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p-DEFORMATION OF A GENERAL CLASS OF POLYNOMIALS AND ITS PROPERTIES

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Abstract. The work incorporates the extension of the Srivastava-Pathan's generalized polynomial by means of p-generalized gamma function: Γ_p and Pochhammer p-symbol $(x)_{n,p}$ due to Rafael Díaz and Eddy Pariguan [Divulgaciones Mathemáticas Vol.15, No. 2(2007), pp. 179-192]. We establish the inverse series relation of this extended polynomial with the aid of general inversion theorem. We also obtain the generating function relations and the differential equation. Certain p-deformed combinatorial identities are illustrated in the last section.

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1. Introduction

In this work, we consider the general polynomial:

$$S_n(l,m,\alpha,\beta:x)=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^{mk}\lambda_k}{\Gamma(1+\beta-n\alpha+lk)(n-mk)!}x^k \tag{1.1}$$

due to Manisha Dalbhide [1] with an objective to provide extension in the light of recently proposed one parameter deformation $\Gamma_p(x)$ of the classical gamma function $\Gamma(x)$ such that $\Gamma_p(x)$ reduces to $\Gamma(x)$ when $p=1$. This introduction is due to Rafael Díaz and Eddy Pariguan [3]. In fact, the occurrence of the product of the form $x(x+p)(x+2p)\cdots(x+(n-1)p)$ in combinatorics of creation and annihilation operators ([2], [4]) and the perturbative computation of Feynman integrals [5] led them to generalize the Gamma function in the form involving the above factors.
The p-Gamma function is given in Euler integral form as follows [3]. For

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$z \in \mathbb{C},\ \mathrm{Re}(z) > 0$ and $p > 0$,

$$\Gamma_p(z) = \int_0^\infty t^{z-1} e^{-\frac{t^p}{p}}\ dt. \tag{1.2}$$

For $z \in \mathbb{C}, p \in \mathbb{R}$ and $n \in \mathbb{N}$, the Pochhammer p-symbol is given by

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p). \tag{1.3}$$

In this notation, we also have for $z \in \mathbb{C} \setminus p\mathbb{Z}_{<0}$ and $p > 0$,

$$\Gamma_p(z) = \lim_{n\to\infty} \frac{n!\ p^n\ (np)^{\frac{z}{p}-1}}{(z)_{n,p}}. \tag{1.4}$$

The following properties follow from (1.3) and (1.4).

$$\Gamma_p(z+p) = z\Gamma_p(z), \tag{1.5}$$

$$\Gamma_p(p) = 1, \tag{1.6}$$

$$(z)_{k,p} = \frac{\Gamma_p(z+kp)}{\Gamma_p(z)}, \tag{1.7}$$

$$(z)_{n-k,p} = \frac{(-1)^k (z)_{n,p}}{(p-z-np)_{k,p}}, \tag{1.8}$$

$$(z)_{mn,p} = m^{mn} \prod_{j=1}^{m} \left(\frac{z+jp-p}{m}\right)_{n,p}. \tag{1.9}$$

When $p = 1$, these identities get reduced to the corresponding properties of the Gamma function and the Pochhammer symbol ([6], [9]). We shall make use the notation

$$\triangle_p(m;n) = \prod_{j=1}^{m} \left(\frac{n+jp-p}{m}\right). \tag{1.10}$$

For $p = 1$, this gives the usual notation $(\triangle_1(m;n) =)\triangle(m;n)$ which indicates the array of m parameters

$$\frac{n}{m}, \frac{n+p}{m}, \ldots, \frac{n+mp-p}{m}.$$

Diaz and Pariguan [3] also proposed the following generalization of the hypergeometric series in the form of Pochhammer p-symbol (cf. [6] with $p = 1$), given by

$${}_rF_s(a,k,b,l)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s} n!} x^n, \tag{1.11}$$

where $a = (a_1, a_2, \cdots, a_r) \in \mathbb{C}^r,\ k = (k_1, k_2, \cdots, k_r) \in (\mathbb{R}^+)^r,\ b = (b_1, b_2, \cdots, b_s) \in \mathbb{C}^s \setminus (k\mathbb{Z}^-)^s$ and $l = (l_1, l_2, \cdots, l_s) \in (\mathbb{R}^+)^s$. This series converges for all x if $r \leq s$, and diverges if $r > s+1,\ x \neq 0$. If $r = s+1$, then the series converges for $|x| < \frac{l_1 l_2 \cdots l_s}{k_1 k_2 \cdots k_r}$. It also satisfies the differential equation [3]:

$$[D\,(l_1 D + b_1 - l_1)\,(l_2 D + b_2 - l_2)\ \cdots\ (l_s D + b_s - l_s)$$
$$- x\,(k_1 D + a_1)\,(k_2 D + a_2)\ \cdots\ (k_r D + a_r)]y \ = \ 0, \qquad (1.12)$$

where $D = x\dfrac{d}{dx}$. For $p > 0,\ a \in \mathbb{C}$ and $|x| < \dfrac{1}{p}$, Diaz and Pariguan [3] showed that

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!} x^n = (1 - px)^{-\frac{a}{p}}. \qquad (1.13)$$

This may be regarded as the p-deformed binomial series. In the present work, we define the following p-deformation of polynomial (1.1).

Definition 1.1. For $0 \leq \alpha \leq 1,\ \beta \in \mathbb{C},\ m \in \mathbb{N}, l\ n, \alpha m \in \mathbb{N} \cup \{0\}$, and $p > 0$,

$$S_{n,p}(l, m, \alpha, \beta : x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk}\ \lambda_k\ x^k}{\Gamma_p(p + \beta - pn\alpha + plk)(n - mk)!}, \qquad (1.14)$$

in which the floor function $\lfloor r \rfloor = floor\ r$, represents the greatest integer $\leq r$.

The extended Jacobi polynomial due to H. M. Srivastava [8] occurs as a special case when $p = 1$ and

$$\lambda_n = \frac{(\alpha_1)_n \cdots (\alpha_c)_n}{(\beta_1)_n \cdots (\beta_d)_n n!}.$$

Thus, if

$$\lambda_n = \frac{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}}{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p} n!}$$

then (1.14) yields the p-deformed extended Jacobi polynomial (or pEJP):

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m,p}[\alpha_1, \ldots, \alpha_c; \beta_1, \ldots, \beta_d : x]$$
$$= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p} \cdots (\alpha_c)_{k,p}}{(\beta + p - pn\alpha)_{lk,p}(\beta_1)_{k,p} \cdots (\beta_d)_{k,p}\ k!}\ x^k. \qquad (1.15)$$

Here if $l = 0$ or (in (1.14)) $\alpha = 0$, then we get the p-deformed Brafman polynomial (pBP):

$$B^m_{n,p}[\alpha_1, \ldots, \alpha_c; \beta_1, \ldots, \beta_d : x] \ = \ \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p} \cdots (\alpha_c)_{k,p}}{(\beta_1)_{k,p} \cdots (\beta_d)_{k,p}\ k!}\ x^k. \qquad (1.16)$$

Next, the polynomial (1.14) generalizes the well known Konhauser polynomial $Z_n^{(\alpha)}(x;l)$ when $\beta = 1,\ \alpha = 0,\ \lambda_n = 1/n!(p+\alpha)_{ln,p}$ and x is replaced by x^l; which is given by

$$Z_{n,m,p}^{(\alpha)}(x;l) \quad = \quad \frac{(p+\alpha)_{ln,p}}{n!} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{kl,p} k!} x^{kl}. \tag{1.17}$$

We call this polynomial as "extended p-deformed Konhauser polynomial" or briefly, *EpKP*. The case $l = 1$ yields the "extended p-deformed Laguerre polynomial", or *EpLP* which we denote as $L_{n,m,p}^{(\alpha)}(x)$.

2. Inverse series relation

In this section, we derive the inverse series relation for the polynomial (1.14) by establishing a general inversion theorem below.

Theorem 2.1. Let $0 \le \alpha \le 1,\ n, l, m \in \mathbb{N} \cup \{0\},$ such that αm is an integer and $\beta \in \mathbb{C} \setminus \{0\},$ then

$$G(n) \quad = \quad \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta + pmk\alpha - pn\alpha + p)(n-mk)!} F(k) \tag{2.1}$$

$$\Rightarrow$$

$$F(n) \quad = \quad \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\ \beta\ \Gamma_p(\beta + pmn\alpha - pk\alpha)}{(mn-k)!} G(k), \tag{2.2}$$

and conversely, the series in (2.2) implies the series (2.1) if for $n \neq mr,\ r \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta + pn\alpha - pk\alpha)}{(n-k)!} G(k) \quad = \quad 0. \tag{2.3}$$

Proof. We prove that (2.1) $\Rightarrow$ (2.2). For that let us denote the right hand side of (2.2) by F_n then on substituting $G(K)$ from (2.1), we have

$$
\begin{aligned}
F_n &= \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\beta\ \Gamma_p(\beta+pmn\alpha-pk\alpha)}{(mn-k)!}\ G(k) \\
&= \sum_{k=0}^{nm}(-1)^{mn-k}\frac{\beta\ \Gamma_p(\beta+pmn\alpha-pk\alpha)}{(mn-k)!} \\
&\qquad\qquad \times \sum_{j=0}^{\lfloor k/m\rfloor} \frac{1}{\Gamma_p(\beta+pmj\alpha-pk\alpha+p)(k-mj)!}\ F(j) \\
&= \sum_{j=0}^{n}\sum_{k=0}^{mn-mj} \frac{(-1)^{mn-k-mj}\beta\ \Gamma_p(\beta+pmn\alpha-p(k+mj)\alpha)}{(mn-k-mj)!\ \Gamma_p(\beta+pmj\alpha-p(k+mj)\alpha+p)\ k!}\ F(j) \\
&= F(n)+\sum_{j=0}^{n-1}\sum_{k=0}^{mn-mj} \frac{(-1)^{mn-k-mj}\beta\ \Gamma_p(\beta+pm(n-j)\alpha-pk\alpha)}{\Gamma_p(\beta-pk\alpha+p)\ (mn-mj-k)!\ k!}\ F(j) \\
&= F(n)+\sum_{j=0}^{n-1}\frac{(-1)^{mn-mj}\beta}{(mn-mj)!}\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k} \\
&\qquad\qquad \times \frac{\Gamma_p(\beta+pmn\alpha-pk\alpha-pmj\alpha)}{\Gamma_p(\beta-pk\alpha+p)}\ F(j) \\
&= F(n)+\sum_{j=0}^{n-1}\frac{(-1)^{mn-mj}\beta}{(mn-mj)!}F(j)\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\sum_{r=0}^{ln-lj-1}A_r\ k^r,
\end{aligned}
$$

where $l = m\alpha$ is an integer $< m$ for $0 \le \alpha \le 1$. Now, if $P(a+bk)$ is a polynomial in k of degree less than N then

$$
\sum_{k=0}^{N}(-1)^k\binom{N}{k}P(a+bk) = 0. \tag{2.4}
$$

Here, the inner sum in the second term vanishes in view of (2.4), giving $F_n = F(n)$.

Next, to show that (2.1) $\Rightarrow$ (2.3), we take

$$
\sum_{k=0}^{n}\frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!}\ G(k) = G_n.
$$

Then substituting from (2.1) for $G(k)$, we get

$$\begin{aligned} G_n &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!} \\ &\qquad\qquad \times \sum_{j=0}^{\lfloor k/m \rfloor} \frac{1}{\Gamma_p(\beta+pmj\alpha-pk\alpha+p)(k-mj)!}\ F(j) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor k/m \rfloor} \frac{(-1)^{n-k}\beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{\Gamma_p(\beta+pmj\alpha-pk\alpha+p)\ (n-k)!\ (k-mj)!}\ F(j) \\ &= \sum_{j=0}^{\lfloor n/m \rfloor}\sum_{k=0}^{n-mj} \frac{(-1)^{n-mj-k}\beta\ \Gamma_p(\beta+pn\alpha-pk\alpha-pmj\alpha)}{(n-mj-k)!\ \Gamma_p(\beta-pk\alpha+p)\ k!}\ F(j) \\ &= \sum_{j=0}^{\lfloor n/m \rfloor} (-1)^{n-mj}\ \frac{\beta\ F(j)}{(n-mj)!} \sum_{k=0}^{n-mj} (-1)^k \binom{n-mj}{k} \sum_{s=0}^{\lfloor n\alpha \rfloor - mj\alpha - 1} B_s\ k^s. \end{aligned}$$

Here also the inner sums express the $(n-mj)^{th}$ difference of polynomial of degree less than $n-mj$ for $0 \le \alpha \le 1$, hence it vanishes in view of (2.4); proving (2.1) implies (2.3).

We now assume (2.2), and (2.3) with $n \neq mj, j = 0, 1, \ldots,$ and show that they together imply (2.1). For that we first note the inverse series relation:

$$\Omega(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!}\ \Psi(k) \tag{2.5}$$

$$\Rightarrow$$

$$\Psi(n) = \sum_{k=0}^{n} \frac{1}{\Gamma_p(\beta+pk\alpha-pn\alpha+p)(n-k)!}\ \Omega(k). \tag{2.6}$$

If $n \neq mj,\ j = 1, 2, \ldots,$ then $\Omega(n) = 0$, whereas for $n = mj$, $\Omega(mj) = F(j)$ from (2.2). In this case, $\Psi(k) = G(k)$ and with these substitutions, (2.6) assumes the form

$$G(n) = \sum_{mk=0}^{n} \frac{1}{\Gamma_p(\beta+pmk\alpha-pn\alpha+p)(n-mk)!}\ \Omega(mk).$$

Thus the inverse pair (2.5) and (2.6) provide us the series relation:

$$F(n) = \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\ \beta\ \Gamma_p(\beta+pmn\alpha-pk\alpha)}{(mn-k)!}\ G(k)$$

$$\Rightarrow$$

$$G(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta+pmk\alpha-pn\alpha+p)(n-mk)!}\ F(k)$$

whenever (2.3) holds. This completes the converse part. □

In order to obtain the inverse series relation of (1.14), we substitute $F(n) = (-1)^{mn}\lambda_n x^n$ and $m\alpha = l$ in this theorem. We then have $G(n) = S_{n,p}(l, m, \alpha, \beta : x)$ and consequently from (2.2), we obtain

$$\lambda_n x^n = \sum_{k=0}^{nm} \frac{(-1)^k \ \beta \ \Gamma_p(\beta + pnl - pk\alpha)}{(mn-k)!} \ S_{k,p}(l, m, \alpha, \beta : x). \tag{2.7}$$

From this, the inverse series of (1.16) occurs in straightforward manner, given by

$$\begin{aligned}\frac{(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_c)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}\cdots(\beta_d)_{n,p}n!}x^n &= \sum_{k=0}^{nm} \frac{(-1)^k}{(mn-k)!k!} \\ &\quad \times B^m_{n,p}[\alpha_1, \ldots, \alpha_c; \beta_1, \ldots, \beta_d : x].\end{aligned}$$

The polynomial EpKP possesses the inverse series:

$$x^{ln} = \sum_{k=0}^{nm} \frac{(-1)^k n!(p+\alpha)_{ln,p}}{(p+\alpha)_{kl,p}(mn-k)!} \ Z^{(\alpha)}_{k,m,p}(x; l) \tag{2.8}$$

and for $l = 1$, it furnishes the inverse of EpLP (cf. [6, Eq.(2), p. 207] with p=1):

$$x^n = \sum_{k=0}^{nm} \frac{(-1)^k n!(p+\alpha)_{n,p}}{(p+\alpha)_{k,p}(mn-k)!} \ L^{(\alpha)}_{k,m,p}(x). \tag{2.9}$$

3. Generating function relations of p-deformed polynomial

By making use of the identity (2.1) of Theorem 2.1, the generating function relations (or GFR) of pEJP and EpKP will be derived here. The special cases namely, the pBP and EpLP respectively will then follow immediately. Since the Bessel function occurs in the generating function relation of Laguerre polynomial (see [6, Eq. (2), p. 201]), we define the p-deformed Bessel function, p-deformed generalized Bessel function as well as p-deformed modified Bessel function respectively, as follows (cf. [6,9] with $p = 1$).

$$J_{n,p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p+np+kp)k!}\left(\frac{x}{2}\right)^{n+2k}, \tag{3.1}$$

$$J^{\mu}_{\nu,p}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma_p(p+\nu p+kp\mu)k!}(-x)^k, \tag{3.2}$$

and

$$I_{n,p}(z) = i^{-n} J_{n,p}(iz). \tag{3.3}$$

Now from (2.1), we consider

$$\sum_{n=0}^{\infty} G(n)\ t^n = \left(\sum_{n=0}^{\infty} \frac{1}{\Gamma_p(\beta - pn\alpha + p)\ n!}\ t^n\right)\left(\sum_{k=0}^{\infty} F(k)\ t^{mk}\right), \quad (3.4)$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(\gamma)_{n,p}\ G(n)\ t^n &= \sum_{k=0}^{\infty}\frac{(\gamma)_{mk,p}}{\Gamma_p(\beta+p)}\ F(k)\ t^{mk}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(-1)^{n\alpha}(\gamma+mkp)_{n,p}(-\beta)_{n\alpha,p}}{n!}\ t^n. \quad (3.5)\end{aligned}$$

By specializing α, β and $F(k)$ appropriately, we obtain generating function relations of the aforementioned polynomials.

