Chapter 9

The p-Deformed classes of Basic Riordan's inverse pairs

9.1 Introduction

This chapter incorporates the q-analogues of the inverse pairs obtained in chapter 8. We make use of the general inversion pairs of chapters 3, 5 and 7. For carrying out the q-extension to these pairs, we use Theorems - 3.2.1, 3.2.3, 5.2.1 and 7.2.1; also, the inverse pair (5.3.1) and (5.3.2). These theorems are re considered here in different order. The following is Theorem - 7.2.1 of chapter 7.

Theorem 9.1.1. *For* $r, \alpha \in \mathbb{C} \setminus \{0\},\ \beta \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\},\ m, r \in \mathbb{N},\ p > 0$ *and* $0 < q < 1$*, then*

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mkr\alpha(mk-2n+1)/2} \frac{(q^{mkr\alpha+\beta+p-n\alpha p}; q^{\alpha})_{\infty,p}}{(q^{r\alpha}; q^{r\alpha})_{n-mk}} G(k) \tag{9.1.1}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1 - q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p}; q^{\alpha})_{\infty,p} (q^{r\alpha}; q^{r\alpha})_{mn-k}} F(k) \tag{9.1.2}$$

and $G(n/m) = 0,\ n \neq mr.$

Next, we re-state the theorems of chapter 3 with $a = n \in \mathbb{N}$ as

Theorem 9.1.2. *Let* $0 < q < 1,\ \alpha,\ r,\ \gamma \in \mathbb{C},\ q_2 = q^{-br-p},\ br \neq -p$ *and* $p > 0$*, then*

$$F(n) = \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-nr-brk-kp}; q)_{\infty,p}}{(q_2; q_2)_k} G(n + bk) \tag{9.1.3}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{N} (-\gamma)^k q_2^{k(k-1)/2} \frac{(1 - q^{\alpha-nr+mrk})}{(q^{\alpha-nr+kp}; q)_{\infty,p} (q_2; q_2)_k} F(n + bk), \tag{9.1.4}$$

where b is (i) a negative integer $-m$ in Theorems - 3.2.1 and, (ii) is a positive integer in Theorems - 3.2.2. In case (i), $N = \lfloor n/m \rfloor$ whereas in case (ii), $N = \infty$. The following are the theorems of chapter 3 with $a = n \in \mathbb{N}$ and $br = -p$.

Theorem 9.1.3. *Let $p > 0,\ 0 < q < 1,\ \alpha,\ r,$ and $\gamma \in \mathbb{C}$, then*

$$F(n) = \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-nr};q)_{\infty,p}}{(q;q)_k} G(n+bk) \tag{9.1.5}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{N} (-\gamma)^k q^{k(k-1)/2} \frac{F(n+bk)}{(q^{p+\alpha-nr+kp};q)_{\infty,p}(q;q)_k}, \tag{9.1.6}$$

where b is (i) a negative integer $-m$ in Theorems - 3.2.3 and, (ii) is a positive integer in Theorems - 3.2.4. In case (i), $N = \lfloor n/m \rfloor$ whereas in case (ii), $N = \infty$. From chapter 5, we consider Theorem - 5.2.1.

Theorem 9.1.4. *For $0 < q < 1,\ \lambda \in \mathbb{C} \setminus \{0\},\ \alpha \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\},\ m, r \in \mathbb{N}$ and $p > 0$,*

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mk\lambda(mk-2n+1)/2} \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{n-mk}} G(k) \tag{9.1.7}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}} F(k) \tag{9.1.8}$$

and $G(n/m) = 0,\ n \neq mr.$

Its alternative form is given by the inverse pair ((5.3.1) and (5.3.2)):

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^n q^{\lambda(2mk-n(n+1))/2)} \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k) \tag{9.1.9}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^{mn} q^{\lambda(2kmn-mn(mn+1))/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}} \times \frac{1}{(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k). \tag{9.1.10}$$

and $G(n/m) = 0,\ n \neq mr.$

9.2 The p-Deformed extended Basic simplest inverse pairs

In order to obtain the *simplest* inverse pair, we take in (9.1.1) and (9.1.2) of Theorem - 9.1.1, $\beta \to \infty$, that is $q^\beta \to 0$, and $r\alpha = 1$, then choosing $F(n),\ G(n)$ appropriately, we obtain the p-deformed extended q-simplest inverse pairs corresponding to the choice $z \equiv q^z = q$ in (1.5.3) which are tabulated below. (cf. [55, Ch.2, Table 2.1, p.49] with $m = p = 1$ and $q \to 1^-$).

TABLE 9.1: The p-Deformed extended Basic simplest inverse pairs-I

$$F(n)=\sum q^{mk(mk+1)/2-mnk}B_{n,k}\ G(k)\ ;\ G(n)=\sum(-1)^{mn+k}q^{k(k-1)/2}A_{n,k}\ F(k)$$

No.	F(n)	G(n)	$A_{n,k}$	$B_{n,k}$
1.	$\frac{a_n}{(q;q)_{n,p}}$	$\frac{b_n}{(q;q)_{n,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$
2.	$\frac{a_n}{(q;q)_{r+n,p}}$	$\frac{b_n}{(q;q)_{r+n,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$
3.	$\frac{a_n}{(q;q)_{n,p}\,(q;q)_{n-1,p}}$	$\frac{b_n}{(q;q)_{n,p}\,(q;q)_{n-1,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$

Next, putting $z=q^p$ in (1.5.3), and substituting $r\alpha=1$ and taking $\beta\to\infty$ in (9.1.1) and (9.1.2), we find yet another set of p-deformed extended q-simplest inverse pairs which are tabulated below (cf. [55, Ch.2, Table 2.1, p.49] with $m=p=1$ and $q\to 1^-$).

