Chapter 8

The p-Deformed classes of Riordan's Inverse pairs

8.1 Introduction

The objective of this chapter is to obtain the p-deformed versions of the inverse series relations due to John Riordan [55]. It appears from the literature that Riordan's work incorporates inverse series of general form of the series most of which are finite series. He also classifies them into several classes namely, the simplest inverse series relations, Gould classes, Simpler Chebyshev inverse relations, Chebyshev classes of inverse relations, Simpler Legendre inverse relations and the Legendre-Chebyshev classes of inverse relations [55, p.49-69].
Here we provide p-deformation to these classes by means of the GIPs obtained in chapters 2 and 4.
We reconsider the inversion theorems of Chapter 2 and Chapter 4 with the objective of obtaining the deformation of the suitable inverse pairs belonging to the aforementioned classes.
In section - 8.2, the general inverse pair of the main theorem of chapter 2 will be transformed in to several alternative forms to deform several inverse pairs of Riordan's classes.
In section - 8.3, some of the p-deformed Riordan's inverse pairs are extended from the alternative forms of Theorem - 4.2.1 of chapter 4.
We first re-state the Theorem - 2.2.1 and Theorem - 2.2.2 of chapter 2 as

Theorem 8.1.1. *For $\alpha,\ r,\ \gamma \in \mathbb{C}$ and $p > 0$,*

$$u(n) = \sum_{k=0}^{N} \frac{\gamma^k}{\Gamma_p(p+\alpha-nr-brk-kp)k!}\, v(n+bk) \tag{8.1.1}$$

$$\Leftrightarrow$$

$$v(n) = \sum_{k=0}^{N} \frac{(-\gamma)^k(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{k!}\, u(n+bk), \tag{8.1.2}$$

where b is (i) a negative integer $-m$ in Theorem - 2.2.1 and, (ii) is a positive integer in Theorem - 2.2.2. In case (i), $N = \lfloor n/m \rfloor$ whereas in case (ii), $N = \infty$.

The following is Theorem - 4.2.1 of chapter 4.

Theorem 8.1.2. *For $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{N}$ and $p > 0$,*

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \frac{1}{\Gamma_p(\alpha + mk\lambda + p - np)(n - mk)!} G(k) \quad (8.1.3)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} F(k), \quad (8.1.4)$$

$$\text{and} \quad G(n/m) = 0, n \neq mr. \quad (8.1.5)$$

8.2 p-Deformed classes of Riordan's inverse pairs

Here, we shall obtain several inverse pairs in the forms of deformed Riordan's inverse series relations. For this, we now derive several pairs of inverse series relations in this section with the help (8.1.1) and (8.1.2) of the general inversion pair as it represents both the Theorem - 8.1.1. We begin with replacing $v(a)$ by $\Gamma_p(p + \alpha - ar)v(a)$ in (8.1.1) and (8.1.2) to get

$$u(a) = \sum_{k=0}^{N} \frac{\gamma^k \Gamma_p(p + \alpha - ar - brk)}{\Gamma_p(p + \alpha - ar - brk - kp)k!} v(a + bk) \quad (8.2.1)$$

$$\Leftrightarrow$$

$$v(a) = \sum_{k=0}^{N} \frac{(-\gamma)^k (\alpha - ar - brk)\Gamma_p(\alpha - ar + kp)}{\Gamma_p(p + \alpha - ar)k!} u(a + bk). \quad (8.2.2)$$

Next, introducing the factor $(\alpha - ar + kp)$ in (8.2.2) and making use of (1.3.3), we get

$$u(a) = \sum_{k=0}^{N} \frac{\gamma^k \Gamma_p(p + \alpha - ar - brk)}{\Gamma_p(p + \alpha - ar - brk - kp)k!} v(a + bk) \quad (8.2.3)$$

$$\Leftrightarrow$$

$$v(a) = \sum_{k=0}^{N} \frac{(-\gamma)^k (\alpha - ar - brk)\Gamma_p(p + \alpha - ar + kp)}{(\alpha - ar + kp)\Gamma_p(p + \alpha - ar)k!} u(a + bk). \quad (8.2.4)$$