(I) Generating function relations of pEJP and pBP.
The substitution

$$F(n) = \frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p}\cdots(\alpha_p)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}\cdots(\beta_q)_{n,p}\ n!}\ x^n$$

in (3.4) implies

$$G(n) = \frac{\mathcal{H}^{(\alpha,\beta)}_{n,m\alpha,m,p}[\alpha_1,\alpha_2,\cdots,\alpha_p;\beta_1,\beta_2,\cdots,\beta_q:x]}{\Gamma_p(p+\beta-pn\alpha)\ n!}$$

and consequently we are led to the generating function relation:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\ \mathcal{H}^{(\alpha,\beta)}_{n,m\alpha,m,p}[\alpha_1,\alpha_2,\cdots,\alpha_p;\beta_1,\beta_2,\cdots,\beta_q:x]\ t^n\\ &= \left(\sum_{n=0}^{\infty}\frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\ t^n\right)\left(\sum_{k=0}^{\infty}\frac{(\alpha_1)_{k,p}(\alpha_2)_{k,p}\cdots(\alpha_p)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p}\cdots(\beta_q)_{k,p}k!}\ x^k\ (-t)^{mk}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\ t^n\ {}_rF_s\left((\alpha_1,\cdots,\alpha_p),p,(\beta_1,\cdots,\beta_q),p\right)(x(-t)^m)\end{aligned} \quad (3.6)$$

in view of (1.11). Here for $\alpha = 0$, the first series on the right hand side converges for $|t| < \infty$ and for $\alpha = 1$, it converges for $|t| < 1/p$. For $\alpha = 1$, the above GFR takes more elegant form:

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{H}^{(1,\beta)}_{n,m,m,p}[\alpha_1,\alpha_2,\cdots,\alpha_p;\beta_1,\beta_2,\cdots,\beta_q:x]\ (-\beta)_{n,p}\frac{(-t)^n}{n!}\\ &= (1+tp)^{\frac{\beta}{p}}\ {}_rF_s\left((\alpha_1,\cdots,\alpha_p),p,(\beta_1,\cdots,\beta_q),p\right)(x(-t)^m). \quad (3.7)\end{aligned}$$

On the other hand, the case $\alpha = 0$ (hence $l = 0$) leads us to the GFR of pBP which is given by

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{B}_{n,p}^{m}[\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q : x] \ \frac{t^n}{n!} \\ = \quad & e^t \ {}_rF_s\left((\alpha_1, \ldots, \alpha_p), p, (\beta_1, \ldots, \beta_q), p\right) \ (x(-t)^m). \end{aligned}$$

The special case $p = 1$ reduces to the generating function relation occurring in [9, Ex. 67, p.199]. Next, taking $\alpha = 0, \beta = 0$, and

$$F(n) = \frac{(-1)^{mn}(\alpha_1)_{n,p}(\alpha_2)_{n,p} \cdots (\alpha_p)_{n,p}}{(\beta_1)_{n,p}(\beta_2)_{n,p}, \cdots, (\beta_q)_{n,p} n!} x^n$$

in (3.5), we get

$$G(n) = \frac{1}{n!} \ B_{n,p}^{m}[\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q : x].$$

Hence in view of (1.13), we find yet another relation:

$$\begin{aligned} &\sum_{n=0}^{\infty} (\gamma)_{n,p} \ B_{n,p}^{m}[\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q : x] \ \frac{t^n}{n!} \\ = \quad & (1-pt)^{-\gamma/p} \sum_{k=0}^{\infty} \frac{(\gamma)_{mk,p}(\alpha_1)_{k,p}(\alpha_2)_{k,p} \cdots (\alpha_p)_{k,p}}{(\beta_1)_{k,p}(\beta_2)_{k,p} \cdots (\beta_q)_{k,p} n!} \\ & \times (1-pt)^{-mk} x^k (-t)^{mk}. \end{aligned} \tag{3.8}$$

By making use of (1.13) together with property (1.10) and (1.9) in (3.8), we get (cf. [9, Eq.(2), p.136] with p=1)

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p}^{m}[\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q : x] \ \frac{(\gamma)_{n,p} \ t^n}{n!} = (1-pt)^{-\gamma/p} \\ {}_{m+p}F_q\left(\left(\triangle_p(m;\gamma), \alpha_1, \alpha_2, \cdots, \alpha_p\right), p, (\beta_1, \beta_2, \cdots, \beta_q)\right) \left(\frac{(-mt)^m x}{(1-pt)^m}\right). \end{aligned} \tag{3.9}$$

For $m + p \leq q + 1$, this yields divergent generating function relations.

(II) Generating function relations of EpKP and EpLP.

If we take $\alpha = 0, \ \beta = 1$ and $F(n) = (-1)^{mn} x^{ln} / \Gamma_p(p + \alpha + pnl) n!$ in (3.4), then we have $G(n) = Z_{n,m,p}^{\alpha}(x; l)/(p + \alpha)_{nl,p} \Gamma_p(p + \alpha)$ which consequently leads us to the GFR:

$$\sum_{n=0}^{\infty} \frac{Z_{n,m,p}^{\alpha}(x;l)}{(p+\alpha)_{nl,p}} t^n \quad = \quad \frac{e^t \Gamma_p(p+\alpha)}{\Gamma_p(1+p)} J_{\frac{\alpha}{p},p}^{l}\left((-1)^{m+1} x^l t^m\right) \tag{3.10}$$

involving the p-deformed generalized Bessel function (3.2). The particular case $p = 1, \ m = 1$ provides us the generating function relation obtained in [9, Ex. 65, p. 198]. Yet another generating function relation is obtainable by making the substitutions $\alpha = 0$, $\beta = 1$, and $F(n) = (-1)^{mn}\ x^{ln}/\Gamma_p(p+\alpha+pnl)n!$ in (3.5). In this case, $G(n) = Z^{\alpha}_{n,m,p}(x;l)/(p+\alpha)_{nl,p}\Gamma_p(p+\alpha)$ and we find that

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p}}{(p+\alpha)_{nl,p}}\ Z^{(\alpha)}_{n,m,p}(x;l)\ t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)}$$
$$\times {}_mF_l\left(\triangle_p(m;\gamma), p, \triangle_p(l;\alpha+p), p\right)\left(\left(\frac{x}{l}\right)^l\left(\frac{-mt}{1-pt}\right)^m\right). \quad (3.11)$$

The divergent generating function relations occur for $m > l+1$. This also reduces to the generating function relation noted in [9, Ex. 66, p. 198] with $p = 1$ and $m = 1$.

As noted in Section-1, the case $l = 1$ of (1.17) is the extended p-deformed Laguerre polynomial $L^{(\alpha)}_{n,m,p}(x)$. Hence the GFRs (3.10) and (3.11) when $l = 1$, will reduce to those corresponding to the EpLP. They are however also directly deducible from (3.4) with $\alpha = 0$, $\beta = 1$, and $F(n) = (-1)^{mn}x^n/(p+\alpha)_{n,p}\ n!$ and thereby $G(n) = L^{(\alpha)}_{n,m,p}(x)/(p+\alpha)_{n,p}$. In either case, we have

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{e^t\Gamma_p(p+\alpha)}{\Gamma_p(1+p)}\ \sum_{k=0}^{\infty} \frac{(-1)^{mk}}{\Gamma_p(p+\alpha+kp)k!}(xt^m)^k. \quad (3.12)$$

Here the cases $m = 2r$ and $m = 2r+1, r \in \mathbb{N}\cup\{0\}$, are worth mentioning. If we take $m = 2r+1$, then we have the following GFR involving (3.1).

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r+1,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{e^t\ \Gamma_p(p+\alpha)\ (t^r\sqrt{xt})^{-\alpha/p}}{\Gamma_p(1+p)}\ J_{\frac{\alpha}{p},p}(2t^r\sqrt{xt}). \quad (3.13)$$

When $p = 1$ and $r = 0$, this further reduces to the GFR as obtained in [6, Eq.(2), p.201], whereas for $m = 2r$, the following GFR occurs which involves p-deformed modified Bessel function (3.3).

$$\sum_{n=0}^{\infty} \frac{L^{(\alpha)}_{n,2r,p}(x)}{(p+\alpha)_{n,p}} t^n = \frac{e^t\ \Gamma_p(p+\alpha)\ (\sqrt{x(t)^{2r}})^{-\alpha/p}}{\Gamma_p(1+p)}\ I_{\frac{\alpha}{p},p}(2t^r\sqrt{x})). \quad (3.14)$$

Similarly, if $\alpha = 0$, $\beta = 1$, and $F(n) = (-1)^{mn}x^n/\Gamma_p(p+\alpha+np)\ n!$, then $G(n) = L^{(\alpha)}_{n,m,p}(x)/(p+\alpha)_{n,p}\Gamma_p(p+\alpha)$ and (3.5) yield the relation:

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p}L^{(\alpha)}_{n,m,p}(x)}{(p+\alpha)_{n,p}}\ t^n$$
$$= \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)}\sum_{k=0}^{\infty} \frac{\prod_{j=1}^{m}\left(\frac{\gamma+jp-p}{m}\right)_{k,p}}{(p+\alpha)_{k,p}\ k!}\ x^k\left(\frac{-mt}{1-pt}\right)^{mk}. \quad (3.15)$$

Here the series on the right hand side converges for $m = 1, 2$. These cases are illustrated below.

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L_{n,1,p}^{(\alpha)}(x)}{(p+\alpha)_{n,p}}\ t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)}\, {}_1F_1(\gamma, p, \alpha+p, p) \left(\frac{-xt}{1-pt}\right), \tag{3.16}$$

and

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,p} L_{n,2,p}^{(\alpha)}(x)}{(p+\alpha)_{n,p}} t^n = \frac{(1-pt)^{-\gamma/p}}{\Gamma_p(1+p)} \times {}_2F_1\left(\triangle_p(2;\gamma), p, \alpha+p, p\right) \left(\frac{-4xt^2}{(1-pt)^2}\right), \tag{3.17}$$

where $\left|\frac{4xt^2}{(1-pt)^2}\right| < 1$. If $p = 1$ and γ is replaced by c, then (3.16) reduces to the GFR given in [6, Eq.(3), p.202]. Further, for $\gamma = p + \alpha$, it reduces to the GFR:

$$\sum_{n=0}^{\infty} L_{n,1,p}^{(\alpha)}(x) t^n = \frac{(1-pt)^{-(p+\alpha)/p}}{\Gamma_p(1+p)}\ exp\left(\frac{-xt}{1-pt}\right) \tag{3.18}$$

whose particular case $p = 1$ appears in [6, Eq.(4), p.202].
In (3.17), the substitution $\gamma = 2\alpha + p$ gives an elegant form:

$$\sum_{n=0}^{\infty} \frac{(2\alpha+p)_{n,p} L_{n,2,p}^{(\alpha)}(x)}{(p+\alpha)_{n,p}} t^n = \frac{(1-pt)^{-(2\alpha+p)/p}}{\Gamma_p(1+p)} \times \left(1 - \frac{4pxt^2}{(pt-1)^2}\right)^{-(2\alpha+p)/2p}. \tag{3.19}$$

4. Differential equation

With the help of (1.12), we derive the differential equation satisfied by the special case of (1.14) corresponding to the choice $\{1/k!\}$ of the sequence $\{\lambda_k\}$. The particular polynomial thus obtained is denoted by $R_{n,p}^m(x; l)$ whose explicit series representation is given by

$$R_{n,p}^m(x; l) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\left\{\prod_{j=1}^{m} \left(\frac{-np+jp-p}{m}\right)_{k,p}\right\}}{k! \left\{\prod_{j=1}^{l} \left(\frac{\beta - pn\alpha + jp}{l}\right)_{k,p}\right\}} \left(\frac{m^m x p^{-m}}{l^l}\right)^k$$

$$= {}_mF_l\left(\triangle_p(m; -np), p, \triangle_p(l; p+\beta-pn\alpha), p\right) \left(\frac{m^m x p^{-m}}{l^l}\right). \tag{4.1}$$

Now comparing this with (1.12), we obtain in straight forward manner, the differential equation:

$$[D\,(lpD+\beta-pn\alpha+p-lp)\,(lpD+\beta-pn\alpha+2p-lp)\cdots(lpD+\beta-pn\alpha)$$

$$-\,x\,(mD-n)\,(mD-n+1)\cdots(mD-n+m-1)]\,R^m_{n,p}(x;l)=0. \qquad (4.2)$$

This may be further reduced to the differential equations satisfied by pEJP and EpKP by specializing the parameters appropriately.

5. Combinatorial identities

It is interesting to note that Theorem 2.1 provides us p-deformed inverse series of certain combinatorial identities studied by John Riordan [7]. It may be noted here that the p-deformation of $n!$ is deducible from (1.3) by putting $z=p$ or $z=1$. In the former case, we have $(p)_{n,p}=p^n\ n!$ whereas in the later case, $(1)_{n,p}=p^n\ (1/p)_n$. In both the cases, $p=1$ yields $(1)_{n,1}=n!$
Now, if we take $\alpha=0,\ \beta=p$ and $m=1$, then choosing $F(n),\ G(n)$ appropriately, we obtain from (2.1) and (2.2) of Theorem 2.1, the inverse pairs corresponding to the choice $z=1$ in (1.3) which are tabulated below. (cf. [7, Ch.2, Table 2.1, p.49] with $p=1$).

Table-1 p-deformed Simplest Inverse pairs-I

$$F(n)=\sum A_{n,k}\ G(k);\ \ G(n)=\sum(-1)^{n+k}B_{n,k}\ F(k)$$

No.	F(n)	p G(n)	$A_{n,k}$	$B_{n,k}$
1.	$\frac{a_n}{(1)_{n,p}}$	$\frac{b_n}{(1)_{n,p}}$	$\frac{(1)_{n,p}}{(n-k)!\,(1)_{k,p}}$	$\frac{(1)_{n,p}}{(n-k)!\,(1)_{k,p}}$
2.	$\frac{a_n}{(1)_{r+n,p}}$	$\frac{b_n}{(1)_{r+n,p}}$	$\frac{(1)_{r+n,p}}{(n-k)!\,(1)_{r+k,p}}$	$\frac{(1)_{r+n,p}}{(n-k)!\,(1)_{r+k,p}}$
3.	$\frac{a_n}{(1)_{n,p}\,(1)_{n-1,p}}$	$\frac{b_n}{(1)_{n,p}\,(1)_{n-1,p}}$	$\frac{(1)_{n,p}\,(1)_{n-1,p}}{(n-k)!\,(1)_{k-1,p}\,(1)_{k,p}}$	$\frac{(1)_{n,p}\,(1)_{n-1,p}}{(n-k)!\,(1)_{k-1,p}\,(1)_{k,p}}$

Further, in [7] the following combinatorial identities are derived in the form of inverse pairs [7, Problem 4, 5, p. 71, 72, 74].

$$\frac{2^{-2n}}{n!\ n!}=\sum_{k=0}^{2n}\frac{(-1)^k\ 2^{-k}}{(2n-k)!\ k!}\binom{2k}{k}\ ;\ \frac{2^{-n}}{n!}\binom{2n}{n}=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{2^{-2k}}{(n-2k)!\ k!\ k!}, \qquad (5.1)$$

$$a_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}b_{2k}\ ;\ b_{2n}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}a_k, \qquad (5.2)$$

and

$$\beta_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!\ k!\ k!}\ ; \frac{1}{n!\ n!} = \sum_{k=0}^{2n} \frac{(-1)^k \beta_k}{(2n-k)!\ k!}. \tag{5.3}$$

Here, with the aid of Theorem 2.1, we obtain their p-deformed versions as follows.
When $\alpha = 0, m = 2, \beta = p\ F(n) = ((1)_{n,p}\ (1)_{n,p}\ 2^n)^{-1}$ and

$$G(n) = \frac{(1)_{2n,p}}{p\ 2^n\ (1)_{n,p}\ (1)_{n,p}\ (1)_{n,p}}$$

then the theorem yields the inverse pair:

$$\frac{2^{-2n}}{(1)_{n,p}\ (1)_{n,p}} = \sum_{k=0}^{2n} \frac{(-1)^k 2^{-k}\,(1)_{2k,p}}{(2n-k)!\,(1)_{k,p}\,(1)_{k,p}\,(1)_{k,p}}; \tag{5.4}$$

$$\frac{2^{-n}\,(1)_{2n,p}}{(1)_{n,p}\,(1)_{n,p}\,(1)_{n,p}} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{-2k}}{(n-2k)!\ (1)_{k,p}\,(1)_{k,p}}. \tag{5.5}$$

When $p = 1$, this gives (5.1). Next, by making use of the substitutions $\alpha = 0,\ m = 2,\ \beta = p,\ pG(n) = a_n/(1)_{n,p}$ and $F(n) = b_{2n}/(1)_{2n,p}$, the theorem reduces to the inversion pair:

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{n,p}}{(n-2k)!\,(1)_{2k,p}} b_{2k}\ ;\ b_{2n} = \sum_{k=0}^{2n} \frac{(-1)^k\,(1)_{2n,p}}{(2n-k)!\,(1)_{k,p}} a_k \tag{5.6}$$

which yields (5.2) when $p = 1$. Similarly, with $\alpha = 0, m = 2, \beta = p,\ pG(n) = \beta_n/(1)_{n,p}$ and $F(n) = 1/(1)_{n,p}\ (1)_{n,p}$, we obtain

$$\beta_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{n,p}}{(n-2k)!\ (1)_{k,p}\ (1)_{k,p}}\ ; \frac{1}{(1)_{n,p}\ (1)_{n,p}} = \sum_{k=0}^{2n} \frac{(-1)^k \beta_k}{(2n-k)!\ (1)_{k,p}}. \tag{5.7}$$

Again for $p = 1$, this coincides with (5.3).
Now exploiting the possibility $z = p$ in (1.3), and substituting $\alpha = 0,\ \beta = p,\ m = 1$ in (2.1) and (2.2) of the theorem, we find yet other p-deformed versions of the Simplest Inverse pairs which are tabulated below (cf. [7, Ch.2, Table 2.1, p.49] with $p = 1$).