TABLE 9.2: The p-Deformed extended Basic simplest inverse pairs-II

$$F(n)=\sum q^{mk(mk+1)/2-mnk}B_{n,k}\ G(k)\ ;\ G(n)=\sum(-1)^{mn+k}q^{k(k-1)/2}A_{n,k}\ F(k)$$

No.	G(n)	F(n)	$A_{n,k}$	$B_{n,k}$
1.	$\frac{a_n}{(q^p;q)_{n,p}}$	$\frac{b_n}{(q^p;q)_{n,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$
2.	$\frac{a_n}{(q^p;q)_{r+n,p}}$	$\frac{b_n}{(q^p;q)_{r+n,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$
3.	$\frac{a_n}{(q^p;q)_{n,p}\,(q^p;q)_{n-1,p}}$	$\frac{b_n}{(q^p;q)_{n,p}\,(q^p;q)_{n-1,p}}$	$\frac{1}{(q;q)_{n-mk}}$	$\frac{1}{(q;q)_{mn-k}}$

9.3 Basic analogues of p-deformed Riordan's inverse pairs

Theorem - 9.1.2 and Theorem - 9.1.3 provide q-analogue of the p-deformed Riordan inverse pairs of Table 8.1 to Table 8.6 with the help of the following alternative inverse pairs of these two theorems. We begin with Theorem - 9.1.3 with the substitution $\gamma=1$, replacement of $F(n)/(q^{p+\alpha-nr};q)_{\infty,p}$ by $F(n)$ and $G(n)$ by $G(n)$ to obtain

∗ **Inverse pair 9.3.a**

$$F(n)=\sum_{k=0}^{M}\frac{G(n+bk)}{(q;q)_k}\ \Leftrightarrow\ G(n)=\sum_{k=0}^{M}(-1)^k q^{k(k-1)/2}\frac{F(n+bk)}{(q;q)_k}.$$

Next, in Theorem - 9.1.2, we replace r by $-r$ and take $q^{br-p}=q_6$, $br\neq p$, to obtain

$$\begin{aligned} F(n) &= \sum_{k=0}^{M}\gamma^k\frac{(q^{p+\alpha+nr+brk-kp};q)_{\infty,p}}{(q_6;q_6)_k}\,G(n+bk) && (9.3.1)\\ &\Leftrightarrow\\ G(n) &= \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(1-q^{\alpha+nr+brk})}{(q^{\alpha+nr+kp};q)_{\infty,p}(q_6;q_6)_k}\,F(n+bk). && (9.3.2)\end{aligned}$$

Here replacing $G(n)$ by $G(n)/(q^{p+\alpha+rn};q)_{\infty,p}$, we get

$$\begin{aligned} F(n) &= \sum_{k=0}^{M}\gamma^k\frac{(q^{p+\alpha+nr+brk-kp};q)_{\infty,p}}{(q^{p+\alpha+rn+brk};q)_{\infty,p}(q_6;q_6)_k}\,G(n+bk) && (9.3.3)\\ &\Leftrightarrow\\ \frac{G(n)}{(q^{p+\alpha+rn};q)_{\infty,p}} &= \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(1-q^{\alpha+nr+brk})}{(q^{\alpha+nr+kp};q)_{\infty,p}(q_6;q_6)_k}\,F(n+bk). && (9.3.4)\end{aligned}$$

This may be simplified to the form:

$$\begin{aligned} F(n) &= \sum_{k=0}^{M}\gamma^k\frac{(q^{p+\alpha+nr+brk-kp};q)_{\infty,p}}{(q^{p+\alpha+rn+brk};q)_{\infty,p}(q_6;q_6)_k}\,G(n+bk)\\ &\Leftrightarrow\\ G(n) &= \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(1-q^{\alpha+nr+brk})(q^{p+\alpha+rn};q)_{\infty,p}}{(1-q^{\alpha+nr+kp})(q^{p+\alpha+nr+kp};q)_{\infty,p}(q_6;q_6)_k}\,F(n+bk).\end{aligned}$$

Alternatively,

$$\begin{aligned} F(n) &= \sum_{k=0}^{M}\gamma^k\frac{1}{(q^{p+\alpha+nr+brk};q)_{-k,p}(q_6;q_6)_k}\,G(n+bk) && (9.3.5)\\ &\Leftrightarrow\\ G(n) &= \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(1-q^{\alpha+nr+brk})(q^{p+\alpha+rn};q)_{k,p}}{(1-q^{\alpha+nr+kp})(q_6;q_6)_k}\,F(n+bk). && (9.3.6)\end{aligned}$$

From this, we obtain

∗ **Inverse pair 9.3.b**

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(q^p;q)_{\frac{\alpha+nr+brk}{p},p}}{(q^p;q)_{\frac{\alpha+nr+brk-kp}{p},p}(q_6;q_6)_k} G(n+bk)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_6^{k(k-1)/2} \frac{(1-q^{\alpha+nr+brk})(q^p;q)_{\frac{\alpha+rn+kp}{p},p}}{(1-q^{\alpha+nr+kp})(q^p;q)_{\frac{\alpha+rn}{p},p}(q_6;q_6)_k} F(n+bk).$$

