha n(n-1)/2} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk} q^{mkr\alpha(mk-2n+1)/2}}{(\beta q^{rlk+p-n\alpha p}; q^{\alpha})_{\frac{rlk}{p\alpha},p} (q^{r\alpha}; q^{r\alpha})_{n-mk}} \lambda_k\, x^k t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk} q^{r\alpha n(n-1)/2 + mkr\alpha(mk-2n+1)/2}}{(\beta q^{p-n\alpha p}; q^{\alpha})_{\frac{rlk}{p\alpha},p} (q^{r\alpha}; q^{r\alpha})_{n-mk}} \lambda_k\, x^k t^n.
\end{aligned}$$

This together with the double series relation

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+mk),$$

gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{r\alpha n(n-1)/2}\, \frac{S_{n,p,r}(l,m,\alpha,\beta : x|q^{\alpha})}{(\beta q^{p-pn\alpha})_{\infty,p}}\, t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{mk} q^{r\alpha(n+mk)(n+mk-1)/2 + mkr\alpha(mk-2n-2mk+1)/2}}{(\beta q^{p-(n+mk)\alpha p}; q^{\alpha})_{\frac{rlk}{p\alpha},p} (q^{r\alpha}; q^{r\alpha})_{n+mk-mk}}\, \lambda_k\, x^k t^{n+mk} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{mk} q^{r\alpha n(n-1)/2}}{(\beta q^{p-(n+mk)\alpha p}; q^{\alpha})_{\frac{rlk}{p\alpha},p} (q^{r\alpha}; q^{r\alpha})_{n}}\, \lambda_k\, x^k t^{n+mk}. \qquad (7.5.1)
\end{aligned}$$

∗ **Generating function relation of pEqJP:**

We substitute

$$\lambda_n = \frac{(\alpha_1; q)_{n,p} (\alpha_2; q)_{n,p} \cdots (\alpha_c; q)_{n,p}}{(\beta_1; q)_{n,p} (\beta_2; q)_{n,p} \cdots (\beta_d; q)_{n,p} (q^{r\alpha^2}; q^{r\alpha^2})_n}$$

and replace r by $r\alpha$ in (7.5.1) to obtain the generating function relation of pEqJP (6.1.3) as given below.

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{r\alpha^2 n(n-1)/2} \frac{\mathscr{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha);(\beta) : xq^{r\alpha l}|q^{\alpha}]}{(q^{r\alpha^2}; q^{r\alpha^2})_n}\, t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{mk} q^{r\alpha^2 n(n-1)/2} (\alpha_1; q)_{k,p} (\alpha_2; q)_{k,p} \cdots (\alpha_c; q)_{k,p}}{(\beta q^{p-pn\alpha-pm\alpha k}; q^{r\alpha})_{\frac{rlk}{p},p} (q^{r\alpha^2}; q^{r\alpha^2})_n (\beta_1; q)_{k,p} (\beta_2; q)_{k,p} \cdots (\beta_d; q)_{k,p}} \\
&\times \frac{x^k\, t^{n+mk}}{(q^{r\alpha^2}; q^{r\alpha^2})_k}. \qquad (7.5.2)
\end{aligned}$$

$*$ **Generating function for the pqBP:**
The further special case $\beta \equiv q^{\beta} \to 0$ in (7.5.2) leads us to

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{r\alpha^2 n(n-1)/2} \frac{B^m_{n,p,r,\alpha}[(\alpha);(\beta): xq^{r\alpha^2 m}|q^{\alpha}]}{(q^{r\alpha^2};q^{r\alpha^2})_n}\ t^n \\
=&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{mk} q^{r\alpha^2 n(n-1)/2}(\alpha_1;q)_{k,p}(\alpha_2;q)_{k,p}\cdots(\alpha_c;q)_{k,p}}{(q^{r\alpha^2};q^{r\alpha^2})_n(\beta_1;q)_{k,p}(\beta_2;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{r\alpha^2};q^{r\alpha^2})_k}\ x^k\ t^{n+mk} \\
=&\ E_{q^{r\alpha^2}}(t)\sum_{k=0}^{\infty} \frac{(-1)^{mk}(\alpha_1;q)_{k,p}(\alpha_2;q)_{k,p}\cdots(\alpha_c;q)_{k,p}}{(\beta_1;q)_{k,p}(\beta_2;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{r\alpha};q^{r\alpha})_k} x^k\ t^{mk},
\end{aligned}$$