Table-2 p-deformed Simplest Inverse pairs-II

$$F(n) = \sum A_{n,k}\ G(k);\ G(n) = \sum (-1)^{n+k} B_{n,k}\ F(k)$$

No.	F(n)	p G(n)	$A_{n,k}$	$B_{n,k}$
1.	$\dfrac{a_n}{(p)_{n,p}}$	$\dfrac{b_n}{(p)_{n,p}}$	$\dfrac{(p)_{n,p}}{(n-k)!\,(p)_{k,p}}$	$\dfrac{(p)_{n,p}}{(n-k)!\,(p)_{k,p}}$
2.	$\dfrac{a_n}{(p)_{r+n,p}}$	$\dfrac{b_n}{(p)_{r+n,p}}$	$\dfrac{(p)_{r+n,p}}{(n-k)!\,(p)_{r+k,p}}$	$\dfrac{(p)_{r+n,p}}{(n-k)!\,(p)_{r+k,p}}$
3.	$\dfrac{a_n}{(p)_{n,p}\,(p)_{n-1,p}}$	$\dfrac{b_n}{(p)_{n,p}\,(p)_{n-1,p}}$	$\dfrac{(p)_{n,p}\,(p)_{n-1,p}}{(n-k)!\,(p)_{k-1,p}\,(p)_{k,p}}$	$\dfrac{(p)_{n,p}\,(p)_{n-1,p}}{(n-k)!\,(p)_{k-1,p}\,(p)_{k,p}}$

Further, when $\alpha = 0, m = 2, \beta = p,\ F(n) = 1/2^n\ (p)_{n,p}(p)_{n,p}$ and

$$G(n) = \frac{(p)_{2n,p}}{p\ 2^n\ (p)_{n,p}\ (p)_{n,p}(p)_{n,p}},$$

then the theorem yields the inverse pair (cf. [7, Problem 4, p. 71], or (5.1) when $p = 1$):

$$\frac{2^{-2n}}{(p)_{n,p}\ (p)_{n,p}} = \sum_{k=0}^{2n} \frac{(-1)^k 2^{-k}\,(p)_{2k,p}}{(2n-k)!\,(p)_{k,p}\,(p)_{k,p}\,(p)_{k,p}}; \tag{5.8}$$

$$\frac{2^{-n}\,(p)_{2n,p}}{(p)_{n,p}\,(p)_{n,p}\,(p)_{n,p}} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{2^{-2k}}{(n-2k)!\ \ (p)_{k,p}\,(p)_{k,p}}. \tag{5.9}$$

Similarly, by making use of substitutions $\alpha = 0,\ m = 2,\ \beta = p,\ pG(n) = a_n/(p)_{n,p}$ and $F(n) = b_{2n}/(p)_{2n,p}$, the theorem reduces to the pair (cf. [7, Problem 4, p. 71-72] or (5.2) with $p = 1$):

$$a_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(p)_{n,p}}{(n-2k)!\,(p)_{2k,p}} b_{2k}\ ;\ b_{2n} = \sum_{k=0}^{2n} \frac{(-1)^k\,(p)_{2n,p}}{(2n-k)!\,(p)_{k,p}} a_k. \tag{5.10}$$

Finally, with $\alpha = 0, m = 2, \beta = p, pG(n) = \beta_n/(p)_{n,p}$ and $F(n) = 1/(p)_{n,p}\,(p)_{n,p}$, we obtain (cf. [7, Problem 5(b), p. 74] or (5.3) with $p = 1$):

$$\beta_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(p)_{n,p}}{(n-2k)!\,(p)_{k,p}\ \ (p)_{k,p}};\ \frac{1}{(p)_{n,p}\,(p)_{n,p}} = \sum_{k=0}^{2n} \frac{(-1)^k \beta_k}{(2n-k)!\,(p)_{k,p}}. \tag{5.11}$$

Many such inversion pairs can be deduced by specializing the parameters $\alpha,\ \beta,\ m$ and the sequences $\{F_n\},\ \{G_n\}$ appropriately.

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A p-DEFORMED q-INVERSE PAIR AND ASSOCIATED POLYNOMIALS INCLUDING ASKEY SCHEME

Rajesh V. Savalia

Abstract. We construct a general bi-basic inverse series relation which provides extension to several q-polynomials including the Askey-Wilson polynomials and the q-Racah polynomials. We introduce a general class of polynomials suggested by this general inverse pair which would unify certain polynomials such as the q-extended Jacobi polynomials and q-Konhauser polynomials. We then emphasize on applications of the general inverse pair and obtain the generating function relations, summation formulas involving the associated polynomials and derive the p-deformation of some of the q-analogues of Riordan's classes of inverse series relations. We also illustrate the companion matrix corresponding to the general class of polynomials; this is followed by a chart showing the reducibility of the extended p-deformed Askey-Wilson polynomials as well as the extended p-deformed q-Racah polynomials.

1. Introduction

Recently, Díaz and Teruel [5] introduced two parameter deformation of the classical gamma function by means of the q,k-Pochhammer symbol which is denoted and defined by [5, Def. 4, p. 121]

$$[t]_{n,k} = \prod_{j=0}^{n-1} [t+jk]_q, \quad t>0,\ k>0, \tag{1}$$

where $[a]_q = 1-q^a$. Using this, the q,k-generalized gamma function was defined in the form [5, Def. 6, p. 122]:

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ k>0,$$

where $(1+x)_{q,k}^t = \frac{(1+x)_{q,k}^\infty}{(1+xq^{kt})_{q,k}^\infty}$ and $(x+y)_{q,k}^n = \prod_{j=0}^{n-1}(x+yq^{jk})$.

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Alternatively [5, Lem-2, p. 122],

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)^{\infty}_{q,k}}{\left(1-q^t\right)^{\infty}_{q,k}\left(1-q\right)^{\frac{t}{k}-1}}, \quad t>0,\ k>0. \tag{2}$$

As $q \to 1^-$ from within the interval $(0,1)$, the defining expressions in (1) and (2) yield the k-generalized Pochhammer symbol $(t)_{n,k}$ and the k-deformed classical gamma function $\Gamma_k(t)$ ([5, p. 119] and [4]).

Having motivated by the works of R. Diaz and C. Teruel [5], and R. Diaz and E. Pariguan [4], we provide here the extension to certain classical q-polynomials in the sense of k-deformation and derive their inverse series relation. Further, we obtain the generating function relations of these polynomials; and using the inverse series, we deduce certain summation formulas involving the corresponding polynomials.

2. Notations, formulas and definitions

We replace in the present work, k by p, and write $(q^t;q)_{n,p}$ in stead of $[t]_{n,k}$, where $t \in \mathbb{C}$. In the notations of (1) and (2), we have

$$(q^t;q)_{n,p} = (1-q^t)(1-q^{t+p})(1-q^{t+2p})\cdots(1-q^{t+(n-1)p}), \tag{3}$$

$$\Gamma_{q,p}(t) = \frac{(q^p;q)_{\infty,p}(1-q)^{1-t/p}}{(q^t;q)_{\infty,p}}, \quad \Re(t)>0,\ p>0, \tag{4}$$

where

$$(q^a;q)_{n,p} = \begin{cases} 1, & \text{if } n=0, \\ (1-a)(1-aq^p)\cdots(1-aq^{p(n-1)}), & \text{if } n \in \mathbb{Z}_{>0}, \\ \left[(1-aq^{-p})(1-aq^{-2p})\cdots(1-aq^{np})\right]^{-1}, & \text{if } n \in \mathbb{Z}_{<0}, \\ (a;q)_{\infty,p}/(aq^{np};q)_{\infty,p}, & \text{if } n \in \mathbb{C}, \end{cases}$$

and

$$(q^\alpha;q)_{\infty,p} = \prod_{n=0}^{\infty}(1-q^{\alpha+np}),\ |q|<1.$$

We shall adopt the convention that for a parameter $\alpha \in \mathbb{C}$, q^α will be denoted as α.

In what follows, the following formulas will be used in the work. For $0 < q < 1$,

$$(a;q)_{n+m,p} = (a;q)_{n,p}(aq^{np};q)_{m,p},\ m,n \in \mathbb{N}, \tag{5}$$

$$(aq^{-np};q)_{n,p} = (-1)^n a^n q^{-pn(n+1)/2}\left(\frac{q^p}{a};q\right)_{n,p}, \tag{6}$$

$$(a;q)_{n-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2-nkp}\frac{(a;q)_{n,p}}{(q^{p-np}/a;q)_{k,p}}, \tag{7}$$

$$(a;q)_{-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2} \frac{1}{(q^p/a;q)_{k,p}}. \tag{8}$$

When $p = 1$, these formulas get reduced to those listed in [6, Appendix I, pp. 233–234]. For $\lambda \neq 0$, a general q-binomial coefficient is given as

$$\begin{bmatrix} u \\ v \end{bmatrix}_\lambda = \frac{(q^\lambda;q^\lambda)_u}{(q^\lambda;q^\lambda)_v (q^\lambda;q^\lambda)_{u-v}}.$$

There are two q-exponential functions [6, Eq. (1.3.16), p. 9]:

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n}, \quad (x \in \mathbb{R}, |q| < 1), \tag{9}$$

and [6, Eq. (1.3.15), p. 9]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} x^n = \frac{1}{(x;q)_\infty}, \quad (|x| < 1, |q| < 1). \tag{10}$$

These two functions are contained in [7, Eq. (3.1), p. 1011]

$$\varepsilon_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{11}$$

where the series converges for $|x| < \frac{1}{|1-q|}$ if $|q| < 1$ and converges for every $x \in \mathbb{C}$ for $|q| > 1$ or $q = 1$. The notation

$$[n]_q! = \prod_{j=1}^{n} \frac{(1-q^j)}{(1-q)^j}$$

for $q \neq 1$ and if $q = 1$, then $[n]_q! = n!$.

In fact [7, Pr. 5.2, p. 1021], $\varepsilon_q(x) = e_q((1-q)x)$ for $|q| < 1$ and $\varepsilon_{1/q}(x) = E_q((1-q)x)$ for $0 < |q| < 1$.

The q-binomial series for $|z| < 1$ and $|q| < 1$ is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = {}_1\phi_0(a;-;q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{12}$$

the summation formula [6, Eq. (II.5), p. 236]:

$${}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}, \tag{13}$$

and the q-binomial theorem [6, Ex. 1.2(vi), p. 20]

$$\sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} z^k = (z;q)_n. \tag{14}$$

Next, Diaz and Pariguan [4] defined the following generalization of the generalized hypergeometric series.

Definition. For $(a) = (a_1, a_2, \dots, a_r) \in \mathbb{C}^r$, $(k) = (k_1, k_2, \dots, k_r) \in (\mathbb{R}^+)^r$, $(b) = (b_1, b_2, \dots, b_s) \in \mathbb{C}^s \setminus (k\mathbb{Z}^-)^s$ and $l = (l_1, l_2, \dots, l_s) \in (\mathbb{R}^+)^s$,

$$ {}_rF_s(a,k,b,l)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s}\, n!} x^n. \tag{15}$$

This infinite series converges for all x if $r \le s$, diverges if $r > s+1$, and if $r = s+1$, it converges for $|x| < (l_1 l_2 \cdots l_s)/(k_1 k_2 \cdots k_r)$.

We define its q-analogue in the form of bi-basic series with $k_1 = k_2 = \cdots = k_r = l_1 = l_2 = \cdots = l_s = p \in \mathbb{R}^+$ as follows.

Definition. If (a) stands for the array of r parameters $a_1, a_2, \dots, a_r \in \mathbb{C}^r$, (b) stands for the array of s parameters $b_1, b_2, \dots, b_s \in C^s \setminus (Z^-)^s$, $p, \alpha \in \mathbb{R}^+$ and $|q| < 1$, then

$$\begin{aligned} &{}_r\phi_s((a);\ (b)\ ; q^p)(x|q, q^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_{n,p}(a_2;q)_{n,p}\cdots(a_r;q)_{n,p}}{(b_1;q)_{n,p}(b_2;q)_{n,p}\cdots(b_s;q)_{n,p}(q^\alpha;q^\alpha)_n} \left((-1)^n q^{\alpha\binom{n}{2}}\right)^{1+s-r} x^n. \end{aligned} \tag{16}$$

Note. The case:

$$\lim_{q\to 1^-} {}_r\phi_s((a);\ (b)\ ; q^p)\left((1-q)^{1+s-r}x \middle| q, q^\alpha\right) = {}_rF_s((a), p, (b), p)(x).$$

The series behaves similarly as the series (15). In fact, if

$$ {}_r\phi_s((a);\ (b)\ ; p)(x|q^\alpha) = \sum_{n=0}^{\infty} A_n x^n,$$

then by d' Alembert's ratio test,

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \left|\frac{(1-a_1q^{np})(1-a_2q^{np})\cdots(1-a_rq^{np})q^{\alpha n(s+1-r)}}{(1-b_1q^{np})(1-b_2q^{np})\cdots(1-b_sq^{np})(1-q^{\alpha(n+1)})}x\right|.$$

From this, it follows that the series converges for all x if $r \le s$, and it diverges when $r > s+1$ and $x \neq 0$. If $r = s+1$, then it converges for $|x| < 1$.

In the present work, we propose a general class of q,p-polynomials involving the function (4) and the symbol (3), as follows.

Definition. For $a \in \mathbb{C}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $0 < q < 1$ and $p > 0$,

$$\mathcal{B}^a_{n,m,p}(x|q;l) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m}; q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\ \gamma_k x^k, \tag{17}$$

in which $l = r - m$, $r \in \mathbb{C}\setminus\{m\}$, and the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\le u$.

This general class extends the q-extended Jacobi polynomials [2, Eq. (3.8)] and hence the q-Brafman polynomials and the little q-Jacobi polynomials [8, Eq. (3.12.1, p. 92)] (also [6, Ex. 1.32, p. 27]). As a limiting case, this general class also extends the q-Konhauser polynomials [1, Eq. (3.1), p. 3] and hence the q-Laguerre polynomials [10]. The main objective of the work is to establish a general inverse series relations (GISR) which would invert the aforesaid polynomials; and furthermore, this GISR would also extend and invert the well known orthogonal polynomials in ${}_4\phi_3$-function forms namely, the Askey-Wilson polynomials [8, Eq. (3.1.1), p. 63] (also [6, Ex. 2.11, p. 51]) and the q-Racah polynomials [8, Eq. (3.2.1), p. 66] (also [6, Ex. 2.10, p. 51]). It is interesting to note that the q-analogues of some of the Riordan's classes of inverse series relations [3] also assume extension by means of this GISR.

The GISR, as a main result, will be stated and proved in Section 3 using Lemma 3.1. Section 4 incorporates several alternative forms of GISR by means of which various particular polynomials will be deduced. In Section 5, we emphasis on applicability of both series of GISR; the one for obtaining generating function relations (GFR) in subsection 5.1 and the other, that is the inverse series, for deducing the summation formulas in Subsection 5.2. Some of the q-analogues of Riordan's inverse pairs [3] admit deformation which are tabulated in Section 6. In Section 7, the companion matrix [9] for the general class (17) is illustrated. A chart showing the reducibility of the p-deformed Askey-Wilson polynomials and the p-deformed q-Racah polynomials to a number of polynomials is given in the last section that is, in Section 8. This also includes the inter-connections amongst these particular polynomials.

3. Inverse series relations

While proving the main theorem, we shall require the following inverse pair.

Lemma 3.1. *For* $0 < q < 1$, $M \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $p > 0$,

$$(18)\quad g(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} \frac{(1 - q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+(M+mj)\lambda-kp-mjp}; q)_{\infty,p}} f(k)$$

$$\Leftrightarrow$$

$$(19)\quad f(M) = \sum_{k=0}^{M} (-1)^k q^{k\lambda(k-2M+1)/2} \begin{bmatrix} M \\ k \end{bmatrix}_{q^\lambda} (q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p}; q)_{\infty,p}\ g(k).$$

Proof. We first note that the diagonal elements of the coefficient matrix of the first series are

$$(-1)^i q^{i\lambda(i-1)/2} (1 - q^{\alpha+i\lambda+mj\lambda-ip-mjp}) / (q^{\alpha+(i+mj)\lambda-ip-mjp}; q)_{\infty,p}$$

and those of the second series are

$$(-1)^i q^{i\lambda(1-i)/2} (q^{\alpha+i\lambda+mj\lambda+p-(i+mj)p}; q)_{\infty,p}.$$

Since these elements are all non zero; it follows that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We prefer to show that (18) implies (19). For that we denote the right hand side of (19) by $\Phi(M)$ and substitute for $g(k)$ from (18) to get

$$\begin{aligned}\Phi(M) = & \sum_{k=0}^{M}(-1)^k q^{k\lambda(k-2M+1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^\lambda}\left(q^{\alpha+k\lambda+mj\lambda+p-(M+mj)p};q\right)_{\infty,p}\\ & \times\sum_{i=0}^{k}(-1)^i q^{i\lambda(i-1)/2}\begin{bmatrix}k\\i\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})}{(q^{\alpha+(k+mj)\lambda-ip-mjp};q)_{\infty,p}}f(i)\\ = & f(M)+\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\sum_{k=0}^{M-i}(-1)^k\\ & \times q^{k\lambda(k+2i-2M+1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^\lambda}\frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}}.\end{aligned}$$

Here, the ratio

$$\frac{(q^{\alpha+(k+i)\lambda+mj\lambda+p-(M+mj)p};q)_{\infty,p}}{(q^{\alpha+(k+i+mj)\lambda-ip-mjp};q)_{\infty,p}}=\sum_{l=0}^{M-i-1}A_l\ q^{\lambda kl}$$

say, represents a polynomial of degree $M-i-1$ in k, hence we further have

$$\begin{aligned}\Phi(M) = & f(M)+\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\\ & \times\sum_{l=0}^{M-i-1}A_l\sum_{k=0}^{M-i}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}M-i\\k\end{bmatrix}_{q^\lambda}q^{\lambda k(l+i-M+1)}.\end{aligned}\tag{20}$$

The inner most series on the right hand side in (20) may be summed up by means of the q-binomial theorem (14), then we have

$$\begin{aligned}\Phi(M) = & f(M)+\sum_{i=0}^{M-1}\begin{bmatrix}M\\i\end{bmatrix}_{q^\lambda}q^{i\lambda(i-M)/2}(1-q^{\alpha+i\lambda+mj\lambda-ip-mjp})f(i)\\ & \times\sum_{l=0}^{M-i-1}A_l\ (q^{\lambda(l+i-M+1)};q^\lambda)_{M-i}\\ = & f(M).\end{aligned}$$

This completes the proof. □

Interestingly, this lemma gives rise to the *q-series orthogonality relation.* In fact, the substitution $\left[\begin{smallmatrix}0\\M\end{smallmatrix}\right]_{q^\lambda}$ for either $f(M)$ or $g(M)$ yields this property. In particular, the following corollary is of our use.