We put $\gamma = 1$ in (8.2.3) and (8.2.4) to get

∗ **Inverse pair 8.2.a**

$$u(a) = \sum_{k=0}^{N} \frac{\Gamma_p(p + \alpha - ar - brk)}{\Gamma_p(p + \alpha - ar - brk - kp)k!} v(a + bk)$$

$$\Leftrightarrow$$

$$v(a) = \sum_{k=0}^{N} \frac{(-1)^k (\alpha - ar - brk)\Gamma_p(p + \alpha - ar + kp)}{(\alpha - ar + kp)\Gamma_p(p + \alpha - ar)k!} u(a + bk).$$

Replace $u(a)$ by $u(a)/(\alpha - ar)$ and $v(a)$ by $v(a)/(\alpha - ar)$, this pair takes the form of

∗ **Inverse pair 8.2.b**

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} \frac{(\alpha - ar)\Gamma_p(\alpha - ar - brk)}{(\alpha - ar - brk - kp)\Gamma_p(\alpha - ar - brk - kp)k!}\, v(a+bk) \\ &\Leftrightarrow \\ v(a) &= \sum_{k=0}^{N} (-1)^k \frac{\Gamma_p(\alpha - ar + kp)}{\Gamma_p(\alpha - ar)k!}\, u(a+bk). \end{aligned}$$

On replacing α and r by $\alpha + p$ and $-r$ respectively in this inverse pair, we get

∗ **Inverse pair 8.2.c**

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} \frac{(\alpha + p + ar)\Gamma_p(p + \alpha + ar + brk)}{(\alpha + p + ar + brk - kp)\Gamma_p(p + \alpha + ar + brk - kp)k!}\, v(a+bk) \\ &\Leftrightarrow \\ v(a) &= \sum_{k=0}^{N} (-1)^k \frac{\Gamma_p(p + \alpha + ar + kp)}{\Gamma_p(p + \alpha + ar)k!}\, u(a+bk). \end{aligned}$$

On the other hand, by applying the binomial identity

$$(-1)^n \frac{\Gamma_p(p-\alpha)}{\Gamma_p(p - \alpha - np)n!} = \frac{\Gamma_p(\alpha + np)}{\Gamma_p(\alpha)n!} = \frac{\alpha\Gamma_p(p + \alpha + np)}{(\alpha + np)\Gamma_p(p+\alpha)n!}, \quad (8.2.5)$$

the inverse pair - 8.2.a holds the form as given below.

∗ **Inverse pair 8.2.d**

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} (-1)^k \frac{\Gamma_p(-\alpha + ar + brk + kp)}{\Gamma_p(-\alpha + ar + brk)k!}\, v(a+bk) \\ &\Leftrightarrow \\ v(a) &= \sum_{k=0}^{N} \frac{(\alpha - ar - brk)\Gamma_p(-\alpha + ar)}{(\alpha - ar + kp)\Gamma_p(-\alpha + ar - kp)k!}\, u(a+bk). \end{aligned}$$

Likewise, using the formula (8.2.5) in inverse pair - 8.2.b, we obtain

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} (-1)^k \frac{(\alpha - ar)\Gamma_p(p - \alpha + ar + brk + kp)}{(\alpha - ar - brk - kp)\Gamma_p(p - \alpha + ar + brk)k!}\, v(a+bk) \\ &\Leftrightarrow \\ v(a) &= \sum_{k=0}^{N} \frac{\Gamma_p(p - \alpha + ar)}{\Gamma_p(p - \alpha + ar - kp)k!}\, u(a+bk). \end{aligned}$$