using the q-exponential function $E_q(x)$ (7.1.9). If $c = d+1$, then in view of the definition (1.5.14) of ${}_r\phi_s$-function, this simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{r\alpha^2 n(n-1)/2} \frac{B^m_{n,p,r,\alpha}[(\alpha);(\beta): xq^{r\alpha^2 m}|q^{\alpha}]}{(q^{r\alpha^2};q^{r\alpha^2})_n}\ t^n \\
=&\ E_{q^{r\alpha^2}}(t)\ {}_{d+1}\phi_d((\alpha),\ (\beta)\ ;p)(x(-t)^m|q,q^{r\alpha^2}).
\end{aligned}$$

The generating function relations of the extended bibasic p-deformed q-Konhauser polynomial and the extended p-deformed q-Laguerre polynomial are derived in chapter 5, hence they are not re-stated here.

7.6 Summation formulas

We now take up the inverse series derived in section - 7.3 and use it to derive certain summation formulas in this section. In fact, from (7.3.1) we have for all $\lambda_n \neq 0$,

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k=0}^{mn} \frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k} S_{k,p,r}(l,m,\alpha,\beta:x|q^{\alpha}) \\
&= x^n. \qquad (7.6.1)
\end{aligned}$$

Now multiplying both sides by $1/(q^{r\alpha};q^{r\alpha})_n$, and taking summation from $n=0$ to ∞, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{\lambda_n\ (q^{r\alpha};q^{r\alpha})_n} \sum_{k=0}^{mn} \frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k} \\
&\times S_{k,p,r}(l,m,\alpha,\beta:x|q^{\alpha}) \\
=&\sum_{n=0}^{\infty} \frac{x^n}{(q^{r\alpha};q^{r\alpha})_n} \\
=&\ e_{q^{r\alpha}}(x),
\end{aligned}$$

in view of (7.1.8) where $|x| < 1$. In this, if we substitute

$$\lambda_n = \frac{(\alpha_1;q)_{n,p}(\alpha_2;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}{(\beta_1;q)_{n,p}(\beta_2;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{r\alpha^2};q^{r\alpha^2})_n},$$

and replace r by $r\alpha$, then it reduces to the summation formula for pEqJP given as

$$\sum_{n=0}^{\infty}\frac{(\beta_1;q)_{n,p}(\beta_2;q)_{n,p}\cdots(\beta_d;q)_{n,p}}{(\alpha_1;q)_{n,p}(\alpha_2;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha^2(k-1)/2}(1-q^{kr\alpha^2+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha^2+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha^2};q^{r\alpha^2})_{mn-k}}$$
$$\times\frac{(\beta q^{p-pn\alpha};q^\alpha)_{\infty,p}}{(q^{r\alpha^2};q^{r\alpha^2})_k}\mathscr{H}^{(\alpha,\beta)}_{k,m,l,p,r}[(\alpha);(\beta):xq^{r\alpha l}|q^\alpha]=e_{r\alpha^2}(x).$$

We omit the special case corresponding to pqBP for the sake of brevity.
Next, the summation formula for the pEqKP is obtained by taking $\beta\to\infty$, $r\alpha = l\in\mathbb{N}$,

$$\lambda_n = \frac{q^{ln(\alpha+1)+ln(ln-1)/2-lmn}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}},$$

and x is replaced by $(xq^n)^l$. With these substitutions, it takes the form:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{mn}\frac{(-1)^k q^{lk(k-1)/2}(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}}{q^{ln(\alpha+1)+ln(ln-1)/2-lmn}(p\alpha;q)_{lk,p}(q^l;q^l)_n(q^l;q^l)_{mn-k}}$$
$$\times Z^{(\alpha)}_{k,m,p}(x;l|q)=e_{q^l}(x^l).$$