Corollary 3.2. *For* $0\le j\le n, m\in\mathbb{N},\lambda\in\mathbb{C}\backslash\{0\}$ *and* $p>0$,

$$\begin{bmatrix}0\\M\end{bmatrix}_{q^\lambda}=\sum_{k=0}^{M}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}M\\k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}\tag{21}$$

$$\times\, (q^{\alpha+mj\lambda+p-kp-mjp}; q)_{\infty,p}.$$

Proof. In (18), the substitution $g(k) = \begin{bmatrix} 0 \\ k \end{bmatrix}_{q^\lambda}$ gives

$$f(k) = (q^{\alpha+mj\lambda+p-kp-mjp}; q)_{\infty,p},$$

and with these $f(k)$ and $g(k)$, (19) yields the series orthogonality relation. □

We now establish the main GISR as:

Theorem 3.3. *For* $0 < q < 1$, $\lambda \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ *and* $p > 0$,

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mk\lambda(mk-2n+1)/2} \frac{(q^{\alpha+mk\lambda+p-np}; q)_{\infty,p}}{(q^\lambda; q^\lambda)_{n-mk}} G(k) \tag{22}$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k q^{k\lambda(k-1)/2} \frac{(1 - q^{\alpha+k\lambda-kp})}{(q^\lambda; q^\lambda)_{mn-k} (q^{\alpha+mn\lambda-kp}; q)_{\infty,p}} F(k) \tag{23}$$

and for $n \neq mr$, $r \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \frac{(1 - q^{\alpha+k\lambda-kp})}{(q^\lambda; q^\lambda)_{n-k} (q^{\alpha+n\lambda-kp}; q)_{\infty,p}} F(k) = 0. \tag{24}$$

Proof. We first show that (22)⇒(23). We denote the right hand side of (23) by $V(n)$ and then substitute for $F(k)$ from (22) to get

$$\begin{aligned} V(n) = &\sum_{k=0}^{mn} (-1)^k q^{k\lambda(k-1)/2} \frac{(1 - q^{\alpha+k\lambda-kp})}{(q^\lambda; q^\lambda)_{mn-k} (q^{\alpha+mn\lambda-kp}; q)_{\infty,p}} \\ &\times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj} q^{mj\lambda(mj-2k+1)/2} \frac{(q^{\alpha+mj\lambda+p-kp}; q)_{\infty,p}}{(q^\lambda; q^\lambda)_{k-mj}} G(j). \end{aligned}$$

Here making use of the double series relation [14]:

$$\sum_{k=0}^{mn} \sum_{j=0}^{\lfloor k/m \rfloor} A(k,j) = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k+mj, j),$$

we further get

$$\begin{aligned} V(n) = &\sum_{j=0}^{n} \sum_{k=0}^{mn-mj} (-1)^k q^{(k+mj)\lambda(k+mj-1)/2 + mj\lambda(mj-2k-2mj+1)/2} \\ &\times \frac{(1 - q^{\alpha+k\lambda+mj\lambda-kp-mjp})(q^{\alpha+mj\lambda+p-kp-mjp}; q)_{\infty,p}}{(q^\lambda; q^\lambda)_{mn-mj-k} (q^{\alpha+mn\lambda-kp-mjp}; q)_{\infty,p} (q^\lambda; q^\lambda)_k} G(j) \\ = &\ G(n) + \sum_{j=0}^{n-1} \frac{G(j)}{(q^\lambda; q^\lambda)_{mn-mj}} \sum_{k=0}^{mn-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} mn-mj \\ k \end{bmatrix}_{q^\lambda} \end{aligned} \tag{25}$$

$$\times (1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})\frac{(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}.$$

We now show that the inner series in this last expression vanishes. For that we replace $(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ by $f(k)$ and denote the inner series by $g(mn-mj)$, then we have

$$g(mn-mj)=\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}mn-mj\\ k\end{bmatrix}_{q^\lambda} \times \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}}f(k). \tag{26}$$

The inverse companion of this series follows at once from Lemma 3.1 in the form:

$$f(mn-mj)=\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-2mn+2mj+1)/2}\begin{bmatrix}mn-mj\\ k\end{bmatrix}_{q^\lambda} \times (q^{\alpha+k\lambda+mj\lambda+p-mnp};q)_{\infty,p}\; g(k). \tag{27}$$

As suggested by Corollary 3.2, we set $g(k)=\begin{bmatrix}0\\ k\end{bmatrix}_{q^\lambda}$ in series (27), we then get $f(k)=(q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}$ back, and with these $f(k)$ and $g(k)$, the series orthogonality relation occurs from (26) as given below.

$$\begin{bmatrix}0\\ mn-mj\end{bmatrix}_{q^\lambda}=\sum_{k=0}^{mn-mj}(-1)^k q^{k\lambda(k-1)/2}\begin{bmatrix}mn-mj\\ k\end{bmatrix}_{q^\lambda}\frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+mn\lambda-kp-mjp};q)_{\infty,p}} \times (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}.$$

Using this in (25), we get

$$V(n)=G(n)+\sum_{j=0}^{n-1}\frac{G(j)}{(q^\lambda;q^\lambda)_{mn-mj}}\begin{bmatrix}0\\ mn-mj\end{bmatrix}_{q^\lambda}=G(n).$$

Thus, (22)⇒(23). We now show that (22)⇒(24). For that let $R(n)$ denote the right hand side of (24) that is,

$$R(n)=\sum_{k=0}^{n}(-1)^k\; q^{k\lambda(k-1)/2}\;\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}}\;F(k). \tag{28}$$

Proceeding as before, that is, substituting for $F(k)$ from (22), we have

$$R(n)=\sum_{k=0}^{n}(-1)^k\; q^{k\lambda(k-1)/2}\;\frac{(1-q^{\alpha+k\lambda-kp})}{(q^\lambda;q^\lambda)_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}} \times \sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}\; q^{mj\lambda(mj-2k+1)/2}\;\frac{(q^{\alpha+mj\lambda+p-kp};q)_{\infty,p}}{(q^\lambda;q^\lambda)_{k-mj}}\;G(j) \tag{29}$$

$$= \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^{\lambda};q^{\lambda})_{n-mj}} \sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^{\lambda}} \times \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p}.$$

We see that the inner series on the right hand side in this last expression differs slightly from the one occurring in (25); that is, instead of $mn-mj$, it is $n-mj$ here. Accordingly, the series orthogonality relation occurs in the form:

$$\sum_{k=0}^{n-mj} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n-mj \\ k \end{bmatrix}_{q^{\lambda}} \frac{(1-q^{\alpha+k\lambda+mj\lambda-kp-mjp})}{(q^{\alpha+n\lambda-kp-mjp};q)_{\infty,p}} (q^{\alpha+mj\lambda+p-kp-mjp};q)_{\infty,p} = \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^{\lambda}}.$$

This leads us to

$$R(n) = \sum_{j=0}^{\lfloor n/m \rfloor} \frac{G(j)}{(q^{\lambda};q^{\lambda})_{n-mj}} \begin{bmatrix} 0 \\ n-mj \end{bmatrix}_{q^{\lambda}}.$$

If $n \neq mr$, $r \in \mathbb{N}$, then the right hand member in (29) vanishes and thus (22)$\Rightarrow$(24); which completes the proof of the first part. For the converse part, assume that (23) and (24) both hold true. In view of (24),

$$R(n) = 0, \; n \neq mr, \; r \in \mathbb{N}, \tag{30}$$

and also,

$$R(mn) = G(n) \tag{31}$$

by comparing (23) with (28). Now, from the inverse pair (26) and (27), taking $j=0$ and $m=1$, we find that

$$R(n) = \sum_{k=0}^{n} (-1)^k \; q^{k\lambda(k-1)/2} \; \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+n\lambda-kp};q)_{\infty,p}(q^{\lambda};q^{\lambda})_{n-k}} \; F_k$$

$$\Rightarrow \quad F_n = \sum_{k=0}^{n} (-1)^k \; q^{k\lambda(k-2n+1)/2} \; \frac{(q^{\alpha+k\lambda+p-np};q)_{\infty,p}}{(q^{\lambda};q^{\lambda})_{n-k}} \; R(k).$$

Hence, in view of the relations (30) and (31), we arrive at

$$R(mn) = \sum_{k=0}^{mn} (-1)^k \; q^{k\lambda(k-1)/2} \; \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^{\lambda};q^{\lambda})_{mn-k}} \; F_k$$

$$\Rightarrow \quad F_n = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \; q^{mk\lambda(mk-2n+1)/2} \; \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^{\lambda};q^{\lambda})_{n-mk}} \; R(mk).$$

Thus, the series in (23) with $R(n)=0, n \neq mr$ for $r \in \mathbb{N}$, implies the series in (22). This proves the converse part and hence the theorem. □

4. Particular cases

In this section, we obtain the alternative forms of Theorem 3.3 by assuming that the condition (24) holds true. Hence for the sake of brevity, we *shall not* mention the condition (24) in each of the following inverse pairs. These alternative forms will be used to deduce the basic analogues of the general class of q,p-polynomials (17) and its particular cases along with their inverse series relations. Besides this, one of such alternative forms will also be used to deduce the *extended p-deformed Askey-Wilson polynomials* and the *extended p-deformed q-Racah polynomials* together with their inverse series.

We begin with Theorem 3.3 and apply the formula

$$(q^{-1};q^{-1})_n = (-1)^n q^{-n(n+1)/2}(q;q)_n,$$

in it to get

$$\begin{aligned}F(n) &= \sum_{k=0}^{\lfloor n/m\rfloor} (-1)^n q^{\lambda(2mk-n(n+1)/2)} \frac{(q^{\alpha+mk\lambda+p-np};q)_{\infty,p}}{(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k)\\ \Leftrightarrow G(n) &= \sum_{k=0}^{mn} (-1)^{mn} q^{\lambda(2kmn-mn(mn+1))/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k).\end{aligned}$$

Next, using the formula

$$(q^{-n};q)_k (q;q)_{n-k} = (-1)^k q^{k(k-2n-1)/2}(q;q)_n$$

with q is replaced by $q^{-\lambda}$ and k by mk in this pair, it transforms to

$$\begin{aligned}F(n) &= \sum_{k=0}^{\lfloor n/m\rfloor} (-1)^{n-mk} q^{-\lambda(n(n+1)-mk(mk-1)+2mnk-2mk)/2} \frac{(q^{n\lambda};q^{-\lambda})_{mk}}{(q^{-\lambda};q^{-\lambda})_n}\\ &\qquad \times (q^{\alpha+mk\lambda+p-np};q)_{\infty,p}\ G(k)\\ \Leftrightarrow G(n) &= \sum_{k=0}^{mn} (-1)^{mn-k} q^{-\lambda(mn(mn+1)-k(k-1))/2} \frac{(q^{mn\lambda};q^{-\lambda})_k (1-q^{\alpha+k\lambda-kp})}{(q^{\alpha+mn\lambda-kp};q)_{\infty,p}(q^{-\lambda};q^{-\lambda})_{mn}} F(k).\end{aligned}$$

Here replacing $F(n)$ by $q^{-\lambda n(n+1)/2}F(n)$ and $G(n)$ by

$$(q^{-\lambda(mn(mn+1)/2)}/(q^{\alpha+mn\lambda+p};q)_{\infty,p})\ G(n),$$

we obtain after little simplification, the pair:

$$\begin{aligned}F(n) &= \sum_{k=0}^{\lfloor n/m\rfloor} q^{-\lambda mnk} \frac{(q^{n\lambda};q^{-\lambda})_{mk}}{(q^{\alpha+mk\lambda+p};q)_{-n,p}(q^{-\lambda};q^{-\lambda})_n} G(k)\\ \Leftrightarrow G(n) &= \sum_{k=0}^{mn} q^{-k\lambda} \frac{(q^{mn\lambda};q^{-\lambda})_k (1-q^{\alpha+k\lambda-kp})(q^{\alpha+mn\lambda+p};q)_{-k,p}}{(q^{-\lambda};q^{-\lambda})_{mn}(1-q^{\alpha+mn\lambda-kp})} F(k).\end{aligned}$$

But since

$$\frac{q^{-\lambda mnk}}{(q^{\alpha+mk\lambda+p};q)_{-n,p}} = \frac{q^{-\lambda mnk}(q^{p-\alpha-mk\lambda-p};q)_{n,p}}{(-1)^n q^{pn(n+1)/2-(\alpha+mk\lambda+p)n}}$$

$$= (-1)^n q^{-pn(n+1)/2+(\alpha+p)n}(q^{-\alpha-mk\lambda};q)_{n,p}$$

and

$$q^{-k\lambda}(q^{\alpha+mn\lambda+p};q)_{-k,p} = \frac{(-1)^k q^{-k\lambda} q^{pk(k+1)/2-(\alpha+mn\lambda+p)k}}{(q^{p-\alpha-mn\lambda-p};q)_{k,p}}$$
$$= \frac{(-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k}}{(q^{-\alpha-mn\lambda};q)_{k,p}},$$

consequently, the above pair changes to

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^n q^{-pn(n+1)/2+(\alpha+p)n} \frac{(q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}}{(q^{-\lambda};q^{-\lambda})_n} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} (-1)^k q^{-k\lambda-mnk\lambda} q^{pk(k+1)/2-(\alpha+p)k} \frac{(q^{mn\lambda};q^{-\lambda})_k(1-q^{\alpha+k\lambda-kp})}{(q^{-\lambda};q^{-\lambda})_{mn}(1-q^{\alpha+mn\lambda-kp})}$$
$$\times \frac{F(k)}{(q^{-\alpha-mn\lambda};q)_{k,p}}.$$

Further, replacing $F(n)$ by $(-1)^n q^{-pn(n+1)/2+(\alpha+p)n} F(n)/(q^{-\lambda};q^{-\lambda})_n$, and noticing that

$$\frac{q^{-k\lambda}(1-q^{\alpha+k\lambda-kp})}{(1-q^{\alpha+mn\lambda-kp})(q^{-\alpha-mn\lambda};q)_{k,p}}$$
$$= \frac{q^{-k\lambda}q^{\alpha+k\lambda-kp}(1-q^{-\alpha-k\lambda+kp})}{q^{\alpha+mn\lambda-kp}(1-q^{-\alpha-mn\lambda+kp})(q^{-\alpha-mn\lambda};q)_{k,p}}$$
$$= \frac{(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{k+1,p}},$$

the above pair assumes the form:

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (q^{n\lambda};q^{-\lambda})_{mk}(q^{-\alpha-mk\lambda};q)_{n,p}\; G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn} \frac{q^{-mnk\lambda}(q^{mn\lambda};q^{-\lambda})_k(1-q^{-\alpha-k\lambda+kp})}{q^{mn\lambda}(q^{-\lambda};q^{-\lambda})_{mn}(q^{-\alpha-mn\lambda};q)_{k+1,p}(q^{-\lambda};q^{-\lambda})_k}\; F(k).$$

Finally, replacing

$$G(n) \text{ by } G(n)/(q^{mn\lambda}(q^{-\alpha-mn\lambda};q)_{\infty,p}), \quad F(n) \text{ by } F(n)/(q^{a+np};q)_{\infty,p}$$

and substituting $\alpha = -a$, $m\lambda = -l$, where $l = r - m$, in this last pair, we obtain

$$(32) \qquad F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p} G(k)$$

$$\Leftrightarrow$$

$$G(n) = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{nl};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} F(k). \tag{33}$$

We first deduce the inverse series of the polynomials (17). In fact, the choice $G(n) = \gamma_n x^n$ in (32) yields the polynomials (17); whereas the same substitution in (33) yields its inverse series:

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} \mathcal{B}^a_{k,m,p}(x|q;l). \tag{34}$$

Next, regarding $l \in \mathbb{C}$, and putting $a = e$ and

$$\gamma_n = (q^{\alpha_1};q)_{n,p} \cdots (q^{\alpha_c};q)_{n,p}/((q^{\beta_1};q)_{n,p} \cdots (q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (17) and (34) provides the basic analogue of the p-deformed extended Jacobi polynomials $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$ and its inverse series:

$$\begin{aligned} &\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(q^{e+np};q)_{\frac{kl}{p},p}(q^{\alpha_1};q)_{k,p}\cdots(q^{\alpha_c};q)_{k,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(q^{l/m};q^{l/m})_k} x^k \end{aligned} \tag{35}$$

$$\Leftrightarrow$$

$$\begin{aligned} &\frac{(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n} x^n \\ &= \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{e+Lk+kp})}{(q^{l/m};q^{l/m})_{mn}(q^{e+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_k} \mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q], \end{aligned} \tag{36}$$

where (α) indicates the array of c parameters $\alpha_1, \alpha_2, \ldots, \alpha_c$ and (β) indicates the array of d parameters $\beta_1, \beta_2, \ldots, \beta_d$. Here the limit $q^e \to 0$ leads us to the bi-basic p-deformed q-Brafman polynomials and its inverse series as follows.