Now in (9.3.5) and (9.3.6), replacing $F(n)$ by $F(n)/(1-q^{\alpha+rn})$ and $G(n)$ by $G(n)/(1-q^{\alpha+rn})$, we get

$$\frac{F(n)}{(1-q^{\alpha+rn})} = \sum_{k=0}^{M} \gamma^k \frac{1}{(q^{p+\alpha+nr+brk};q)_{-k,p}(q_6;q_6)_k} \frac{G(n+bk)}{(1-q^{\alpha+rn+rbk})}$$

$$\Leftrightarrow$$

$$\frac{G(n)}{(1-q^{\alpha+rn})} = \sum_{k=0}^{M} (-\gamma)^k q_6^{k(k-1)/2} \frac{(1-q^{\alpha+nr+brk})(q^{p+\alpha+rn};q)_{k,p}}{(1-q^{\alpha+nr+kp})(q_6;q_6)_k} \frac{F(n+bk)}{(1-q^{\alpha+rn+rbk})},$$

that is,

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(1-q^{\alpha+rn})}{(q^{p+\alpha+nr+brk};q)_{-k,p}(q_6;q_6)_k} \frac{G(n+bk)}{(1-q^{\alpha+rn+rbk})}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_6^{k(k-1)/2} \frac{(1-q^{\alpha+rn})(q^{p+\alpha+rn};q)_{k,p}}{(1-q^{\alpha+nr+kp})(q_6;q_6)_k} F(n+bk),$$

that is,

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(1-q^{\alpha+rn})}{(q^{\alpha+nr+brk};q)_{-k+1,p}(q_6;q_6)_k} G(n+bk)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_6^{k(k-1)/2} \frac{(q^{\alpha+rn};q)_{k+1,p}}{(1-q^{\alpha+nr+kp})(q_6;q_6)_k} F(n+bk).$$

This pair may be written as

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(1-q^{\alpha+rn})}{(1-q^{\alpha+nr+brk-kp})(q^{\alpha+nr+brk};q)_{-k,p}(q_6;q_6)_k} G(n+bk) \tag{9.3.7}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(q^{\alpha+rn};q)_{k,p}}{(q_6;q_6)_k}\ F(n+bk). \tag{9.3.8}$$

And further,

$$F(n) = \sum_{k=0}^{M}\gamma^k \frac{(1-q^{\alpha+rn})(q^p;q)_{\frac{\alpha+nr+brk-p}{p},p}}{(1-q^{\alpha+nr+brk-kp})(q^p;q)_{\frac{\alpha+nr+brk-p-kp}{p},p}(q_6;q_6)_k}\ G(n+bk) \tag{9.3.9}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(q^p;q)_{\frac{\alpha+rn-p+kp}{p},p}}{(q^p;q)_{\frac{\alpha+rn-p}{p},p}}\ F(n+bk). \tag{9.3.10}$$

In this last pair, replacing α by $\alpha+p$, gives

∗ **Inverse pair 9.3.c**

$$F(n) = \sum_{k=0}^{M}\gamma^k \frac{(1-q^{\alpha+rn+p})(q^p;q)_{\frac{\alpha+nr+brk}{p},p}}{(1-q^{\alpha+nr+brk-kp+p})(q^p;q)_{\frac{\alpha+nr+brk-kp}{p},p}(q_6;q_6)_k}\ G(n+bk)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M}(-\gamma)^k q_6^{k(k-1)/2}\frac{(q^p;q)_{\frac{\alpha+rn+kp}{p},p}}{(q^p;q)_{\frac{\alpha+rn}{p},p}(q_6;q_6)_k}\ F(n+bk).$$

In Theorem - 9.1.2, inverting the base q, we get

$$F(n) = \sum_{k=0}^{M}\gamma^k \frac{(q^{-p-\alpha+nr+brk+kp};q^{-1})_{\infty,p}}{(q_7;q_7)_k}\ G(n+bk) \tag{9.3.11}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M}(-\gamma)^k q_7^{k(k-1)/2}\frac{(1-q^{-\alpha+nr+brk})}{(q^{-\alpha+nr-kp};q^{-1})_{\infty,p}(q_7;q_7)_k}\ F(n+bk), \tag{9.3.12}$$

where $q_7 = q^{p+br}$ and $-br \neq p$. In this inverse pair, replacing $G(n)$ by $G(n)/(q^{-\alpha+rn};q^{-1})_{\infty,p}$ and $F(n)$ by $F(n)/(1-q^{-\alpha+rn})$, we get

$$\frac{F(n)}{(1-q^{-\alpha+rn})} = \sum_{k=0}^{M}\gamma^k \frac{(q^{-p-\alpha+nr+brk+kp};q^{-1})_{\infty,p}}{(q^{-\alpha+rn+rbk};q^{-1})_{\infty,p}(q_7;q_7)_k}\ G(n+bk)$$

$$\Leftrightarrow$$

$$\frac{G(n)}{(q^{-\alpha+rn};q^{-1})_{\infty,p}} = \sum_{k=0}^{M}(-\gamma)^k q_7^{k(k-1)/2}\frac{(1-q^{-\alpha+nr+brk})\ F(n+bk)}{(1-q^{-\alpha+rn+rbk})(q^{-\alpha+nr-kp};q^{-1})_{\infty,p}(q_7;q_7)_k}.$$

Its simplified form is given by

$$\begin{aligned} F(n) &= \sum_{k=0}^{M} \gamma^k \frac{(1-q^{-\alpha+rn})(q^{-p-\alpha+nr+brk+kp};q^{-1})_{\infty,p}}{(q^{-\alpha+rn+rbk};q^{-1})_{\infty,p}(q_7;q_7)_k} G(n+bk) \\ \Leftrightarrow & \\ G(n) &= \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(q^{-\alpha+rn};q^{-1})_{\infty,p}}{(q^{-\alpha+nr-kp};q^{-1})_{\infty,p}(q_7;q_7)_k} F(n+bk). \end{aligned}$$