Further, for $l = 1$, this immediately yields the formula involving the pEqLP. Next, from (7.6.1) with $|x| < 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{\lambda_n(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}$$
$$\times S_{k,p,r}(l,m,\alpha,\beta:x|q^\alpha)=\frac{1}{1-x}. \quad (7.6.2)$$

By assigning different values to x from $(-1, 1)$, a number of particular summation formulas can be derived. For example, $x = 1/2$ in this formula gives the following one.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{\lambda_n(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}$$
$$\times S_{k,p,r}\left(l,m,\alpha,\beta:\frac{1}{2}\Big|q^\alpha\right) = 2. \quad (7.6.3)$$

The sum of ${}_1\phi_1[*]$ in (5.6.3) enables us to obtain one more summation formula by multiplying

$$\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{(c;q)_n(q;q)_n}$$

to both sides of (7.6.1), replacing x by c/a and then summing-up from $n=0$ to ∞. We then find

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(a;q)_n}{\lambda_n(c;q)_n(q;q)_n}\left(-\frac{c}{a}\right)^n\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}}$$
$$\times\frac{1}{(q^{r\alpha};q^{r\alpha})_k}S_{k,p,r}\left(l,m,\alpha,\beta:\frac{c}{a}\middle|q^\alpha\right)=\frac{(c/a;q)_\infty}{(c;q)_\infty}. \quad (7.6.4)$$

Next, multiply (7.6.1) by $(a;q)_n/(q;q)_n$ and take the sum $n=0$ to ∞ to get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{\lambda_n(q;q)_n}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}$$
$$\times S_{k,p,r}(l,m,\alpha,\beta:x|q^\alpha) = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad (7.6.5)$$

for $|x|<1$ in view of (7.1.7). Lastly, multiplying (7.6.1) by $q^{\binom{n}{2}}/(q;q)_n$ and taking sum $n=0$ to ∞ yields

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{\lambda_n(q;q)_n}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^\alpha)_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}$$
$$\times S_{k,p,r}(l,m,\alpha,\beta:x|q^\alpha)=E_q(x). \quad (7.6.6)$$

The reducibility of summation formulas (7.6.2) to (7.6.6) corresponding to the particular cases pEqJP and pEqKP may be obtained by the suitable substitutions of the parameters involved and the sequence $\{\lambda_n\}$ as stated above sections.

7.7 Companion matrix

Taking $\lfloor n/m\rfloor = N$ in (7.1.2) and converting it to the monic form as denoted by $\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^\alpha)$, we get

$$\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^\alpha)=\sum_{j=0}^{N}\delta_j\ x^j,$$

where

$$\delta_j=(-1)^{\sigma m}\ q^{r\alpha m^2\sigma(j+N)/2+m\sigma/2-mn\sigma}\ \frac{(\beta q^{lj-pn\alpha+p};q^{r\alpha})_{\infty,p}\ (q^{r\alpha};q^{r\alpha})_{n-mN}\lambda_j}{(\beta q^{lN-pn\alpha+p};q^{r\alpha})_{\infty,p}\ (q^{r\alpha};q^{r\alpha})_{n-mj}\lambda_N},$$

and $\sigma = j - N$. With this δ_j, $C\left(\widetilde{S}_{n,p,r}(l, m, \alpha, \beta : x|q^{\alpha})\right)$ assumes the form as stated in Definition 1.3.1. The eigen values of this matrix will be then precisely the zeros of $\widetilde{S}_{n,p,r}(l, m, \alpha, \beta : x|q^{\alpha})$ (see [48, p. 39]).

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