$$\begin{aligned} &B^m_{n,p}[(\alpha);(\beta):xq^l|q] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{k,p}\cdots(q^{\beta_d};q)_{k,p}(q^{l/m};q^{l/m})_k} x^k \\ \Leftrightarrow\ &\frac{(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}}{(q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_n} x^n \\ &= \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} B^m_{k,p}[(\alpha);(\beta):q^l x|q]. \end{aligned}$$

the bi-basic p-deformed q-Brafman polynomials and its inverse series tends to the p-deformed Brafman polynomial [13, Eq. (1.16), p. 228] and its inverse series relation [13, p. 232] as $q \to 1$ with $l = m$. The extended p-deformed little q-Jacobi polynomials (cf. [6, p. 27] with $m = l = p = 1$) and its inverse series may be deduced from (17) and (34) by replacing a by $a+b+p$ and taking $\gamma_n = 1/((aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$ which are stated below.

$$\begin{aligned}
&p_{n,m,p,l}(x;a,b;q)\\
&= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}(abq^{np+p};q)_{\frac{kl}{p},p}}{(aq^p;q)_{k,p}(q^{l/m};q^{l/m})_k}\, x^k\\
\Leftrightarrow\ & \frac{x^n}{(aq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}\\
&= \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{k(l/m)+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} p_{k,m,p,l}(x;a,b;q).
\end{aligned}$$

Next, in (17) and (34), making the limit $q^a \to 0$, putting

$$\gamma_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/(p\alpha;q)_{nl,p}(q^l;q^l)_{mn},$$

and replacing l and x by lm and $(xq^n)^l$ respectively, lead us to the inverse pair of the extended p-deformed q-Konhauser polynomial (cf. [1] with $p=1$ and $m=1$) and its inverse series:

$$Z^{(\alpha)}_{n,m,p}(x;l|q) = \frac{(p\alpha;q)_{nl,p}}{(q^l;q^l)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{q^{kl(\alpha+n+1)+kl(kl-1)/2}\ (q^{-nl};q^l)_{mk}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}}\, x^{kl} \tag{37}$$

$\Leftrightarrow$

$$\frac{q^{ln(\alpha+1)-lmn+ln(ln-1)/2}}{(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}}\, x^{ln} = \sum_{k=0}^{mn} \frac{(-1)^k q^{kl(kl-1)/2}\ Z^{(\alpha)}_{k,m,p}(x;l|q)}{(\alpha q^p;q)_{kl,p}(q^l;q^l)_{mn-k}}. \tag{38}$$

The series in (37) and (38) provide basic analogues of the extended p-deformed Konhauser polynomial [13, Eq. (1.17), p. 229] and its inverse series [13, Eq. (2.8), p. 232]. The instance $l=1$ is the pair of inverse series relations involving the extended p-deformed q-Laguerre polynomials (cf. [10] with $m=p=1$):

$$\begin{aligned}
L^{(\alpha)}_{n,m,p}(x|q) &= \frac{(p\alpha;q)_{n,p}}{(q;q)_n} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-n};q)_{mk}\ q^{k(\alpha+n+1)+k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mk}}\, x^k\\
\Leftrightarrow \frac{q^{n(\alpha+1)-mn+n(n-1)/2}}{(p\alpha;q)_{n,p}(q;q)_{mn}}\, x^n &= \sum_{k=0}^{mn} \frac{(-1)^k q^{k(k-1)/2}}{(p\alpha;q)_{k,p}(q;q)_{mn-k}}\ L^{(\alpha)}_{k,m,p}(x|q).
\end{aligned}$$

It is noteworthy that the inverse pair (32) and (33) provides the extension to the Askey-Wilson polynomials and q-Racah polynomials to which we call the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials; and denote them by

$$p_{n,l,m,p}(\cos\theta;a,b,c,d\,|\,q) \text{ and } R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d\,|\,q),$$

respectively. These polynomials may be deduced from (32) and (33) as follows. First replacing a by $a+b+c+d-p$ and then choosing

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n),$$
$$F(n) = p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p})$$

yield the pair:

$$\begin{aligned}(39)\qquad &\frac{p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}\\ &= \sum_{k=0}^{\lfloor n/m\rfloor} q^{kl}\ \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k}\\ &\quad\times \frac{(abcdq^{np-p};q)_{kl/p,p}\ (ae^{i\theta};q)_{k,p}\ (ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}}\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}(40)\qquad &\frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}\\ &= \sum_{k=0}^{mn} q^{nkl}\ \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k}\\ &\quad\times \frac{(1-abcdq^{kL+kp-p})\ a^k\ p_{k,l,m,p}(\cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}}.\end{aligned}$$

Likewise, in the inverse pair (32) and (33) if a is replaced by $a+b+p$ and $G(n)$ is chosen as

$$(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n),$$

then $F(n) = R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)$ yields the inverse pair:

$$\begin{aligned}(41)\qquad &R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)\\ &= \sum_{k=0}^{\lfloor n/m\rfloor} q^{kl}\ \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k}\frac{(abq^{np+p};q)_{\frac{kl}{p},p}(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}}\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}(42)\qquad &\frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}\\ &= \sum_{k=0}^{mn} q^{nkl}\frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k}\frac{(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{ln/p+1,p}(q^{l/m};q^{l/m})_{mn}}\\ &\quad\times R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q),\end{aligned}$$

These p-deformed q-polynomials provide p-extension to a number of particular q-polynomials (see [8, pp. 61–62] for complete reducibility chart and [8, Ch. 3]). They include the q-Hahn, dual q-Hahn, continuous q-Hahn, continuous dual

q-Hahn, Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials together with their inverse series relations.

5. Application

We now apply the first series of Theorem 3.3 to derive the generating function relations for the particular q-polynomials; and then apply the second series that is, the inverse series to obtain the summation formulas involving these q-polynomials.

5.1. Generating function relations

The generating function relations for the general class of q-polynomials (17), the extended p-deformed Askey-Wilson polynomials and the extended p-deformed q-Racah polynomials will be derived with the help of the alternative form (32) as follows.

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor} q^{ln(n-1)/2m}\frac{(q^{-ln/m};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_n}\ (a;q)_{n+\frac{kl}{p},p}\ q^{klp}G(k)\ t^n.\end{aligned}$$

Now, in formula (7), replacing k by mk and taking $p=1$, it changes to

$$(q^{l/m};q^{l/m})_{n-mk} = (-1)^{mk}q^{lk(mk+1)/2-lk-lnk}\ \frac{(q^{l/m};q^{l/m})_n}{(q^{-ln/m};q^{l/m})_{mk}},$$

from which we have

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}q^{ln(n-1)/2m+lk(mk-1)/2-lnk}\frac{(a;q)_{n+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_{n-mk}}\ q^{kl}G(k)\ t^n.\end{aligned}$$

Here the double sum may be replaced by means of the identity [14, Eq. (5), p. 101] (also [11, Eq. (7), p. 57] for $m=2$):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk),$$

to get

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{ln(n-1)/2m}\frac{(a;q)_{n+mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_n}\ G(k)\ t^{n+mk}.\end{aligned} \tag{43}$$

From this, we deduce the GFR of the q-polynomials with the assumption that $|t| < 1$.

(i) **GFR of $\mathcal{B}^a_{n,m,p}(x|q;l)$.**
In (43), the choice $G(n) = \gamma_n x^n \Rightarrow F(n) = \mathcal{B}^a_{n,m,p}(x|q;l)$ which yields a general generating function relation:

$$(44)\quad \sum_{n=0}^{\infty} q^{ln(n-1)/2m} \frac{(a;q)_{n,p}}{(q^{l/m};q^{l/m})_n} \mathcal{B}^a_{n,m,p}(x|q;l)\ t^n$$
$$= \sum_{k=0}^{\infty} (-1)^{mk} (a;q)_{mk+\frac{kl}{p},p}\ {}_1\phi_1\left(aq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right)\ \gamma_k\ \left((-t)^m x\right)^k,$$

in which we have used (5). This relation when further specialized appropriately, provides us the GFR of the particular polynomials which are illustrated below.

(ii) **GFR of $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$.**
Next, choosing $l \in \mathbb{C}$, $a = e$ and

$$\gamma_n = (\alpha_1;q)_{n,p}\cdots(\alpha_c;q)_{n,p}/((\beta_1;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (44), we immediately obtain the GFR

$$\sum_{n=0}^{\infty} q^{(l/m)n(n-1)/2} \mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]\ \frac{(e;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\ t^n$$
$$= \sum_{k=0}^{\infty}\ {}_1\phi_1\left(eq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right) \frac{(\alpha_1;q)_{k,p}\cdots(\alpha_c;q)_{k,p}\left((-t)^m x\right)^k}{(\beta_1;q)_{k,p}\cdots(\beta_d;q)_{k,p}(q^{l/m};q^{l/m})_n}.$$

The limiting case $e \equiv q^e \to 0$ of this yields

(iii) **GFR of $B^m_{n,p}[(\alpha);(\beta):xq^l|q]$.**
We find using (10), that

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\ B^m_{n,p}[(\alpha);(\beta):xq^l|q]\ \frac{t^n}{(q^{l/m};q^{l/m})_n}$$
$$= \varepsilon_{q^{-l/m}}(t)\ {}_c\phi_d((\alpha);\ (\beta)\ ;p)\left(x(-t)^m q^{-l}x\middle|q,q^{l/m}\right),$$

wherein $c = d+1$ for convergence as well as the validity of the function notation.

(iv) **GFR of $p_{n,m,p,l}(x;a,b;q)$.**
In (44), we replace a by $a+b+p$ and substitute $\gamma_n = 1/((ap;q)_{n,p}(q^{l/m};q^{l/m})_n)$ to get

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2m} \frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n} p_{n,m,p,l}(x;a,b;q)\ t^n$$
$$= \sum_{k=0}^{\infty} \frac{(abq^p;q)_{mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_k}\ {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m}).$$

(v) **GFR of $Z^{(\alpha)}_{n,m,p}(x;l|q)$.**
In (44), taking limit $q^a \to 0$, replacing l and x by lm and $(xq^n)^l$, $l \in \mathbb{N}$, and putting $\gamma_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/(p\alpha;q)_{nl,p}(q^l;q^l)_{mn}$, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2}\frac{Z^{(\alpha)}_{n,m,p}(x;l|q)}{(p\alpha;q)_{nl,p}}\ t^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{nl(n-1)/2-klm}\frac{q^{kl(\alpha+n+1)-klm+kl(kl-1)/2}}{(p\alpha;q)_{kl,p}(q^l;q^l)_{mk}(q^l;q^l)_n}\ q^{mkl}x^{kl}\ t^{n+mk}.\end{aligned}$$

Here we may put $kl=s$ to get elegant form:

$$\sum_{n=0}^{\infty} q^{ln(n-1)/2}\frac{Z^{(\alpha)}_{n,m,p}(x;l|q)}{(p\alpha;q)_{nl,p}}t^n = \varepsilon_{q^{-l}}(t)\sum_{s=0}^{\infty}\frac{q^{(\alpha+n+1)s+s(s-1)/2-ms}}{(p\alpha;q)_{s,p}(q^l;q^l)_{ms}}x^s(-t)^{ms/l}.$$

(vi) **GFR of $L^{(\alpha)}_{n,m,p}(x|q)$.**
It is the straightforward case $l=1$ the GFR 5.

The GFR of the polynomials (39) and (41) follow from (43). Here also we assume that $|t|<1$. Now, if a is replaced by $a+b+c+d-p$ and if

$$G(n) = (ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n),$$

then $F(n) = p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n/((ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p})$ leads us to

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}(abcdq^{-p};q)_{n,p}\frac{p_{n,l,m,p}(\cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}\ t^n\\ &=\sum_{k=0}^{\infty}\frac{(abcdq^{-p};q)_{mk+\frac{kl}{p},p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_k}\\ &\quad\times {}_1\phi_1\left(abcdq^{mkp+kl-p};0;p\right)(t|q,q^{l/m})(-t)^{mk}.\end{aligned}$$

Likewise, replacing a by $a+b+p$ and choosing

$$G(n) = (q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}/((aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n)$$

in (43) implies $F(n) = R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)$. We then find the following GFR.

$$\begin{aligned}&\sum_{n=0}^{\infty} q^{ln(n-1)/2m}\frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n}R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)\ t^n\\ &=\sum_{k=0}^{\infty}\frac{(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}(abq^p;q)_{mk+\frac{kl}{p},p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}(q^{l/m};q^{l/m})_k}\\ &\quad\times {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m})\ (-t)^{mk}.\end{aligned}$$

5.2. Summation formulas

In this section, we illustrate the application of the inverse series of the GISR and in particular the inverse series of the general class (17), to deduce certain summation formulas. The inverse series (23) of Theorem 3.3 provides the sums involving the p-Askey-Wilson polynomials (39) and p-q-Racah polynomials (41); whereas the inverse series (34) takes care of the sums involving the other polynomials.

We begin with the inverse series (34) with the assumption that $\gamma_n \neq 0, \forall n = 0, 1, 2, \ldots$, then we have

$$\frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = x^n. \tag{45}$$

In this, multiplying both sides by $(a;q)_n/(q;q)_n$ and taking summation from $n=0$ to ∞ and then using (12) with $|x|<1$, we get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l)$$
$$= \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

If $x=0$, then $\mathcal{B}^a_{k,m,p}(0|q;l) = \gamma_0$ simplifies this sum to the form:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n\gamma_0}{(q;q)_n\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = 1.$$

Next, multiplying both sides by $1/[n]_q! = 1/(q;q)_n$ and taking summation from $n=0$ to ∞, in (45) provides

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{\gamma_n(q;q)_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = \varepsilon_q(x),$$

using (11). Taking summation $n=0$ to ∞ and assuming $|x|<1$ in (45) yields

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) = \frac{1}{1-x}.$$

By assigning different values to x from $(-1,1)$, a number of particular summation formulas can be derived. For example, $x=1/2$ in this formula gives the following one.

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}\left(\frac{1}{2}\Big|q;l\right) = 2.$$

The sum of ${}_1\phi_1[*]$ in (13) enables us to obtain one more summation formula by multiplying $\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{(c;q)_n(q;q)_n}$ to both sides of (45), replacing x by c/a and then summing-up from $n=0$ to ∞. We then obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{\gamma_n(c;q)_n(q;q)_n}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}\mathcal{B}^a_{k,m,p}\left(\frac{c}{a}\middle|q;l\right)$$
$$=\frac{(c/a;q)_\infty}{(c;q)_\infty}.$$

The reducibility to all these summation formulas to the particular polynomials may be obtained by making the substitutions as specified in Section 4.

Illustration. Taking $a=e$ and

$$\gamma_n=(q^{\alpha_1};q)_{n,p}\cdots(q^{\alpha_c};q)_{n,p}/((q^{\beta_1};q)_{n,p}\cdots(q^{\beta_d};q)_{n,p}(q^{l/m};q^{l/m})_{n,p})$$

in (45) yields the summation formula involving the p-deformed extended q-Jacobi polynomials as follows.

$$\frac{(\beta_1;q)_{n,p}\cdots(\beta_d;q)_{n,p}(q^{l/m};q^{l/m})_{n,p}}{(\alpha_1;q)_{n,p}\cdots(\alpha_c;q)_{n,p}}\sum_{k=0}^{mn}q^{kln}\frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_{mn}}$$
$$\times\frac{(1-q^{e+Lk+kp})}{(q^{e+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_k}\mathcal{F}^{(e)}_{k,m,p,l}[(\alpha);(\beta):x|q]=x^n.$$

We now obtain summation formulas involving the extended p-deformed Askey-Wilson polynomials and p-deformed extended q-Racah polynomials. While illustrating the sums, we require the deformed versions of the q-Gauss sum [6, Eq. (1.5.1), p. 10] and corresponding q-Vandermonde's sum [6, Eq. (1.5.2), p. 11] :

$${}_2\phi_1\left(a,b;c;q,\frac{c}{ab}\right)=\frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty}$$

and

$${}_2\phi_1\left(q^{-n},b;c;q,\frac{cq^n}{b}\right)=\frac{(c/b;q)_n}{(c;q)_n},$$

respectively. For that we notice the relation $(a;q)_{n,p}=(a;q^p)_n$, $p>0$, thereby transform these sums to the forms:

$$(46)\qquad {}_2\phi_1\left(a,b;c;q^p,\frac{c}{ab}\right)=\frac{(c/a;q)_{\infty,p}\,(c/b;q)_{\infty,p}}{(c;q)_{\infty,p}\,(c/ab;q)_{\infty,p}}$$

and

$$(47)\qquad {}_2\phi_1\left(q^{-np},b;c;q^p,\frac{cq^{np}}{b}\right)=\frac{(c/b;q)_{n,p}}{(c;q)_{n,p}}.$$

We rewrite the inverse series (40) by introducing $(q^p;q)_{n,p}$ to get

$$(48)\qquad \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$

$$\times \frac{a^k p_{k,l,m,p}(\cos\theta; a, b, c, d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = \frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}.$$

We intend to use (46), and for that we multiply both side of (48) by

$$q^{n(b-a-2\cos\theta)}$$

and then take sum from $n = 0$ to ∞, then after little simplification, we find

$$\sum_{n=0}^{\infty} \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta; a, b, c, d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} q^{n(b-a-2\cos\theta)}$$
$$= \frac{\left(be^{-i\theta};q\right)_{\infty,p}\left(be^{i\theta};q\right)_{\infty,p}}{(ab;q)_{\infty,p}\left(q^{b-a-2\cos\theta};q\right)_{\infty,p}}.$$

In (48), we transfer $(ae^{-i\theta};q)_{n,p}$ to the other side to get

$$\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta; a, b, c, d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} = \frac{(ae^{i\theta};q)_{n,p}}{(ab;q)_{n,p}(q^p;q)_{n,p}}.$$

In this, multiplying both sides by $(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n$ and then taking the sum from $n = 0$ to j, then using (47) on the right hand side, we obtain

$$\sum_{n=0}^{j} \frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$
$$\times \frac{a^k p_{k,l,m,p}(\cos\theta; a, b, c, d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n = \frac{\left(be^{-i\theta};q\right)_{j,p}}{(ab;q)_{j,p}}.$$