Now replace α by $-\alpha$ here to get

$*$ **Inverse pair 8.2.e**

$$\begin{aligned} u(a) &= \sum_{k=0}^{N} (-1)^k \frac{(\alpha+ar)\Gamma_p(p+\alpha+ar+brk+kp)}{(\alpha+ar+brk+kp)\Gamma_p(p+\alpha+ar+brk)k!} v(a+bk) \\ \Leftrightarrow & \\ v(a) &= \sum_{k=0}^{N} \frac{\Gamma_p(p+\alpha+ar)}{\Gamma_p(p+\alpha+ar-kp)k!} u(a+bk). \end{aligned}$$

Now, on substituting $br=-p$ and $a=n$ in the GIP (8.1.1) and (8.1.2), it takes the form:

$$\begin{aligned} u(n) &= \sum_{k=0}^{N} \frac{\gamma^k}{k!\ \Gamma_p(\alpha-nr+p)}\ v(n+bk) \\ \Leftrightarrow & \\ v(n) &= \sum_{k=0}^{N} \frac{(-\gamma)^k(\alpha-nr+kp)\Gamma_p(\alpha-nr+kp)}{k!}\ u(n+bk). \end{aligned}$$

Here replacing $u(n)$ by $u(n)/\Gamma_p(\alpha-nr+p)$ and putting $br=-p$, this pair reduces to the

$*$ **Inverse pair 8.2.f**

$$u(n) = \sum_{k=0}^{N} \frac{1}{k!}\ v(n+bk) \ \Leftrightarrow\ v(n) = \sum_{k=0}^{N} \frac{(-1)^k}{k!}\ u(n+bk).$$

The inverse pairs 8.2.a to 8.2.f with $a=n$, $u(n)=A_n$, $v(n)=B_n$ and the appropriate value of b will be used for extending classes of Riordan's inverse pairs. They are tabulated in Table 8.1 to Table 8.6.

Table 8.1: The simplest inverse relations

$A_n = \sum \frac{1}{k!}\ B_{n+bk};\ B_n = \sum \frac{(-1)^k}{k!}\ A_{n+bk}$

Inv. pair No.	b	A_n	B_n
8.2.f	-1	$a_n/n!$	$b_n/n!$
8.2.f	1	$a_n n!$	$b_n n!$
8.2.f	-1	$a_n(\alpha-n)!$	$b_n(\alpha-n)!$
8.2.f	-1	$a_n/(\alpha+n)!$	$b_n/(\alpha+n)!$
8.2.f	1	$a_n(\alpha+n)!$	$b_n(\alpha+n)!$
8.2.f	-1	$a_n/n!(n-1)!$	$b_n/n!(n-1)!$

TABLE 8.2: The p-deformed Gould classes of inverse relations

$$A_n = \sum a_{n,k}\ B_k;\ B_n = \sum (-1)^{n+k}\ b_{n,k}\ A_k$$

Inv. pair No.	b	r	α	$a_{n,k}$	$b_{n,k}$
8.2.a	-1	$p-l$	α	$\frac{\Gamma_p(p+\alpha+lk-kp)}{\Gamma_p(p+\alpha+lk-np)(n-k)!}$	$\frac{(\alpha+lk-kp)}{(\alpha+ln-kp)} \times \frac{\Gamma_p(p+\alpha+ln-kp)}{\Gamma_p(p+\alpha+ln-np)(n-k)!}$
8.2.c	-1	$l-p$	α	$\frac{(p+\alpha+ln-np)}{(p+\alpha-np+lk)} \times \frac{\Gamma_p(p+\alpha+lk-kp)}{\Gamma_p(p+\alpha+lk-np)(n-k)!}$	$\frac{\Gamma_p(p+\alpha+ln-kp)}{\Gamma_p(p+\alpha+ln-np)(n-k)!}$
8.2.e	1	$l-p$	α	$\frac{\Gamma_p(p+\alpha+ln-np)}{\Gamma_p(p+\alpha+ln-kp)(k-n)!}$	$\frac{(\alpha+ln-np)}{(\alpha+lk-np)} \times \frac{\Gamma_p(p+\alpha+lk-np)}{\Gamma_p(p+\alpha+lk-kp)(k-n)!}$
8.2.d	1	$l-p$	$-\alpha-p$	$\frac{(p+\alpha+lk-kp)}{(p+\alpha+ln-kp)} \times \frac{\Gamma_p(p+\alpha+ln-np)}{\Gamma_p(p+\alpha+ln-kp)(k-n)!}$	$\frac{\Gamma_p(p+\alpha+kl-np)}{\Gamma_p(p+\alpha+kl-kp)(k-n)!}$