Further,

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(1-q^{-\alpha+rn})(q^{-\alpha+nr+brk+kp};q^{-1})_{\infty,p}}{(1-q^{-\alpha+nr+brk+kp})(q^{-\alpha+rn+rbk};q^{-1})_{\infty,p}(q_7;q_7)_k} G(n+bk) \tag{9.3.13}$$

$\Leftrightarrow$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(q^{-\alpha+rn};q^{-1})_{\infty,p}}{(q^{-\alpha+nr-kp};q^{-1})_{\infty,p}(q_7;q_7)_k} F(n+bk). \tag{9.3.14}$$

Now, for $k \in \mathbb{N} \cup \{0\}$, we have

$$\frac{(q^{-\alpha+nr+brk+kp};q^{-1})_{\infty,p}}{(q^{-\alpha+rn+rbk};q^{-1})_{\infty,p}} = (q^{p-\alpha+nr+brk};q)_{k,p}, \tag{9.3.15}$$

and

$$\begin{aligned} \frac{(q^{-\alpha+rn};q^{-1})_{\infty,p}}{(q^{-\alpha+nr-kp};q^{-1})_{\infty,p}} &= (1-q^{-\alpha+rn})(1-q^{-\alpha+rn-p})\cdots(1-q^{-\alpha+rn-(k-1)p}) \\ &= (-1)^k q^{-pk(k+1)/2+(-\alpha+nr)k}(q^{p+\alpha-nr-p};q)_{k,p} \\ &= \frac{1}{(q^{p-\alpha+nr};q)_{-k,p}}. \end{aligned} \tag{9.3.16}$$

In view of (9.3.15) and (9.3.16), the inverse pair (9.3.13) and (9.3.14) takes the form:

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(1-q^{-\alpha+rn})(q^{p-\alpha+nr+brk};q)_{k,p}}{(1-q^{-\alpha+nr+brk+kp})(q_7;q_7)_k} G(n+bk) \tag{9.3.17}$$

$\Leftrightarrow$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{1}{(q^{p-\alpha+nr};q)_{-k,p}(q_7;q_7)_k} F(n+bk). \tag{9.3.18}$$

Thus, we obtain

∗ **Inverse pair 9.3.d**

$$\begin{aligned} F(n) &= \sum_{k=0}^{M} \gamma^k \frac{(1-q^{-\alpha+rn})(q^p;q)_{\frac{-\alpha+nr+brk+kp}{p},p}}{(1-q^{-\alpha+nr+brk+kp})(q^p;q)_{\frac{-\alpha+nr+brk}{p},p}(q_7;q_7)_k} G(n+bk) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(q^p;q)_{\frac{-\alpha+nr}{p},p}}{(q^p;q)_{\frac{-\alpha+nr-kp}{p},p}(q_7;q_7)_k} F(n+bk). \end{aligned}$$

Next, in the pair (9.3.17) and (9.3.18), replacing $F(n)$ by $(1-q^{-\alpha+rn})F(n)$ and $G(n)$ by $(1-q^{-\alpha+rn})G(n)$, we find

$$\begin{aligned} F(n) &= \sum_{k=0}^{M} \gamma^k \frac{(q^{-\alpha+nr+brk};q)_{k+1,p}}{(1-q^{-\alpha+nr+brk+kp})(q_7;q_7)_k} G(n+bk) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(1-q^{-\alpha+rn+rbk})}{(q^{-\alpha+nr};q)_{-k+1,p}(q_7;q_7)_k} F(n+bk), \end{aligned}$$

that is,

$$F(n) = \sum_{k=0}^{M} \gamma^k \frac{(q^{-\alpha+nr+brk};q)_{k,p}}{(q_7;q_7)_k} G(n+bk) \tag{9.3.19}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(1-q^{-\alpha+rn+rbk})}{(1-q^{-\alpha+rn-kp})(q^{-\alpha+nr};q)_{-k,p}(q_7;q_7)_k} F(n+bk). \tag{9.3.20}$$

This may be put in the following form.

∗ **Inverse pair 9.3.e**

$$\begin{aligned} F(n) &= \sum_{k=0}^{M} \gamma^k \frac{(q^p;q)_{\frac{-\alpha+nr+brk-p+kp}{p},p}}{(q^p;q)_{\frac{-\alpha+nr+brk-p}{p},p}(q_7;q_7)_k} G(n+bk) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{M} (-\gamma)^k q_7^{k(k-1)/2} \frac{(1-q^{-\alpha+rn+rbk})(q^p;q)_{\frac{-\alpha+nr-p}{p},p}}{(1-q^{-\alpha+rn-kp})(q^p;q)_{\frac{-\alpha+nr-p-kp}{p},p}(q_7;q_7)_k} F(n+bk). \end{aligned}$$

The inverse pairs in 9.3.a to 9.3.e with $F(n) = A_n$ and $G(n) = B_n$ when specialized suitably, yield the p-deformed basic analogues of the inverse series relations tabulated in chapter 8. as shown in following table 9.3 to table 9.9.