We proceed in a similar manner to derive summation formulas from the inverse series (42). We rewrite it by introducing the factor $(q^p;q)_{n,p}$ and transfer the factors $(cdq^{x+p};q)_{n,p}$, $(bdq^p;q)_{n,p}$, $(cq^p;q)_{n,p}$ and $(q^{l/m};q^{l/m})_n$ to the other side to get

$$\frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}}$$
$$\times \frac{R_{k,m,p,l}(q^{-x}+cdq^{x+1}; a, b, c, d|q)}{(q^{l/m};q^{l/m})_k} = \frac{(q^{-x};q)_{n,p}}{(aq^p;q)_{n,p}(q^p;q)_{n,p}}.$$

Now multiplying both sides by $(q^{-jp};q)_{n,p}(axq^{jp+p})^n$ and then taking the summation from $n = o$ to j, we obtain

$$\sum_{n=0}^{j} \frac{(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}} \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}}$$

$$\times \frac{(q^{-jp};q)_{n,p}(axq^{jp+p})^n}{(q^{l/m};q^{l/m})_k} R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q) = \frac{(aq^{x+p};q)_{j,p}}{(aq^p;q)_{j,p}}.$$

6. Extension of Riordan's q-inverse pairs

Apart from yielding the extended p-deformed q-polynomials, Theorem 3.3 and its alternative forms also provide an effective tool for carrying out the extension of certain inverse series relations belonging to q-Riordan's classification [3, Tables 2, 5, 7, pp. 17–20] in the sense of p-deformation (also see [12] for *ordinary* forms). For instance, replacing $G(n)$ and $F(n)$ by

$$q^{-\lambda mn(mn+1)/2}(q^p;q)_{\alpha/p+\lambda mn-mn,p}G(n)/(q^p;q)_{\infty,p} \text{ and } q^{-\lambda n(n+1)/2}(-1)^n F(n)$$

in (32) and (33), yield the
* Inverse pair - 1

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor} q^{-\lambda(mk(mk-1))/2} \frac{(q^p;q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+mk\lambda-np}{p},p}(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn}(-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(1-q^{\alpha+k\lambda-kp})(q^p;q)_{\frac{\alpha+mn\lambda-kp-p}{p},p}}{(q^p;q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k).$$

Next, replacing α by $\alpha+p, F(n)$ by $F(n)/(1-q^{\alpha+\lambda n-np+p})$ and $G(n)$ by $G(n)/(1-q^{\alpha+\lambda mn-mnp+p})$ in inverse pair - 1, yields
* Inverse pair - 2

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor} q^{-\lambda mk(mk-1)/2} \frac{(1-q^{\alpha+\lambda n-np+p})(q^p;q)_{\frac{\alpha+\lambda mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+mk\lambda-np+p}{p},p}(q^{-\lambda};q^{-\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn}(-1)^{mn+k} q^{-\lambda k(k-2mn+1)/2} \frac{(q^p;q)_{\frac{\alpha+mn\lambda-kp}{p},p}}{(q^p;q)_{\frac{\alpha+\lambda mn-mnp}{p},p}(q^{-\lambda};q^{-\lambda})_{mn-k}} F(k).$$

Here inverting the base q, and then replacing $G(n)$ by

$$G(n)/(q^{-\alpha-\lambda mn+mn-1};q^{-1})_{\infty,p},$$

this pair transforms to the
* Inverse pair - 3

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor} q^{\lambda mk(mk-1)/2} \frac{(q^p;q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p;q)_{\frac{-\alpha-mk\lambda+mkp-p}{p},p}(q^{\lambda};q^{\lambda})_{n-mk}} G(k)$$
$$\Leftrightarrow G(n) = \sum_{k=0}^{mn}(-1)^{mn+k} q^{\lambda k(k-2mn+1)/2} \frac{(1-q^{-\alpha-\lambda k+kp})(q^p;q)_{\frac{-\alpha-\lambda mn+mnp-p}{p},p}}{(q^p;q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^{\lambda};q^{\lambda})_{mn-k}} F(k).$$

Finally, replacing $F(n)$ by $F(n)/(1-q^{-\alpha-\lambda np+np})$ in this last pair, we get
$*$ Inverse pair - 4

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor} q^{\lambda mk(mk-1))/2}\frac{(1-q^{-\alpha-\lambda n+np})(q^p;q)_{\frac{-\alpha+np-\lambda mk-p}{p},p}}{(q^p;q)_{\frac{-\alpha-mk\lambda+mkp}{p},p}(q^\lambda;q^\lambda)_{n-mk}}G(k)$$

$$\Leftrightarrow G(n)=\sum_{k=0}^{mn}(-1)^{mn+k}q^{\lambda k(k-2mn+1)/2}\frac{(q^p;q)_{\frac{-\alpha-mn\lambda+mnp}{p},p}}{(q^p;q)_{\frac{-\alpha-\lambda mn+kp}{p},p}(q^\lambda;q^\lambda)_{mn-k}}F(k).$$

Inverse pairs - 1 to 4 lead us to the p-deformed versions of certain inverse pairs of q-Riordan classes as shown in the following table.

TABLE 1. p-deformed extension of certain q-Riordan inverse pairs

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor} q^{\beta mk(mk-1)/2}A_{n,k}G(k);$$

$$G(n)=\sum_{k=0}^{mn}(-1)^{mn+k}q^{\beta k(k-2mn+1)/2}B_{n,k}F(k)$$

Inverse pair -	β	α	λ	$A_{n,k}$	$B_{n,k}$	p-deformed extension of q-class (inverse pair no.) as in Table 2, 5 and 7 [3]
1	$-l$	α	l	$\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+lmk-np}{p},p}} \times\frac{1}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(1-q^{\alpha+lk-kp})}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+lmn-kp-p}{p},p}}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold class(1) Table 2
2	$-l$	α	l	$\frac{(1-q^{\alpha+ln-np+p})}{(q^p;q)_{\frac{\alpha+lmk-np+p}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+lmn-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times\frac{1}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold class(2) Table 2
3	$p-2$	$-\alpha-p$	$p-2$	$\frac{(q^p;q)_{\frac{\alpha+np+2mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times\frac{1}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(1-q^{\alpha+2k+p})}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp+p}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class(1) Table 5
4	$p-2$	$-\alpha$	$p-2$	$\frac{(1-q^{\alpha+2n})}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+np-mkp+2mk-p}{p},p}}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp}{p},p}} \times\frac{1}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class(2) Table 5
4	$p-c$	$-\alpha$	$p-c$	$\frac{(1-q^{\alpha+cn})}{(q^p;q)_{\frac{\alpha+cmk}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+np+cmk-mkp-p}{p},p}}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp}{p},p}} \times\frac{1}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	q-Legendre-Chebyshev Class(1) Table 7
1	$-p-c$	α	$p+c$	$\frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^p;q)_{\frac{\alpha+cmk+mkp-np}{p},p}} \times\frac{1}{(q^{-(c+p)};q^{-(c+p)})_{n-mk}}$	$\frac{(1-q^{\alpha+ck})}{(q^p;q)_{\frac{\alpha+cmn}{p},p}} \times\frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp-p}{p},p}}{(q^{-(c+p)};q^{-(c+p)})_{mn-k}}$	q-Legendre -Chebyshev Class(3) Table 7
3	$p-c$	$-\alpha-p$	$p-c$	$\frac{(q^p;q)_{\frac{\alpha+np+cmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+cmk}{p},p}}$	$\frac{(1-q^{\alpha+ck+p})}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp+p}{p},p}}$	q-Legendre-Chebyshev

Table 1. – *Continue*

Inverse pair -	β	α	λ	$A_{n,k}$	$B_{n,k}$	p-deformed extension of q-class (inverse pair no.) as in Table 2, 5 and 7[3]
				$\times \frac{1}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\times \frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	Class(5) Table 7
2	$-p-c$	α	$p+c$	$\frac{(1-q^{\alpha+cn+p})}{(q^p;q)_{\frac{\alpha+cmk+mkp-np+p}{p},p}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp}{p},p}}{(q^p;q)_{\frac{\alpha+cmn}{p},p}}$	q-Legendre-Chebyshev
				$\times \frac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^{(c+p)};q^{(c+p)})_{n-mk}}$	$\times \frac{1}{(q^{(c+p)};q^{(c+p)})_{mn-k}}$	Class(7) Table 7

7. Companion matrix

The companion matrix of a polynomial is defined as follows [9, p. 39].

Definition. If a polynomial $f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_{j-1} x^{j-1} + x^j \in K[X], K$ is a field, then to f there is associated the $j \times j$ matrix $C(f(x))$, called the companion matrix of $f(x)$, is denoted and defined by

$$C(f(x)) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \ldots & -\delta_{j-1} \end{bmatrix}.$$

Note. The eigenvalues of the companion matrix $C(f(x))$ are precisely the zeros of $f(x)$, counting the multiplicity. The characteristic polynomial of $C(f(x))$ is therefore, $f(x)$.

We have the following [9, Prop. 1.5.14, p. 39].

Proposition. *If $f(x) \in K[X]$ is non constant and $A = C(f(x))$ is the companion matrix of $f(x)$, then $f(A) = O$, where O is a zero matrix.*

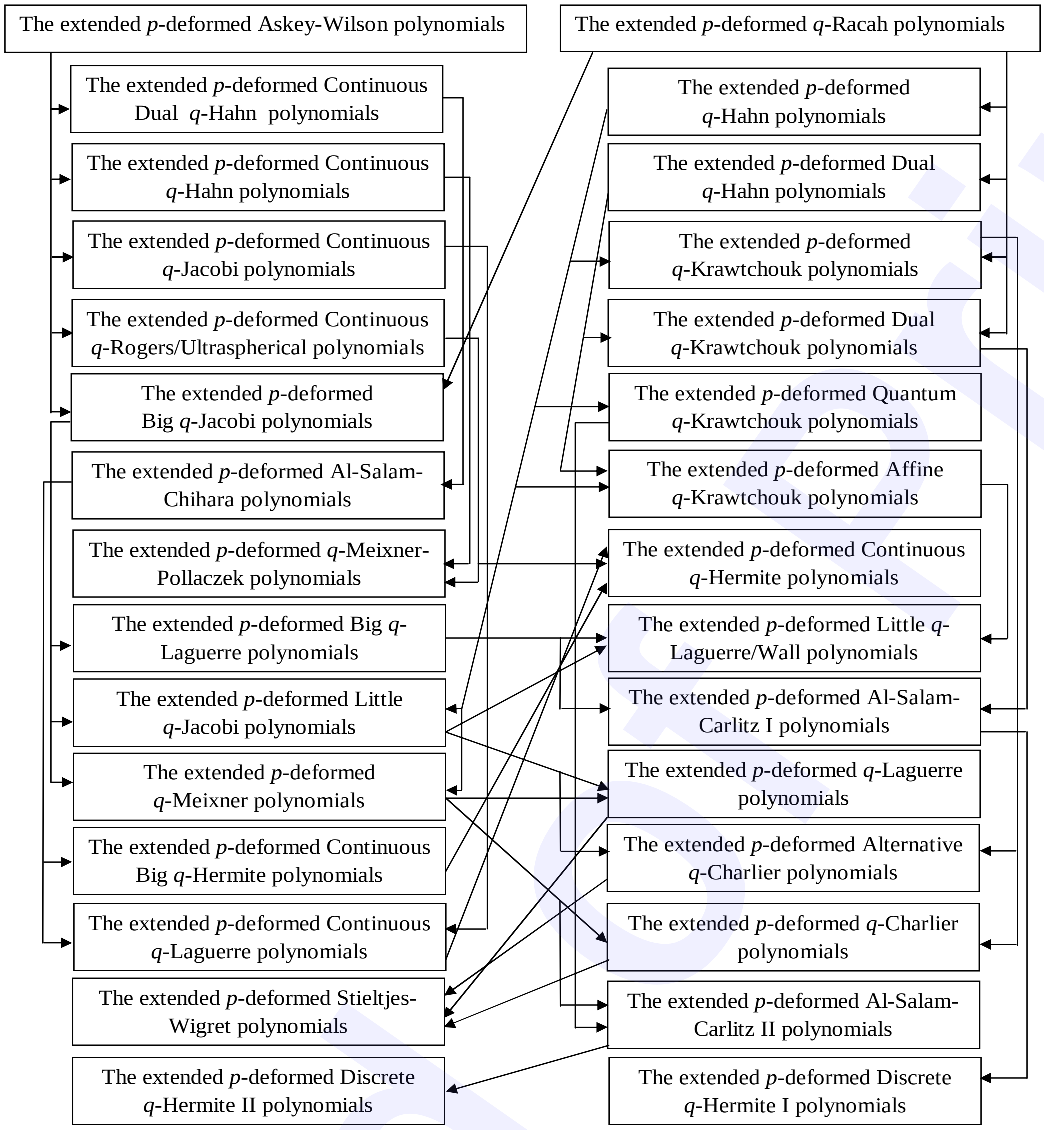
Taking $\lfloor n/m \rfloor = N$ in (17) and converting it to the monic form $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$, we get

$$\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l) = \sum_{k=0}^{N} \delta_k \; x^k,$$

where

$$\delta_k = \frac{q^{kl}(q^{-n(l/m)};q^{l/m})_{mk}(q^{a+np};q)_{\frac{kl}{p},p}\gamma_k}{q^{lN}(q^{-(l/m)n};q^{(l/m)})_n(q^{a+np};q)_{\frac{Nl}{p},p}\gamma_N}.$$

Thus, $\widetilde{B}^a_{n,m,p}(x|q;l)$ is of the form as stated in Definition 7. The eigen values of this matrix will be then precisely the zeros of the polynomial $\widetilde{\mathcal{B}}^a_{n,m,p}(x|q;l)$.



8. p-deformed polynomials' reducibility chart

Scheme of p-deformed extended basic hypergeometric orthogonal polynomials

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A GENERAL INVERSION PAIR AND p-DEFORMATION OF ASKEY SCHEME

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ABSTRACT.The present work incorporates the general inverse series relations involving p-Pochhammer symbol and p-Gamma function. A general class of p-polynomials is introduced by means of this general inverse pair which is used to derive the generating function relations and summation formulas for certain p-polynomials belonging to this general class. This includes the p-deformation of Jacobi polynomials, the Brafman polynomials and Konhauser polynomials. Moreover, the orthogonal polynomials of Racah and those of Wilson are also provided p-deformation by means of the general inversion pair. The generating function relations and summation formulas for these polynomials are also derived. We then emphasize on the combinatorial identities and obtain their p-deformed versions.

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1. INTRODUCTION

In 2007, Rafael and Pariguan [3] introduced one parameter deformation of the classical Gamma function in the form:

$$\Gamma_p(z) = \int_0^\infty t^{z-1} e^{-\frac{t^p}{p}}\, dt,$$

where $z \in \mathbb{C}$, $\Re(z) > 0$ and $p > 0$. In fact, the occurrence of the product of the form $x(x+p)(x+2p)\cdots(x+(n-1)p)$ in combinatorics of creation and annihilation operators [2, 4] and the perturbative computation of Feynman integrals [1] led them to generalize along with the Gamma function, the Pochhammer p-symbol in the form:

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p), \tag{1.1}$$

in which $z \in \mathbb{C}, p \in \mathbb{R}$ and $n \in \mathbb{N}$. These generalizations lead us to the following properties.

$$\Gamma_p(z+p) = z\Gamma_p(z),$$

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$$\begin{aligned}
\Gamma_p(p) &= 1, \\
(z)_{k,p} &= \frac{\Gamma_p(z+kp)}{\Gamma_p(z)}, \\
(z)_{n-k,p} &= \frac{(-1)^k (z)_{n,p}}{(p-z-np)_{k,p}}, \\
(z)_{mn,p} &= m^{mn} \prod_{j=1}^{m} \left(\frac{z+jp-p}{m} \right)_{n,p}.
\end{aligned} \tag{1.2}$$

When $p=1$, these identities reduce to the corresponding properties of the classical Gamma function and the Pochhammer symbol $(z)_n$ [8, 13]. In addition, we shall make use of the notation:

$$\triangle_p(m;n) = \prod_{j=1}^{m} \left(\frac{n+jp-p}{m} \right).$$

which stands for the array of m parameters:

$$\frac{n}{m}, \frac{n+p}{m}, \ldots, \frac{n+mp-p}{m}.$$

In [3], the following generalization of the hypergeometric series is introduced.

$${}_rF_s(a,k,b,l)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s} n!} x^n, \tag{1.3}$$

where $a=(a_1,a_2,\cdots,a_r)\in\mathbb{C}^r,\ k=(k_1,k_2,\cdots,k_r)\in(\mathbb{R}^+)^r,\ b=(b_1,b_2,\cdots,b_s)\in\mathbb{C}^s\setminus(k\mathbb{Z}^-)^s$ and $l=(l_1,l_2,\cdots,l_s)\in(\mathbb{R}^+)^s$. This series converges for all x if $r\le s$, and diverges if $r>s+1,\ x\neq 0$. If $r=s+1$, then the series converges for $|x|<\frac{l_1l_2\cdots l_s}{k_1k_2\cdots k_r}$. Further, for $p>0,\ a\in\mathbb{C}$ and $|x|<\dfrac{1}{p}$,

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!} x^n = (1-px)^{-\frac{a}{p}}. \tag{1.4}$$

This may be regarded as the p-deformed binomial series. Interestingly, the interval (or disk) of convergence is not the unit length interval (or unit disk) but it can be magnified or diminished by choosing p appropriately. Motivated by such remarkable feature, we make an attempt to provide deformation of certain polynomials by means of a general inverse series relation. We take up the properties such as generating function relations (GFR) and summation formulas. Moreover, we derive the deformed versions of the combinatorial identities due to John Riordan [9].

2. MAIN RESULT

While proving the main result, we shall require the following inverse pair.