TABLE 8.3: The p-deformed simpler Chebyshev inverse relation

$$A_n = \sum a_{n,k}\ B_{n+bk};\ B_n = \sum b_{n,k}\ A_{n+bk}$$

Inv. pair No.	b	r	α	$a_{n,k}$	$b_{n,k}$
8.2.e	-2	1	0	$\frac{\Gamma_p(p+n)}{\Gamma_p(p+n-kp)k!}$	$(-1)^k \frac{(n)}{(n-2k+kp)} \times \frac{\Gamma_p(p+n-2k+kp)}{\Gamma_p(p+n-2k)k!}$
8.2.d	-2	1	$-p$	$(-1)^k \frac{\Gamma_p(p+n-2k+kp)}{\Gamma_p(p+n-2k)k!}$	$\frac{(p+n-2k)}{(p+n-kp)} \times \frac{\Gamma_p(p+n)}{\Gamma_p(p+n-kp)k!}$
8.2.a	2	-1	0	$\frac{\Gamma_p(p+n+2k)}{\Gamma_p(p+n+2k-kp)k!}$	$(-1)^k \frac{(n+2k)}{(n+kp)} \times \frac{\Gamma_p(p+n+kp)}{\Gamma_p(p+n)k!}$
8.2.c	2	-1	0	$\frac{(p+n)}{(p+n+2k-kp)} \times \frac{\Gamma_p(p+n+2k)}{\Gamma_p(p+n+2k-kp)k!}$	$(-1)^k \frac{\Gamma_p(p+n+kp)}{\Gamma_p(p+n)k!}$
8.2.a	-1	-1	0	$\frac{\Gamma_p(p+n-k)}{\Gamma_p(p+n-k-kp)k!}$	$(-1)^k \frac{(n-k)}{(n+kp)} \times \frac{\Gamma_p(p+n+kp)}{\Gamma_p(p+n)k!}$
8.2.c	-1	1	0	$\frac{(p+n)}{(p+n-k-kp)} \times \frac{\Gamma_p(p+n-k)}{\Gamma_p(p+n-k-kp)k!}$	$(-1)^k \frac{\Gamma_p(p+n+kp)}{\Gamma_p(p+n)k!}$

TABLE 8.4: p-Deformed Chebyshev classes of inverse relations

$A_n = \sum a_{n,k}\ B_{n+ck};\ B_n = \sum (-1)^k b_{n,k}\ A_{n+ck}$

Inv. pair No.	b	r	α	$a_{n,k}$	$b_{n,k}$
8.2.e	c	1	0	$\frac{\Gamma_p(p+n)}{\Gamma_p(p+n-kp)k!}$	$\frac{n\Gamma_p(p+n+ck+kp)}{(n+ck+kp)\Gamma_p(p+n+ck)k!}$
8.2.d	c	1	$-p$	$\frac{(p+n+ck)\Gamma_p(p+n)}{(p+n-kp)\Gamma_p(p+n-kp)k!}$	$\frac{\Gamma_p(p+n+ck+kp)}{\Gamma_p(p+n+ck)k!}$
8.2.a	c	-1	0	$\frac{\Gamma_p(p+n+ck)}{\Gamma_p(p+n+ck-kp)k!}$	$\frac{(n+ck)\Gamma_p(p+n+kp)}{(n+kp)\Gamma_p(p+n)k!}$
8.2.c	c	1	0	$\frac{(p+n)\Gamma_p(p+n+ck)}{(p+n+ck-kp)\Gamma_p(p+n+ck-kp)k!}$	$\frac{\Gamma_p(p+n+kp)}{\Gamma_p(p+n)k!}$