TABLE 9.3: p-Deformation of the Basic simplest inverse pairs

$$A_n = \sum_{k=0}^{n} q^{k(k-1)/2} a_{n,k}\ B_k;\ B_n = \sum_{k=0}^{n} (-1)^k q^{k(k-2n-1)/2} b_{n,k}\ A_k$$

Inv. pair No.	b	$a_{n,k}$	$b_{n,k}$
9.3.a	-1	$\frac{(q;q)_{n,p}}{(q;q)_{n-k}(q;q)_{k,p}}$	$\frac{(q;q)_{n,p}}{(q;q)_{n-k}(q;q)_{k,p}}$
9.3.a	1	$\frac{q^k(q;q)_{k,p}}{(q;q)_{k-n}(q;q)_{n,p}}$	$\frac{q^k(q;q)_{k,p}}{(q;q)_{k-n}(q;q)_{n,p}}$
9.3.a	-1	$\frac{(q;q)_{\alpha-k,p}}{(q;q)_{n-k}(q;q)_{\alpha-n,p}}$	$\frac{(q;q)_{\alpha-k,p}}{(q;q)_{n-k}(q;q)_{\alpha-n,p}}$
9.3.a	-1	$\frac{(q;q)_{\alpha+n,p}}{(q;q)_{n-k}(q;q)_{\alpha+k,p}}$	$\frac{(q;q)_{\alpha+n,p}}{(q;q)_{n-k}(q;q)_{\alpha+k,p}}$
9.3.a	1	$\frac{q^k(q;q)_{\alpha+k,p}}{(q;q)_{n-k}(q;q)_{\alpha+n,p}}$	$\frac{q^k(q;q)_{\alpha+k,p}}{(q;q)_{n-k}(q;q)_{\alpha+n,p}}$
9.3.a	-1	$\frac{(q;q)_{n,p}(q;q)_{n-1,p}}{(q;q)_{n-k}(q;q)_{k-1,p}(q;q)_{k,p}}$	$\frac{(q;q)_{n,p}(q;q)_{n-1,p}}{(q;q)_{n-k}(q;q)_{k-1,p}(q;q)_{k,p}}$

TABLE 9.4: p-Deformation of the Basic Gould Classes

$$A_n = \sum_{k=0}^{n} q^{\beta k(k-1)/2}\ a_{n,k}\ B_k;\ B_n = \sum_{k=0}^{n} (-1)^{k+n} q^{\beta k(k-2n-1)/2}\ b_{n,k}\ B_k$$

Inv. pair No. with $\gamma = 1$ and $r = l - p$	b	α	β	$a_{n,k}$	$b_{n,k}$
9.3.b	-1	α	$-l$	$\frac{(q^p;q)_{\frac{\alpha+lk-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lk-np}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{q_6^{-lk}(1-q^{\alpha+lk-kp})(q^p;q)_{\frac{\alpha+nl-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+nl-np}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.c	-1	α	$-l$	$\frac{(1-q^{\alpha+p+ln-np})(q^p;q)_{\frac{\alpha+lk-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lk-np+p}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{q_6^{-lk}(q^p;q)_{\frac{\alpha+ln-kp}{p},p}}{(q^p;q)_{\frac{\alpha+nl-np}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.d	1	$-\alpha$	l	$\frac{q_7^{lk}(q^p;q)_{\frac{\alpha+nl-np}{p},p}}{(q^p;q)_{\frac{\alpha+nl-kp}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+nl-np})(q^p;q)_{\frac{\alpha+lk-np-p}{p},p}}{(q^p;q)_{\frac{\alpha+lk-kp}{p},p}(q^\beta;q^\beta)_{k-n}}$
9.3.e	1	$-\alpha-p$	l	$\frac{q_7^{lk}(1-q^{\alpha+p+lk-kp})(q^p;q)_{\frac{\alpha+ln-np}{p},p}}{(q^p;q)_{\frac{\alpha+nl-kp+p}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q^p;q)_{\frac{\alpha+lk-np}{p},p}}{(q^p;q)_{\frac{\alpha+lk-kp}{p},p}(q^\beta;q^\beta)_{k-n}}$

TABLE 9.5: p-Deformation of the Basic simpler Chebyshev Classes

$$A_n = \sum \gamma^k a_{n,k}\ B_{n+bk};\ B_n = \sum (-\gamma)^k q^{\beta k(k-1)/2}\ b_{n,k}\ A_{n+bk}$$

Inv. pair No. with $r=1$	b	α	β	γ	$a_{n,k}$	$b_{n,k}$
9.3.d	-2	0	$p-2$	-1	$\frac{(q^p;q)_{\frac{n}{p},p}}{(q^p;q)_{\frac{n-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^n)(q^p;q)_{\frac{n-2k+kp-p}{p},p}}{(q^p;q)_{\frac{n-2k}{p},p}(q^\beta;q^\beta)_k}$
9.3.e	-2	$-p$	$p-2$	-1	$\frac{(q^p;q)_{\frac{n-2k+kp}{p},p}}{(q^p;q)_{\frac{n-2k}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{p+n-2k})(q^p;q)_{\frac{n}{p},p}}{(q^p;q)_{\frac{n-kp+p}{p},p}(q^\beta;q^\beta)_k}$
9.3.b	2	0	$2-p$	1	$\frac{(q^p;q)_{\frac{n+2k}{p},p}}{(q^p;q)_{\frac{n+2k-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+2k})(q^p;q)_{\frac{n+kp-p}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$
9.3.c	2	0	$2-p$	1	$\frac{(q^p;q)_{\frac{n+kp}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+p})(q^p;q)_{\frac{n+2k}{p},p}}{(q^p;q)_{\frac{n+2k-kp+p}{p},p}(q^\beta;q^\beta)_k}$
9.3.b	-1	0	$-p-1$	1	$\frac{(q^p;q)_{\frac{n-k}{p},p}}{(q^p;q)_{\frac{n-k-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n-k})(q^p;q)_{\frac{n+kp-p}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$
9.3.c	-1	0	$-p-1$	1	$\frac{(q^p;q)_{\frac{n+kp}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{p+n})(q^p;q)_{\frac{n-k}{p},p}}{(q^p;q)_{\frac{n-k-kp+p}{p},p}(q^\beta;q^\beta)_k}$