Lemma 2.1. For $M \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $p > 0$,

$$\begin{aligned} g(M) &= \sum_{k=0}^{M} (-1)^k \binom{M}{k} (\alpha + k\lambda + mj\lambda - kp - mjp) \\ &\quad \times \Gamma_p(\alpha + (M+mj)\lambda - kp - mjp)\, f(k) \end{aligned} \tag{2.1}$$

if and only if

$$f(M) = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \frac{1}{\Gamma_p(\alpha + k\lambda + mj\lambda + p - (M+mj)p)}\, g(k). \tag{2.2}$$

Proof. We first note that the diagonal elements of the coefficient matrix of the first series are $(-1)^i(\alpha + i\lambda + mj\lambda - ip - mjp)\Gamma_p(\alpha + (i+mj)\lambda - kp - mjp)$ and those of the second series are

$$(-1)^i \frac{1}{\Gamma_p(\alpha + i\lambda + mj\lambda + p - (i+mj)p)}.$$

Since these elements are all non zero; it follows that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We prefer to show that (2.1) implies (2.2). For that we denote the right hand side of (2.2) by $\Phi(M)$ and substitute for $g(k)$ from (2.1) to get

$$\begin{aligned} \Phi(M) &= \sum_{k=0}^{M} (-1)^k \binom{M}{k} \frac{1}{\Gamma_p(\alpha + k\lambda + mj\lambda + p - (M+mj)p)} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \\ &\quad \times (\alpha + i\lambda + mj\lambda - ip - mjp)\Gamma_p(\alpha + (k+mj)\lambda - ip - mjp) f(i) \\ &= \sum_{i=0}^{M} \binom{M}{i} (\alpha + i\lambda + mj\lambda - ip - mjp) f(i) \sum_{k=0}^{M-i} (-1)^k \binom{M-i}{k} \\ &\quad \times \frac{\Gamma_p(\alpha + (k+i+mj)\lambda - ip - mjp)}{\Gamma_p(\alpha + (k+i)\lambda + mj\lambda + p - (M+mj)p)}. \end{aligned}$$

Here, the ratio

$$\frac{\Gamma_p(\alpha + (k+i+mj)\lambda - ip - mjp)}{\Gamma_p(\alpha + (k+i)\lambda + mj\lambda + p - (M+mj)p)} = \sum_{l=0}^{M-i-1} A_l\, k^l$$

say, which represents a polynomial of degree $M - i - 1$ in k, hence we further have

$$\begin{aligned} \Phi(M) &= f(M) + \sum_{i=0}^{M} \binom{M}{i} (\alpha + i\lambda + mj\lambda - ip - mjp) f(i) \sum_{l=0}^{M-i-1} A_l \\ &\quad \times \sum_{k=0}^{M-i} (-1)^k \binom{M-i}{k} k^l. \end{aligned}$$

Now, if $P(a + bk)$ is a polynomial in k of degree less than N, then

$$\sum_{k=0}^{N} (-1)^k \binom{N}{k} P(a + bk) = 0.$$

Thus, we get $\Phi(M) = f(M)$. This completes the proof of the inverse pair. □

This lemma gives rise to the *orthogonality relation*. In fact, the substitution $\binom{0}{M}$ for either $f(M)$ or $g(M)$ yields this property. One of the orthogonality relations which we later need, is

Corollary 2.2. For $0 \le j \le n, m \in \mathbb{N}, \lambda \in \mathbb{C}$ and $p > 0$,

$$\binom{0}{M} = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \frac{(\alpha + k\lambda + mj\lambda - kp - mjp)}{\Gamma_p(\alpha + mj\lambda + p - kp - mjp)} \Gamma_p(\alpha + mn\lambda - kp - mjp).$$

Proof. In (2.1), the substitution $g(k) = \binom{0}{k}$ gives $f(k) = 1/(\Gamma_p(\alpha + mj\lambda + p - kp - mjp))$, and with these $f(k)$ and $g(k)$, (2.2) yields the series orthogonality relation. $\square$

As a main result, we establish the general inverse series relation as

Theorem 2.3. For $\lambda \in \mathbb{C},\ \alpha \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N}$ and $p > 0$,

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} \frac{1}{\Gamma_p(\alpha + mk\lambda + p - np)(n - mk)!} G(k) \tag{2.3}$$

implies

$$G(n) = \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} F(k) \tag{2.4}$$

and conversely, the series in (2.4) implies the series (2.3) if for $n \ne mr,\ r \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + n\lambda - kp)}{(n - k)!} F(k) = 0. \tag{2.5}$$

Proof. We first show that (2.3) implies (2.4). We denote the right hand side of (2.4) by $V(n)$ and then substitute for $F(k)$ from (2.3) to get

$$\begin{aligned} V(n) &= \sum_{k=0}^{mn} (-1)^k \frac{(\alpha + k\lambda - kp)\Gamma_p(\alpha + mn\lambda - kp)}{(mn - k)!} \\ &\quad \times \sum_{j=0}^{\lfloor k/m \rfloor} (-1)^{mj} \frac{1}{\Gamma_p(\alpha + mj\lambda + p - kp)(k - mj)!} G(j). \end{aligned}$$

Here making use of the double series relation ([13] also [8, Lemma 10 and 11, p.56-57]):

$$\sum_{k=0}^{mn} \sum_{j=0}^{\lfloor k/m \rfloor} A(k, j) = \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k + mj, j),$$

we further get

$$\begin{aligned} V(n) &= \sum_{j=0}^{n-1} \frac{G(j)}{(mn - mj)!} \sum_{k=0}^{mn-mj} (-1)^k \binom{mn - mj}{k} \frac{\Gamma_p(\alpha + mn\lambda - kp - mjp)}{\Gamma_p(\alpha + mj\lambda + p - kp - mjp)} \\ &\quad \times (\alpha + (k + mj)\lambda - kp - mjp) + G(n). \end{aligned} \tag{2.6}$$

We now show that the inner series in this last expression vanishes. For that we replace $1/\Gamma_p(\alpha+mj\lambda+p-kp-mjp)$ by $f(k)$ and denote the inner series by $g(mn-mj)$, then we have

$$\begin{aligned} g(mn-mj) &= \sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}(\alpha+k\lambda+mj\lambda-kp-mjp) \\ &\quad \times\Gamma_p(\alpha+mn\lambda-kp-mjp)f(k). \end{aligned} \tag{2.7}$$

The inverse series of this series follows from Lemma 2.1 in the form:

$$f(mn-mj)=\sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{1}{\Gamma_p(\alpha+k\lambda+mj\lambda+p-mnp)}\,g(k). \tag{2.8}$$

According to Corollary 2.2, we set $g(k)=\binom{0}{k}$ in series (2.8), we then get $f(k)=1/\Gamma_p(\alpha+mj\lambda+p-kp-mjp)$ back, and with these $f(k)$ and $g(k)$, the series orthogonality relation occurs from (2.9) as given below.

$$\begin{aligned} \binom{0}{mn-mj} &= \sum_{k=0}^{mn-mj}(-1)^k\binom{mn-mj}{k}\frac{(\alpha+k\lambda+mj\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)} \\ &\quad \times\Gamma_P(\alpha+mn\lambda-kp-mjp). \end{aligned} \tag{2.9}$$

Using this in (2.6), we get

$$V(n)=G(n)+\sum_{j=0}^{n-1}\frac{G(j)}{(mn-mj)!}\binom{0}{mn-mj}=G(n).$$

Thus, (2.3) implies (2.4). Our next aim is to show that (2.3) implies (2.5). For that let $R(n)$ denote the right hand side of (2.5) that is,

$$R(n) = \sum_{k=0}^{n}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!}\,F(k). \tag{2.10}$$

Proceeding as before, that is, substituting for $F(k)$ from (2.3) in (2.10), we have

$$\begin{aligned} R(n) &= \sum_{k=0}^{n}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!} \\ &\quad \times\sum_{j=0}^{\lfloor k/m\rfloor}(-1)^{mj}\frac{1}{\Gamma_p(\alpha+mj\lambda+p-kp)(k-mj)!}\,G(j) \\ &= \sum_{j=0}^{\lfloor n/m\rfloor}\frac{G(j)}{(n-mj)!}\sum_{k=0}^{n-mj}(-1)^k\binom{n-mj}{k}\frac{\Gamma_p(\alpha+n\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}. \\ &\quad \times(\alpha+k\lambda+mj\lambda-kp-mjp). \end{aligned} \tag{2.11}$$

We see that the inner series on the right hand side in this last expression differs slightly from the one occurring in (2.6); that is, instead of $mn-mj$, it is $n-mj$ here. Accordingly, the series orthogonality relation occurs in the form:

$$\sum_{k=0}^{n-mj}(-1)^k\binom{n-mj}{k}\frac{(\alpha+k\lambda+mj\lambda-kp-mjp)}{\Gamma_p(\alpha+mj\lambda+p-kp-mjp)}\Gamma_p(\alpha+n\lambda-kp-mjp) = \binom{0}{n-mj}.$$

This leads us to $R(n)=0$.

If $n\neq mr, r\in\mathbb{N}$, then the right hand member in (2.11) vanishes and thus (2.3) implies (2.5); which completes the proof of the first part. For the converse part, assume that (2.4) and (2.5) both hold true. In view of (2.5),

$$R(n)=0,\ n\neq mr,\ r\in\mathbb{N},\ \text{and}\ R(mn)=G(n). \tag{2.12}$$

by comparing (2.4) with (2.10). Now, from the inverse pair (2.7) and (2.8), taking $j=0$ and $m=1$, we find that

$$R(n) = \sum_{k=0}^{n}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!}F_k$$

implies

$$F_n = \sum_{k=0}^{n}(-1)^k\frac{1}{\Gamma_p(\alpha+k\lambda+p-np)(n-k)!}R(k).$$

Hence, in view of the relations in (2.12), we arrive at

$$R(mn) = \sum_{k=0}^{mn}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+mn\lambda-kp)}{(mn-k)!}F_k$$

implies

$$F_n = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}\frac{1}{\Gamma_p(\alpha+mk\lambda+p-np)(n-mk)!}R(mk).$$

Thus, the series in (2.4) with $R(n)=0, n\neq mr$ for $r\in\mathbb{N}$, implies the series in (2.3). This proves the converse part and hence the theorem. □

3. ALTERNATE FORMS

The general inverse series relation of Theorem 2.3 helps us to construct a general class of p-polynomials together with their inverse series. Besides this, Theorem 2.3 also provides extension to certain inverse series relations belonging to the Riordan's classified inverse pairs appearing in Table 1 to Table 6 [9, Ch 2] in the sense of p-deformation. These inverse pairs are deduced by means of the following alternative versions of the theorem. The first alternative form is deduced by taking $\alpha=-c,\ \lambda=-(r-m)p/m$ and replacing $G(n)$ by $G(n)\Gamma_p(-c-rnp+p)$ in Theorem 2.3 to get

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{n-mk}\frac{\Gamma_p(p-c-rkp)}{\Gamma_p(p-c-rkp-np-mkp)(n-mk)!}G(k), \quad (3.1)$$

$$G(n)=\sum_{k=0}^{mn}\frac{(c+rkp/m)\Gamma_p(p-c-(r-m)np-kp)}{(c+(r-m)np+kp)\Gamma_p(p-c-(r-m)np-mnp)(mn-k)!}F(k). \quad (3.2)$$

On making property (1.2) with appropriate values of n,k and z in (3.2) leads us to

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(c+rkp)_{n-mk,p}}{(n-mk)!}G(k), \quad (3.3)$$

$$G(n) = \sum_{k=0}^{mn}\frac{(-1)^{mn-k}(c+rkp/m)(c)_{rn,p}}{(c)_{(r-m)n+k+1,p}(mn-k)!}F(k). \quad (3.4)$$

Now, if we put $r=l+m$, replace $G(n)$ by $(-1)^{mn}(c)_{ln+mn,p}G(n)$ and $F(n)$ by $F(n)(c)_{n,p}/n!$ in (3.3), (3.4) respectively, then we get the general inversion pair :
Inverse pair 1.

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor}(-n)_{mk}(c+np)_{lk,p}G(k), \quad (3.5)$$

$$G(n) = \sum_{k=0}^{mn}\frac{(-mn)_k(c+kp+lkp/m)}{(c+kp)_{ln+1,p}(mn)!k!}F(k). \quad (3.6)$$

This pair will be used to derive extended p-deformed polynomials and its inverse series relation. Now taking $\alpha=a+p$ and replacing $F(n)$ by $(-1)^nF(n)/(a+n\lambda-np+p)$, $G(n)$ by $\Gamma_p(a+mn\lambda-mnp+p)G(n)/(a+mn\lambda-mnp)$ in Theorem 2.3, one gets
inverse pair 2.

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{n-mk}\frac{(a+n\lambda-np+p)\Gamma_p(a+mk\lambda-mkp+p)}{(a+mk\lambda-np+p)\Gamma_p(a+mk\lambda-np+p)(n-mk)!}G(k),$$

$$G(n) = \sum_{k=0}^{mn}\frac{\Gamma_p(a+mn\lambda-kp+p)}{\Gamma_p(a+mn\lambda-mnp+p)(mn-k)!}F(k).$$

Next, in Theorem 2.3, replacing α by $-a-p$, $F(n)$ by $(-1)^nF(n)$ and $G(n)$ by $G(n)/\Gamma_p(a-mn\lambda+mnp+p)$, we get
inverse pair 3.

$$F(n) = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{\Gamma_p(a+np-mk\lambda+p)}{\Gamma_p(a-mk\lambda+mkp+p)(n-mk)!}G(k),$$

$$G(n) = \sum_{k=0}^{mn}(-1)^{mn-k}\frac{(a-k\lambda+kp+p)\Gamma_p(a-mn\lambda+mnp+p)}{(a-mn\lambda+kp+p)\Gamma_p(a-mn\lambda+kp+p)(mn-k)!}F(k).$$

Here, if we replace $F(n)$ by $F(n)/(a-n\lambda+np)$, $G(n)$ by $G(n)/(a-mn\lambda+mnp)$ and a by $a-p$, then we obtain
inverse pair 4.

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(a-n\lambda+np)\Gamma_p(a+np-mk\lambda+p)}{(a-mk\lambda+np)\Gamma_p(a-mk\lambda+mkp+p)(n-mk)!} G(k), \\ G(n) &= \sum_{k=0}^{mn} (-1)^{mn-k} \frac{\Gamma_p(a-mn\lambda+mnp+p)}{\Gamma_p(a-mn\lambda+kp+p)(mn-k)!} F(k). \end{aligned}$$

4. GENERAL CLASS OF P-POLYNOMIALS

We propose here a general class of polynomials which would yield as its particular cases the p-deformation of the Racah polynomials and the Wilson polynomials along with their inverse series relations. The proposed general class is defined as follows.

Definition 4.1. For $a,\ l \in \mathbb{C},\ m \in \mathbb{N},\ n \in \{0\} \cup \mathbb{N},\ x \in \mathbb{R}$ and $p > 0$,

$$\mathscr{B}^{a}_{n,m,p}(x;l) = \sum_{k=0}^{\lfloor n/m \rfloor} (-n)_{mk}\, (a+np)_{lk,p}\, \gamma_k\, x^k, \tag{4.1}$$

where the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\le u$.

This polynomial contains the extended Jacobi polynomials [12], the Brafman polynomials, the Konhauser polynomials and the Laguerre polynomials as the special cases. In fact, it provides the extension to them in the sense of p-deformation which are deduced below. The inverse series of this polynomial occurs from the Inverse pair - 1. To see this, let us substitute $c = a$ and $G(n) = \gamma_n\, x^n$ in the first series of inverse pair - 1. Then with $F(n) = \mathscr{B}^{a}_{n,m,p}(x;l)$, it reduces to the polynomial (4.1). On the other hand, the second series with these substitutions, yields the inverse series:

$$\gamma_n\, x^n = \sum_{k=0}^{mn} \frac{(-mn)_k\, (a+kp+lkp/m)}{(a+kp)_{ln+1,p}\, (mn)!\, k!}\, \mathscr{B}^{a}_{k,m,p}(x;l). \tag{4.2}$$

The p-deformation of the extended Jacobi polynomials along with the inverse series relation can be obtained from (4.1) and (4.2) by taking $\gamma_n = (\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p}/((\beta_1)_{n,p} \cdots (\beta_d)_{n,p} n!)$. The inverse pair thus formed, is given by

$$\mathscr{F}^{(a)}_{n,l,m,p}[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(a+np)_{lk,p}}{(\beta_1)_{k,p} \cdots (\beta_d)_{k,p}\, k!} (\alpha_1)_{k,p} \cdots (\alpha_c)_{k,p}\, x^k, \tag{4.3}$$

$$\frac{(\alpha_1)_{n,p} \cdots (\alpha_c)_{n,p} (mn)!}{(\beta_1)_{n,p} \cdots (\beta_d)_{n,p} n!} x^n = \sum_{k=0}^{mn} \frac{(-mn)_k (a+kp+lkp/m)}{(a+kp)_{ln+1,p} k!}\, \mathscr{F}^{(a)}_{k,l,m,p}[(\alpha);(\beta):x], \tag{4.4}$$

where (α) stands for the array of the parameters $\alpha_1, \ldots, \alpha_c$ and (β) stands for the array of the parameters $\beta_1, \ldots, \beta_d$. The polynomials $\mathscr{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]$ [12, Eq.(3.16)] occur when $p = 1$. Here we have abbreviated the notation $\mathscr{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]$ for

$\mathscr{F}^{(a)}_{n,l,m}[\alpha_1,\ldots,\alpha_c;\beta_1,\ldots,\beta_d : x]$. In this pair, putting $l=0$, we obtain the p-deformed Brafman polynomials together with the inverse series as given below [10, Eq.(1.16), p.228].

$$\begin{aligned} B^m_{n,p}[\alpha_1,\ldots,\alpha_c;\beta_1,\ldots,\beta_d : x] &= \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\, k!}\, x^k, \\ \frac{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}(mn)!}{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p} n!} x^n &= \sum_{k=0}^{mn} \frac{(-mn)_k}{k!}\, B^m_{k,p}[\alpha_1,\ldots,\alpha_c;\beta_1,\ldots,\beta_d : x]. \end{aligned}$$

The pair of inverse series relations of the extended p-deformed Konhauser polynomials can be obtained from (4.1) and (4.2) by taking $l=0, \gamma_n = 1/((p+\alpha)_{sn,p}\ n!)$ and replacing x by $x^s, s\in\mathbb{N}$. In this case, $\mathscr{B}^a_{n,m,p}(x;0) = n!\ Z^{(\alpha)}_{n,m,p}(x;s)/(p+\alpha)_{sn,p}$ gives [10, Eq.(1.17), (2.8), p.229, 232]

$$\begin{aligned} Z^{(\alpha)}_{n,m,p}(x;s) &= \frac{(p+\alpha)_{sn,p}}{n!} \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{sk,p}\ k!}\, x^{sk}, \qquad (4.5) \\ x^{sn} &= \sum_{k=0}^{mn} \frac{(-1)^k (p+\alpha)_{sn,p}\ n!}{(p+\alpha)_{sk,p}\ (mn-k)!}\, Z^{(\alpha)}_{k,m,p}(x;s). \end{aligned}$$

In this last inverse pair, taking $l=1$, we readily get the inverse pair involving the p-deformed extended Laguerre polynomials as given below.

$$L^{(\alpha)}_{n,m,p}(x) = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(p+\alpha)_{n,p}}{(p+\alpha)_{k,p}\ n!\ k!}\, x^k, \quad x^n = \sum_{k=0}^{mn} \frac{(-1)^k (p+\alpha)_{n,p}\ n!}{(p+\alpha)_{k,p}\ (mn-k)!}\, L^{(\alpha)}_{k,m,p}(x). \qquad (4.6)$$

The well known generalized orthogonal polynomials possessing the higher order hypergeometric function form ${}_4F_3[*]$ are the Wilson polynomials [7, 14, 15] and the Racah polynomials (or Racah coefficient or 6-j symbols) [5, Eq.(7.2.16), p. 165] (also [7, 14]). These polynomials encompass several particular polynomials such as the polynomials of Jacobi, Hahn, continuous Hahn, continuous dual Hahn, Meixner, Meixner-Pollaczek, Krawtchouk and Charlier (see [7, Askey-Scheme, p.23]). The inter connections amongst these polynomials are shown at the end. It is interesting to see that both these polynomials are contained in the Inverse pair - 1 together with their inverse series.