TABLE 8.5: The p-deformed simpler Legendre inverse relations

Inv. pair No.	b	r	α	$A_n =$	$B_n =$
8.2.d	-1	2	$-\alpha-p$	$\sum \frac{\Gamma_p(p+\alpha+2k+np-kp)}{\Gamma_p(p+\alpha+2k)(n-k)!} B_k$	$\sum \frac{(-1)^{k+n}(p+\alpha+2k)}{(p+\alpha+2n-np+kp)}$ $\times \frac{\Gamma_p(p+\alpha+2n)}{\Gamma_p(p+\alpha+2n-np+kp)(n-k)!} A_k$
8.2.e	-1	2	α	$\sum \frac{\Gamma_p(p+\alpha+2n)}{\Gamma_p(p+\alpha+2n-np+kp)(n-k)!} B_k$	$\sum \frac{(-1)^{k+n}(\alpha+2n)}{(\alpha+2k+np-kp)}$ $\times \frac{\Gamma_p(p+\alpha+2k+np-kp)}{\Gamma_p(p+\alpha+2k)(n-k)!} A_k$
8.2.c	1	2	α	$\sum \frac{\Gamma_p(p+\alpha+2n+kp-np)}{\Gamma_p(p+\alpha+2n)(k-n)!} B_k$	$\sum \frac{(-1)^{k+n}(p+\alpha+2n)}{(p+\alpha+2k-kp+np)}$ $\times \frac{\Gamma_p(p+\alpha+2k)}{\Gamma_p(p+\alpha+2k-kp+np)(k-n)!} A_k$
8.2.a	1	-2	α	$\sum \frac{\Gamma_p(p+\alpha+2k)}{\Gamma_p(p+\alpha+2k-kp+np)(k-n)!} B_k$	$\sum \frac{(-1)^{k+n}(\alpha+2k)}{(\alpha+2n+kp-np)}$ $\times \frac{\Gamma_p(p+\alpha+2n+kp-np)}{\Gamma_p(p+\alpha+2n)(k-n)!} A_k$
8.2.e	-2	2	α	$\sum \frac{\Gamma_p(p+\alpha+2n)}{\Gamma_p(p+\alpha+2n-kp)k!} B_{n-2k}$	$\sum \frac{(-1)^k(\alpha+2n)}{(\alpha+2n-4k+kp)}$ $\times \frac{\Gamma_p(p+\alpha+2n-4k+kp)}{\Gamma_p(p+\alpha+2n-4k)k!} A_{n-2k}$
8.2.d	-2	2	$-\alpha-p$	$\sum \frac{(p+\alpha+2n-4k)}{(p+\alpha+2n-kp)}$ $\times \frac{\Gamma_p(p+\alpha+2n)}{\Gamma_p(p+\alpha+2n-kp)k!} B_{n-2k}$	$\sum \frac{(-1)^k\Gamma_p(p+\alpha+2n-4k+kp)}{\Gamma_p(p+\alpha+2n-4k)k!}$ $\times A_{n-2k}$

TABLE 8.6: p-Deformed Legendre-Chebyshev classes of inverse relations
$A_n = \sum a_{n,k}\ B_k;\ B_n = \sum(-1)^{n+k}\ b_{n,k}\ A_k$