TABLE 9.6: p-Deformation of the Basic Chebyshev Classes

$$A_n = \sum a_{n,k}\ B_{n+bk};\ B_n = \sum (-1)^k q^{\beta k(k-1)/2}\ b_{n,k}\ A_{n+bk}$$

Inv. pair No. with $r=1$ and $b=c$	α	β	$a_{n,k}$	$b_{n,k}$
9.3.d	0	$c+p$	$\frac{(q^p;q)_{\frac{n}{p},p}}{(q^p;q)_{\frac{n-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^n)(q^p;q)_{\frac{n+ck+kp-p}{p},p}}{(q^p;q)_{\frac{n+ck}{p},p}(q^\beta;q^\beta)_k}$
9.3.e	$-p$	$c+p$	$\frac{(1-q^{p+n+ck})(q^p;q)_{\frac{n}{p},p}}{(q^p;q)_{\frac{n-kp+p}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(q^p;q)_{\frac{n+ck+kp}{p},p}}{(q^p;q)_{\frac{n+ck}{p},p}(q^\beta;q^\beta)_k}$
9.3.b	0	$c-p$	$\frac{(q^p;q)_{\frac{n+ck}{p},p}}{(q^p;q)_{\frac{n+ck-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+ck})(q^p;q)_{\frac{n+kp-p}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$
9.3.c	0	$c-p$	$\frac{(1-q^{p+n})(q^p;q)_{\frac{n+ck}{p},p}}{(q^p;q)_{\frac{n+ck-kp+p}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(q^p;q)_{\frac{n+kp}{p},p}}{(q^p;q)_{\frac{n}{p},p}(q^\beta;q^\beta)_k}$

TABLE 9.7: p-Deformation of the Basic simpler Legendre classes I

$$A_n = \sum q^{\beta k(k-1)/2}\ a_{n,k}\ B_k;\ B_n = \sum (-1)^{k+n} q^{\beta k(k-2n+1)/2}\ b_{n,k}\ A_k$$

Inv. pair No. with $r=2$ and $\gamma=1$	b	α	β	$a_{n,k}$	$b_{n,k}$
9.3.e	-1	$-\alpha-p$	$p-2$	$\frac{(q^p;q)_{\frac{\alpha+2k+np-kp}{p},p}}{(q^p;q)_{\frac{\alpha+2k}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+p+2k})(q^p;q)_{\frac{\alpha+2n}{p},p}}{(q^p;q)_{\frac{\alpha+2n-np+kp+p}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.d	-1	$-\alpha$	$p-2$	$\frac{(q^p;q)_{\frac{\alpha+2n}{p},p}}{(q^p;q)_{\frac{\alpha+2n-np+kp}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+2n})(q^p;q)_{\frac{\alpha+2k+np-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+2k}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.c	1	α	$2-p$	$\frac{(q^p;q)_{\frac{\alpha+2n+kp-np}{p},p}}{(q^p;q)_{\frac{\alpha+2n}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+p+2n})(q^p;q)_{\frac{\alpha+2k}{p},p}}{(q^p;q)_{\frac{\alpha+2k-kp+np+p}{p},p}(q^\beta;q^\beta)_{k-n}}$
9.3.b	1	α	$2-p$	$\frac{(q^p;q)_{\frac{\alpha+2k}{p},p}}{(q^p;q)_{\frac{\alpha+2k-kp+np}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+2k})(q^p;q)_{\frac{\alpha+2n+kp-np-p}{p},p}}{(q^p;q)_{\frac{\alpha+2n}{p},p}(q^\beta;q^\beta)_{k-n}}$

TABLE 9.8: p-Deformation of the Basic simpler Legendre classes II

$$A_n = \sum_{k=0}^{\lfloor n/2\rfloor} a_{n,k}\ B_{n-2k};\ B_n = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^{k+n} q^{\beta k(k-1)/2}\ b_{n,k}\ B_{n-2k}$$

Inv. pair No. with $r=2$ and $\gamma=1$	b	α	β	$a_{n,k}$	$b_{n,k}$
9.3.d	-2	$-\alpha$	$p-4$	$\frac{(q^p;q)_{\frac{\alpha+2n}{p},p}}{(q^p;q)_{\frac{\alpha+2n-kp}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{\alpha+2n})(q^p;q)_{\frac{\alpha+2n-4k+kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+2n-4k}{p},p}(q^\beta;q^\beta)_k}$
9.3.e	-2	$-\alpha-1$	$p-4$	$\frac{(1-q^{\alpha+p+2n-4k})(q^p;q)_{\frac{\alpha+2n}{p},p}}{(q^p;q)_{\frac{\alpha+2n-kp+p}{p},p}(q^\beta;q^\beta)_k}$	$\frac{(q^p;q)_{\frac{\alpha+2n-4k+kp}{p},p}}{(q^p;q)_{\frac{\alpha+2n-4k}{p},p}(q^\beta;q^\beta)_k}$

TABLE 9.9: p-Deformation of the Basic Legendre-Chebyshev classes

$$A_n = \sum(\gamma)^k q^{\beta k(k-1)/2}\ a_{n,k}\ B_k;\ B_n = \sum(-\gamma)^k q^{\beta k(k-2n+1)/2}\ b_{n,k}\ B_k$$