Inv. pair No.	b	r	α	$a_{n,k}$	$b_{n,k}$
8.2.e	-1	c	α	$\frac{\Gamma_p(p+\alpha+cn)}{\Gamma_p(p+\alpha+cn-np+kp)(n-k)!}$	$\frac{(\alpha+cn)}{(\alpha+ck+np-kp)} \times \frac{\Gamma_p(p+\alpha+ck+np-kp)}{\Gamma_p(p+\alpha+ck)(n-k)!}$
8.2.e	1	c	α	$\frac{\Gamma_p(p+\alpha+cn)}{\Gamma_p(p+\alpha+cn-kp+np)(k-n)!}$	$\frac{(\alpha+cn)}{(\alpha+ck+kp-np)} \times \frac{\Gamma_p(p+\alpha+ck+kp-np)}{\Gamma_p(\alpha+ck)(k-n)!}$
8.2.a	-1	$-c$	α	$\frac{\Gamma_p(p+\alpha+ck)}{\Gamma_p(p+\alpha+ck-np+kp)(n-k)!}$	$\frac{(\alpha+ck)}{(\alpha+cn+np-kp)} \times \frac{\Gamma_p(p+\alpha+cn+np-kp)}{\Gamma_p(p+\alpha+cn)(n-k)!}$
8.2.a	1	$-c$	α	$\frac{\Gamma_p(p+\alpha+ck)}{\Gamma_p(p+\alpha+ck-kp+np)(k-n)!}$	$\frac{(\alpha+ck)}{(\alpha+cn+kp-np)} \times \frac{\Gamma_p(p+\alpha+cn+kp-np)}{\Gamma_p(p+\alpha+cn)(k-n)!}$
8.2.d	-1	c	$-\alpha-p$	$\frac{(p+\alpha+ck)}{(p+\alpha+cn-np+kp)} \times \frac{\Gamma_p(p+\alpha+cn)}{\Gamma_p(p+\alpha+cn-np+kp)(n-k)!}$	$\frac{\Gamma_p(p+\alpha+ck+np-kp)}{\Gamma_p(p+\alpha+ck)(n-k)!}$
8.2.d	1	c	$-\alpha-p$	$\frac{(p+\alpha+ck)}{(p+\alpha+cn+kp-np)} \times \frac{\Gamma_p(p+\alpha+cn)}{\Gamma_p(p+\alpha+cn-kp+np)(k-n)!}$	$\frac{\Gamma_p(p+\alpha+ck+kp-np)}{\Gamma_p(p+\alpha+ck)(k-n)!}$
8.2.c	-1	c	α	$\frac{(p+\alpha+cn)}{(p+\alpha+ck-np+kp)} \times \frac{\Gamma_p(p+\alpha+ck)}{\Gamma_p(p+\alpha+ck-np+kp)(n-k)!}$	$\frac{\Gamma_p(p+\alpha+cn+np-kp)}{\Gamma_p(p+\alpha+cn)(n-k)!}$
8.2.c	1	c	α	$\frac{(\alpha+p+cn)}{(\alpha+p+ck+np-kp)} \times \frac{\Gamma_p(p+\alpha+ck)}{\Gamma_p(p+\alpha+ck-kp+np)(k-n)!}$	$\frac{\Gamma_p(p+\alpha+cn+kp-np)}{\Gamma_p(p+\alpha+cn)(k-n)!}$

All these pairs get reduced to the Riordan's pairs for $p = 1$ and α is replaced by p; which are tabulated through Table 1.1 to Table 1.6 stated in chapter 1.

8.3 Extension of certain p-deformed Riordan's inverse pairs

We now illustrate Theorem - 8.1.2 for carrying out the extension of certain inverse series relations belonging to the p-deformed Riordan's inverse pairs appearing in Table 8.1 to Table 8.6. All these inverse pairs are however directly deducible from Theorem - 8.1.2, the following alternative versions of the theorem readily yield them. They are deduced below. We put $\alpha = a$ in the theorem and get the following pair.

$*$ **Inverse pair 8.3.a**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \frac{1}{\Gamma_p(a+mk\lambda+p-np)(n-mk)!} G(k), \\ \Leftrightarrow & \\ G(n) &= \sum_{k=0}^{mn} (-1)^k \frac{(a+k\lambda-kp)\Gamma_p(a+mn\lambda-kp)}{(mn-k)!} F(k). \end{aligned}$$

Now taking $\alpha = a+p$, replacing $F(n)$ by $(-1)^n F(n)/(a+n\lambda-np+p)$ and $G(n)$ by $\Gamma_p(a+mn\lambda-mnp+p)G(n)/(a+mn\lambda-mnp)$ in Theorem - 8.1.2, we get

$*$ **Inverse pair 8.3.b**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{n-mk} \frac{(a+n\lambda-np+p)\Gamma_p(a+mk\lambda-mkp+p)}{(a+mk\lambda-np+p)\Gamma_p(a+mk\lambda-np+p)(n-mk)!} G(k), \\ \Leftrightarrow & \\ G(n) &= \sum_{k=0}^{mn} \frac{\Gamma_p(a+mn\lambda-kp+p)}{\Gamma_p(a+mn\lambda-mnp+p)(mn-k)!} F(k). \end{aligned}$$

Next, in Theorem - 8.1.2, replacing α by $-a-p$, $F(n)$ by $(-1)^n F(n)$ and $G(n)$ by $G(n)/\Gamma_p(a-mn\lambda+mnp+p)$, the we obtain

$*$ **Inverse pair 8.3.c**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(a+np-mk\lambda+p)}{\Gamma_p(a-mk\lambda+mkp+p)(n-mk)!} G(k), \\ \Leftrightarrow & \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn-k} \frac{(a-k\lambda+kp+p)\Gamma_p(a-mn\lambda+mnp+p)}{(a-mn\lambda+kp+p)\Gamma_p(a-mn\lambda+kp+p)(mn-k)!} F(k). \end{aligned}$$

Here, if we replace $F(n)$ by $F(n)/(a-n\lambda+np)$, $G(n)$ by $G(n)/(a-mn\lambda+mnp)$ and a by $a-p$, then we find

$*$ **Inverse pair 8.3.d**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(a-n\lambda+np)\Gamma_p(a+np-mk\lambda+p)}{(a-mk\lambda+np)\Gamma_p(a-mk\lambda+mkp+p)(n-mk)!} G(k), \\ \Leftrightarrow & \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn-k} \frac{\Gamma_p(a-mn\lambda+mnp+p)}{\Gamma_p(a-mn\lambda+kp+p)(mn-k)!} F(k). \end{aligned}$$

The inverse pairs 8.3.a to 8.3.d with $F(n) = A_n$ and $G(n) = B_n$ provide extension to certain p-deformed Riordan's inverse pairs which are tabulated below.

TABLE 8.7: The p-deformed extension of Riordan's inverse series

$$A_n = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{a_{n,k}}{(n-mk)!} B_k \; ; \; B_n = \sum_{k=0}^{mn} (-1)^{mn-k} \frac{b_{n,k}}{(mn-k)!} A_k$$

Inv. pair No.	λ	$a_{n,k}$	$b_{n,k}$	p-deformed extension of class(inverse pair no.) as in Table 8.2, 8.5 and 8.6
8.3.a	l	$\frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{a+lkp-kp}{a+lmn-kp} \times \frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	p-Gould class (1), Table 8.2
8.3.b	l	$\frac{a+ln-np+p}{a+lmk-np+p} \times \frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	p-Gould class (2), Table 8.2
8.3.c	$p-2$	$\frac{\Gamma_p(a+np+2mk-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{a+2k+p}{a+2mn-mnp+kp+p} \times \frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	p-Simpler Legendre, class (1), Table 8.5
8.3.d	$p-2$	$\frac{a+2n}{a+2mk+np-mkp} \times \frac{\Gamma_p(a+2mk+np-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	p-Simpler Legendre, class (2), Table 8.5
8.3.d	$p-c$	$\frac{a+cn}{a+cmk+np-mkp} \times \frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	p-Legendre-Chebyshev, class (1), Table 8.6
8.3.a	$p+c$	$\frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{a+ck}{a+cmn+mnp-kp} \times \frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	p-Legendre-Chebyshev, class (3), Table 8.6
8.3.c	$p-c$	$\frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{a+ck+p}{a+cmn-mnp+kp+p} \times \frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	p-Legendre-Chebyshev, class (5), Table 8.6
8.3.b	$p+c$	$\frac{a+cn+p}{a+cmk+mkp-np+p} \times \frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	p-Legendre-Chebyshev, class (7), Table 8.6