Inv. pair No. with $r = c$	b	α	β	γ	$A_{n,k}$	$B_{n,k}$
9.3.d	-1	$-\alpha$	$p-c$	-1	$\frac{(q^p;q)_{\frac{\alpha+cn}{p},p}}{(q^p;q)_{\frac{\alpha+cn-np+kp}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+cn})(q^p;q)_{\frac{\alpha+ck+np-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+ck}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.d	1	$-\alpha$	$p+c$	-1	$\frac{(q^p;q)_{\frac{\alpha+cn}{p},p}}{(q^p;q)_{\frac{\alpha+cn-kp+np}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+cn})(q^p;q)_{\frac{\alpha+ck+kp-np-p}{p},p}}{(q^p;q)_{\frac{\alpha+ck}{p},p}(q^\beta;q^\beta)_{k-n}}$
9.3.b	-1	α	$-p-c$	1	$\frac{(q^p;q)_{\frac{\alpha+ck}{p},p}}{(q^p;q)_{\frac{\alpha+ck-np+kp}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+ck})(q^p;q)_{\frac{\alpha+cn+np-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+cn}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.b	1	α	$c-p$	1	$\frac{(q^p;q)_{\frac{\alpha+ck}{p},p}}{(q^p;q)_{\frac{\alpha+ck-kp+np}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+ck})(q^p;q)_{\frac{\alpha+cn+kp-np-p}{p},p}}{(q^p;q)_{\frac{\alpha+cn}{p},p}(q^\beta;q^\beta)_{k-n}}$
9.3.e	-1	$-\alpha-p$	$p-c$	-1	$\frac{(1-q^{\alpha+p+ck})(q^p;q)_{\frac{\alpha+cn}{p},p}}{(q^p;q)_{\frac{\alpha+cn-np+kp+p}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(q^p;q)_{\frac{\alpha+kc+np-kp}{p},p}}{(q^p;q)_{\frac{\alpha+kc}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.e	1	$-\alpha-p$	$p+c$	-1	$\frac{(1-q^{\alpha+p+ck})(q^p;q)_{\frac{\alpha+cn}{p},p}}{(q^p;q)_{\frac{\alpha+cn-kp+np+p}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q^p;q)_{\frac{\alpha+kc+kp-np}{p},p}}{(q^p;q)_{\frac{\alpha+kc}{p},p}(q^\beta;q^\beta)_{k-n}}$
9.3.c	-1	α	$-p-c$	1	$\frac{(1-q^{\alpha+p+cn})(q^p;q)_{\frac{\alpha+ck}{p},p}}{(q^p;q)_{\frac{\alpha+ck-np+kp+p}{p},p}(q^\beta;q^\beta)_{n-k}}$	$\frac{(q^p;q)_{\frac{\alpha+cn+np-kp}{p},p}}{(q^p;q)_{\frac{\alpha+cn}{p},p}(q^\beta;q^\beta)_{n-k}}$
9.3.c	1	α	$c+p$	-1	$\frac{(1-q^{\alpha+p+cn})(q^p;q)_{\frac{\alpha+ck}{p},p}}{(q^p;q)_{\frac{\alpha+ck-kp+np+p}{p},p}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q^p;q)_{\frac{\alpha+cn+kp-np}{p},p}}{(q^p;q)_{\frac{\alpha+cn}{p},p}(q^\beta;q^\beta)_{k-n}}$

When $p = 1$, these inverse pairs get reduced to those deduced in [11, Table 1 - 7].

9.4 Extension of certain q, p-deformed Riordan's inverse pairs

In this section, certain p-deformed q-Riordan inverse pair are extended with the help of alternative form stated as series (9.1.9) and (9.1.10). In these series, replacing $G(n)$ and $F(n)$ by $q^{-\lambda mn(mn+1)/2}(q^p; q)_{\alpha/p+\lambda mn-mn,p}G(n)/(q^p; q)_{\infty,p}$ and $q^{-\lambda n(n+1)/2}(-1)^n F(n)$ yield the

$*$ **Inverse pair 9.4.a**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda(mk(mk-1))/2)} \frac{(q^p; q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p; q)_{\frac{\alpha+mk\lambda-np}{p},p}(q^{-\lambda}; q^{-\lambda})_{n-mk}} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(1-q^{\alpha+k\lambda-kp})(q^p; q)_{\frac{\alpha+mn\lambda-kp-p}{p},p}}{(q^p; q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda}; q^{-\lambda})_{mn-k}} F(k). \end{aligned}$$

This inverse pair, with $F(n), G(n)$ and α replaced by $F(n)/(1-q^{\alpha+n(\lambda-p)+p}), G(n)/(1-q^{\alpha+mn(\lambda-p)+p})$ and $\alpha+p$, yields

$*$ **Inverse pair 9.4.b**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{-\lambda mk(mk-1)/2)} \frac{(1-q^{\alpha+\lambda n-np+p})(q^p; q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p; q)_{\frac{\alpha+mk\lambda-np+p}{p},p}(q^{-\lambda}; q^{-\lambda})_{n-mk}} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(q^p; q)_{\frac{\alpha+mn\lambda-kp}{p},p}}{(q^p; q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda}; q^{-\lambda})_{mn-k}} F(k). \end{aligned}$$

Here, if the base q is inverted, and then $G(n)$ is replaced by $G(n)/(q^{-\alpha-\lambda mn+mn-1}; q^{-1})_{\infty,p}$, then it transforms to the form:

$*$ **Inverse pair 9.4.c**

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{\lambda mk(mk-1)/2)} \frac{(q^p; q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p; q)_{\frac{-\alpha-mk\lambda+mkp-p}{p},p}(q^{\lambda}; q^{\lambda})_{n-mk}} G(k) \\ &\Leftrightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn+k} q^{\lambda k(k-2mn+1)/2} \frac{(1-q^{-\alpha-\lambda k+kp})(q^p; q)_{\frac{-\alpha-\lambda mn+mnp-p}{p},p}}{(q^p; q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^{\lambda}; q^{\lambda})_{mn-k}} F(k). \end{aligned}$$

Finally, replacing $F(n)$ by $F(n)/(1-q^{-\alpha-\lambda np+np})$ in this last pair, gives

$*$ **Inverse pair 9.4.d**

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{\lambda mk(mk-1))/2} \frac{(1-q^{-\alpha-\lambda n+np})(q^p;q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p;q)_{\frac{-\alpha-mk\lambda+mkp}{p},p}(q^\lambda;q^\lambda)_{n-mk}} G(k)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^{mn+k} q^{\lambda k(k-2mn+1)/2} \frac{(q^p;q)_{\frac{-\alpha-mn\lambda+mnp}{p},p}}{(q^p;q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^\lambda;q^\lambda)_{mn-k}} F(k).$$

Inverse pairs - 9.4.a to 9.4.d with $F(n) = A_n$ and $G(n) = B_n$ lead us to the extension of certain inverse pairs of the p-deformed q-Riordan classes as shown in the following table.

TABLE 9.10: p-Deformed extension of certain q, p-Riordan inverse pairs

$$A_n = \sum_{k=0}^{\lfloor n/m \rfloor} q^{\beta mk(mk-1)/2}\ a_{n,k}\ B_k;\ B_n = \sum_{k=0}^{mn} (-1)^{mn+k} q^{\beta k(k-2mn+1)/2}\ b_{n,k}\ B_k$$

Inv. pair No.	β	α	λ	$a_{n,k}$	$b_{n,k}$	p-Deformed extension of q, p-class (inverse pair no.) as in Tables 9.4, 9.7 and 9.9
9.4.a	$-l$	α	l	$\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+lmk-np}{p},p}} \times \frac{1}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(1-q^{\alpha+lk-kp})}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+lmn-kp-p}{p},p}}{(q^{-l};q^{-l})_{mn-k}}$	q, p-Gold class(1) Table 9.4
9.4.b	$-l$	α	l	$\frac{(1-q^{\alpha+ln-np+p})}{(q^p;q)_{\frac{\alpha+lmk-np+p}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+lmn-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times \frac{1}{(q^{-l};q^{-l})_{mn-k}}$	q, p-Gold class(2) Table 9.4

Table 9.10: – *Continue*

Inv. pair No.	β	α	λ	$a_{n,k}$	$b_{n,k}$	p-Deformed extension of q,p-class (inverse pair no.) as in Tables 9.4, 9.7 and 9.9
9.4.c	$p-2$	$-\alpha-p$	$p-2$	$\dfrac{(q^p;q)_{\frac{\alpha+np+2mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times \dfrac{1}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\dfrac{(1-q^{\alpha+2k+p})}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp+p}{p},p}} \times \dfrac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q,p-Simpler Legendre Class(1) Table 9.7
9.4.d	$p-2$	$-\alpha$	$p-2$	$\dfrac{(1-q^{\alpha+2n})}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times \dfrac{(q^p;q)_{\frac{\alpha+np-mkp+2mk-p}{p},p}}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\dfrac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp}{p},p}} \times \dfrac{1}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q,p-Simpler Legendre Class(2) Table 9.7
9.4.d	$p-c$	$-\alpha$	$p-c$	$\dfrac{(1-q^{\alpha+cn})}{(q^p;q)_{\frac{\alpha+cmk}{p},p}} \times \dfrac{(q^p;q)_{\frac{\alpha+np+cmk-mkp-p}{p},p}}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\dfrac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp}{p},p}} \times \dfrac{1}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	q,p-Legendre-Chebyshev Class(1) Table 9.9

Table 9.10: – *Continue*

Inv. pair No.	β	α	λ	$a_{n,k}$	$b_{n,k}$	p-Deformed extension of q,p-class (inverse pair no.) as in Tables 9.4, 9.7 and 9.9
9.4.a	$-p-c$	α	$p+c$	$\frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^p;q)_{\frac{\alpha+cmk+mkp-np}{p},p}} \times \frac{1}{(q^{-(c+p)};q^{-(c+p)})_{n-mk}}$	$\frac{(1-q^{\alpha+ck})}{(q^p;q)_{\frac{\alpha+cmn}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp-p}{p},p}}{(q^{-(c+p)};q^{-(c+p)})_{mn-k}}$	q,p-Legendre -Chebyshev Class(3) Table 9.9
9.4.c	$p-c$	$-\alpha-p$	$p-c$	$\frac{(q^p;q)_{\frac{\alpha+np+cmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+cmk}{p},p}} \times \frac{1}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\frac{(1-q^{\alpha+ck+p})}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp+p}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	q,p-Legendre-Chebyshev Class(5) Table 9.9
9.4.b	$-p-c$	α	$p+c$	$\frac{(1-q^{\alpha+cn+p})}{(q^p;q)_{\frac{\alpha+cmk+mkp-np+p}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^{(c+p)};q^{(c+p)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp}{p},p}}{(q^p;q)_{\frac{\alpha+cmn}{p},p}} \times \frac{1}{(q^{(c+p)};q^{(c+p)})_{mn-k}}$	q,p-Legendre-Chebyshev Class(7) Table 9.9