The *extended p-deformed* Racah polynomials and the corresponding inverse series occur from the Inverse pair - 1 by replacing c by $a+b+p$ and making the substitution $G(n) = (-x)_{n,p}(x+c+d+p)_{n,p}/((a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}n!)$. Then with $F(n) = R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)$, the Inverse pair - 1 yields the pair

$$\begin{aligned} R_{n,l,m,p}(x(x+c+d+p);a,b,c,d) &= \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}\ (a+b+np+p)_{lk,p}\ (-x)_{k,p}}{(a+p)_{k,p}\ (b+d+p)_{k,p}(c+p)_{k,p}\ k!} \\ &\quad \times (x+c+d+p)_{k,p}, \qquad (4.7) \end{aligned}$$

$$\frac{(-x)_{n,p}(x+c+d+p)_{n,p}(mn)!}{(a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}\ n!} = \sum_{k=0}^{mn}\frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}k!} \times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d). \quad (4.8)$$

The *extended p-deformed* Wilson polynomials and the inverse series occur by choosing $G(n) = (a+ix)_{n,p}\ (a-ix)_{n,p}/((a+b)_{n,p}\ (a+c)_{n,p}\ (a+d)_{n,p}n!)$ and replacing c by $a+b+c+d-p$ suggest $F(n) = W_{n,l,m,p}(x^2;a,b,c,d)/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p})$ in inverse pair - 1. We then find the inverse pair :

$$\frac{W_{n,l,m,p}(x^2;a,b,c,d)}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}} = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-n)_{mk}(a+b+c+d+np-p)_{lk,p}}{(a+b)_{k,p}(a+c)_{k,p}} \times\frac{(a+ix)_{k,p}(a-ix)_{k,p}}{(a+d)_{k,p}k!}, \quad (4.9)$$

$$\frac{(a+ix)_{n,p}(a-ix)_{n,p}(mn)!}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!} = \sum_{k=0}^{mn}\frac{(-mn)_k(a+b+c+d-p+kp+lkp/m)}{(a+b+c+d-p+kp)_{ln+1,p}} \times\frac{W_{k,l,m,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!}. \quad (4.10)$$

Both, the p-Racah and the p-Wilson polynomials extend all the above mentioned particular polynomials in p-gamma function and p-Pochhammer symbol.
In the following sections, we exploit the series identities (3.5) and (3.6) in turn, to derive certain generating function relations and summation formulas.

5. GENERATING FUNCTION RELATION

We recall the series identity (3.5) of inverse pair - 1 and derive the generating function relations (or GFR) of the general class $\{\mathscr{B}^a_{n,m,p}(x;l); n = 0,1,2,\ldots\}$ defined by (4.1). We require the generalized p-Wright function due to K. Gehlot et al. [6] which is defined by

$${}_q\Psi^p_r\left[\begin{array}{ll}(a_i,\alpha_i)_{1,q}; & z\\ (b_j,\beta_j)_{1,r}; & \end{array}\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma_p(a_i+\alpha_i k)}{\prod_{j=1}^{r}\Gamma_p(b_j+\beta_j k)k!}z^k,$$

where $z\in\mathbb{C},\ p>0,\ \alpha_i,\ \beta_j\in\mathbb{R}\setminus\{0\}$ and $a_i+\alpha_i k,\ b_j+\beta_j k\in\mathbb{C}\setminus p\mathbb{Z}^-$ for $1\le i\le q$ and $1\le j\le r$. Following the notations

$$\Delta = \sum_{j=1}^{r}\frac{\beta_j}{p} - \sum_{i=1}^{q}\frac{\alpha_i}{p}\ ;\ \delta = \prod_{j=1}^{r}\left|\frac{\beta_j}{p}\right|^{\frac{\beta_j}{p}}\prod_{i=1}^{q}\left|\frac{\alpha_i}{p}\right|^{-\frac{\alpha_i}{p}}\ ;\ \mu = \sum_{j=1}^{r}\frac{b_j}{p} - \sum_{i=1}^{q}\frac{a_i}{p} + \frac{q-r}{2},$$

the series converges for all $z \in \mathbb{C}$ if $\Delta > -1$. If $\Delta = -1$ then the converges absolutely for $|z| < \delta$ and if $|z| = \delta$, then $\Re(\mu) > 1/2$. We shall also use the generalized p-Mittag-Leffler function:

$$E^{\gamma,\,\tau}_{p,\,\alpha,\,\beta}(z) \quad = \quad \sum_{k=0}^{\infty} \frac{(\gamma)_{\tau k,p}}{\Gamma_p(\alpha k+\beta)k!}\, z^k, \tag{5.1}$$

given by R. K. Saxena et al. [11], wherein $p \in \mathbb{R}^+,\ \alpha,\ \beta,\ \gamma \in \mathbb{C}$; $\Re(\alpha,\beta) > 0$ and $\tau \in \mathbb{C}$. This will be needed while deriving the generating function relation of the extended p-deformed Konhauser polynomials. Now, to begin with first GFR, we multiply (3.5) by $(a)_{n,p}\, t^n/n!$ where $|t| < 1/p$, and then take the summation n from 0 to ∞ to get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\, F(n)\, t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-1)^{mk}\, n!\, (a)_{n+lk,p}}{(n-mk)!\, n!} G(k)t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{mk}(a)_{n+mk+lk,p}}{n!} G(k)t^{n+mk} \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{(a+mkp+lkp)_{n,p}}{n!} t^n\right)(a)_{mk+lk,p}G(k)(-t)^{mk} \\
&= (1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} (a)_{mk+lk,p}G(k)\left(\frac{(-t)^m}{(1-tp)^{m+l}}\right)^k .
\end{aligned}$$

Here if $G(n) = \gamma_n x^n$, then $F(n) = \mathscr{B}^a_{n,m,p}(x;l)$ leads us to the GFR of (4.1) with $l+m = r$, in the form:

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\, \mathscr{B}^a_{n,m,p}(x;l)\, t^n = (1-tp)^{\frac{-a}{p}} \sum_{k=0}^{\infty} (a)_{rk,p}\gamma_k \left(\frac{x(-t)^m}{(1-tp)^r}\right)^k . \tag{5.2}$$

From this, the GFR of the p-deformed extended Jacobi polynomials can be obtained by substituting $\gamma_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}/((\beta_1)_{n,p}\cdots\,(\beta_d)_{n,p}\, n!)$ and taking $r = m+l$ as follows.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\, \mathscr{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]\, t^n \\
= \quad &(1-tp)^{-\frac{a}{p}} \sum_{k=0}^{\infty} \frac{(a)_{rk,p}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}k!}\left(\frac{x(-t)^m}{(1-tp)^r}\right)^k \\
= \quad &\frac{(1-tp)^{-\frac{a}{p}} \prod_{i=1}^{d}\Gamma_p(\beta_i)}{\Gamma_p(a)\prod_{j=1}^{c}\Gamma_p(\alpha_c)}\; {}_{c+1}\Psi^p_d\left[\begin{matrix}(a,rp),(\alpha_1,p),\cdots,(\alpha_c,p); & \dfrac{x(-t)^m}{(1-tp)^r} \\ (\beta_1,p),\cdots,(\beta_d,p); & \end{matrix}\right].
\end{aligned}$$

If $l \in \mathbb{N}\bigcup\{0\}$, then this further reduces to

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\, \mathscr{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]\, t^n$$

$$= \quad (1-tp)^{-\frac{a}{p}}\ {}_{r+c}F_d\left((\triangle_p(r,a),(\alpha),p,(\beta)),p)\left(\frac{xr^r(-t)^m}{(1-tp)^r}\right)\right).$$

For $l=0$, this yields the GFR of the p-deformed Brafman polynomials [10, Eq.(3.9), p.234]. Next, the GFR of the extended p-deformed Konhauser polynomials can be derived from (5.2) by taking $l=0, \gamma_n = 1/((p+\alpha)_{nl,p}n!)$ and replacing x by $x^s, s\in\mathbb{N}$. It occurs in the form:

$$\sum_{n=0}^{\infty}\frac{(a)_{n,p}\, Z_{n,m,p}^{(\alpha)}(x;s)}{(p+\alpha)_{sn,p}}\ t^n \quad = \quad (1-tp)^{-\frac{a}{p}}\frac{\Gamma_p(p+\alpha)}{\Gamma_p(a)}\,{}_1\Psi_1^p\left[\begin{array}{ll}(a,mp); & \dfrac{x^s\ (-t)^m}{(1-tp)^m}\\ (p+\alpha,sp); & \end{array}\right].$$

Alternatively, incorporating p-Mittag-Leffler function (5.1), this may be written as

$$\sum_{n=0}^{\infty}\frac{(a)_{n,p}}{\Gamma_p(p+\alpha+snp)}\ Z_{n,m,p}^{(\alpha)}(x;s)\ t^n \quad = \quad (1-tp)^{-\frac{a}{p}}E_{p,\,sp,\,\alpha+p}^{a,m}\left(\frac{x^s(-t)^m}{(1-tp)^m}\right).$$

The instance $s=1$ in both these GFRs yield the GFR of the extended p-deformed Laguerre polynomials.

Next, the GFR of the *extended p-deformed* Racah polynomials is obtained by taking $G(n) = (-x)_{n,p}(x+c+d+p)_{n,p}/((a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}n!)$, and replacing a by $a+b+p$ in (3.5), then $F(n) = R_{n,l,m,p}(x(x+c+d+p)$ leads us to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(a+b+p)_{n,p}}{n!}\ R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)\ t^n\\
= \quad & (1-tp)^{\frac{-a-b-p}{p}}\sum_{k=0}^{\infty}\frac{(a+b+p)_{mk+lk,p}(-x)_{k,p}(x+c+d+p)_{k,p}}{(a+p)_{k,p}(b+d+p)_{k,p}(c+p)_{k,p}k!}\left(\frac{(-t)^m}{(1-tp)^{m+l}}\right)^k\\
= \quad & (1-tp)^{\frac{-a-b-p}{p}}\ \frac{\Gamma_p(a+p)\ \Gamma_p(b+d+p)\ \Gamma_p(c+p)}{\Gamma_p(a+b+p)\ \Gamma_p(-x)\ \Gamma_p(x+c+d+p)}\\
& \times{}_3\Psi_3^p\left[\begin{array}{cc}(a+b+p,mp+lp),(-x,p),(x+c+d+p,p); & \dfrac{(-t)^m}{(1-tp)^{m+l}}\\ (a+p,p),(b+d+p,p),(c+p,p); & \end{array}\right].
\end{aligned}$$

If $l\in\mathbb{N}\bigcup\{0\}$, then this further reduces to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(a+b+p)_{n,p}}{n!}\ R_{n,l,m,p}(x(x+c+d+p);a,b,c,d)\ t^n\\
= \quad & (1-tp)^{\frac{-a-b-p}{p}}\ {}_{m+l+2}F_3\left((\triangle_p(m+l,a+b+p),-x,x+c+d+p\right),p,\\
& \left(a+p,b+d+p,c+p\right),p)\left(\frac{(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}}\right),
\end{aligned}$$

in which $m+l=1,2$ for convergence. In a similar manner, the GFR of the *extended p-deformed* Wilson polynomials occurs from (3.5) by replacing a by $a+b+c+d-p$ and choosing $G(n) = (a+ix)_{n,p}(a-ix)_{n,p}/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!)$. Then

$F(n) = W_{n,l,m,p}(x^2;a,b,c,d)/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p})$ and with the notations $a+b+c+d = h, l+m = r$, gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(h-p)_{n,p}\, W_{n,l,m,p}(x^2;a,b,c,d)}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} n!}\ t^n \\
&= (1-tp)^{\frac{-h+p}{p}} \frac{\Gamma_p(a+b)\Gamma_p(a+c)\Gamma_p(a+d)}{\Gamma_p(h-p)\Gamma_p(a+ix)\Gamma_p(a-ix)} \\
&\times {}_3\Psi_3^p \left[\begin{array}{cc} (h-p, mp+lp), (a+ix,p), (a-ix,p); & \dfrac{(-t)^m}{(1-tp)^r} \\ (a+b,p), (a+c,p), (a+d,p); & \end{array} \right].
\end{aligned}$$

As in the case of the *extended p-deformed* Racah polynomials, here also, if $l \in \mathbb{N}\bigcup\{0\}$, then with $a+b+c+d = h$ and $m+l = r$, this reduces to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(h-p)_{n,p} W_{n,l,m,p}(x^2;a,b,c,d)}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} n!}\ t^n = (1-tp)^{\frac{-h+p}{p}} \\
&\times {}_{r+2}F_3\left(\left(\triangle_p\left(r,h-p\right), a+ix, a-ix\right), p, \left(a+b, a+c, a+d\right), p\right) \left(\frac{r^r(-t)^m}{(1-tp)^r} \right),
\end{aligned}$$

wherein $r = 1,2$, for convergence.

6. SUMMATION FORMULAS

We use the series identity (3.6) in this section to derive certain summation formulas involving the polynomials :$\mathscr{B}^a_{n,m,p}(x;l)$, the *extended p-deformed* Racah as well as the *extended p-deformed* Wilson polynomials. For that we shall require the p-deformation of the well known Gauss summation formula [8, Theorem 18, p.49]. We derive it by first obtaining the p-deformed Euler's integral formula in

Lemma 6.1. If $p > 0,\ a,\ b,\ c \in \mathbb{C}$ with both $c,\ c-b \neq sp,\ s \in \mathbb{N}$, then

$$\frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)} \int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b}{p}-1}(1-xt)^{-\frac{a}{p}} dx = \ {}_2F_1((a,b),p,(c),p)(x/p).$$

Proof. The following p-Beta function is due to Rafael and Pariguan [3].

$$B_p(a,b) = \frac{1}{p}\int_0^1 t^{\frac{a}{p}-1}(1-t)^{\frac{b}{p}-1} dt = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)},$$

where $p > 0,\ a,\ b \in \mathbb{C},\ \Re(a,b) \neq 0, -p, -2p, \cdots$.
Now, if we replace x by xt/p in (1.4) then with $|x\,t| < p$, it becomes

$$(1-xt)^{-\frac{a}{p}} \quad = \quad \sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!\ p^n}\ (xt)^n.$$

On multiplying both sides by $t^{\frac{b}{p}-1}(1-t)^{\frac{c-b}{p}-1}$ and then integrating from $t=0$ to $t=1$, we get

$$\begin{aligned}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b}{p}-1}(1-xt)^{-\frac{a}{p}}dx &= \sum_{n=0}^{\infty}\frac{(a)_{n,p}}{n!\;p^n}\,x^n\int_0^1 t^{\frac{b}{p}+n-1}(1-t)^{\frac{c-b}{p}-1}dt\\ &= p\sum_{n=0}^{\infty}\frac{(a)_{n,p}}{n!\;p^n}\,x^n\,B_p(b+np,c-b)\\ &= p\sum_{n=0}^{\infty}\frac{(a)_{n,p}}{n!\;p^n}\,x^n\,\frac{\Gamma_p(b+np)\Gamma_p(c-b)}{\Gamma_p(c+np)}.\end{aligned}$$

Since the series on the right hand side is p-deformed series (1.3), we are led to the integral representation of the deformed hypergeometric function ${}_2F_1[*]$ in the form:

$$\begin{aligned}&\frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b}{p}-1}(1-xt)^{-\frac{a}{p}}dx\\ &= \sum_{n=0}^{\infty}\frac{(a)_{n,p}(b)_{n,p}}{(c)_{n,p}n!p^n}x^n = {}_2F_1((a,b),p,(c),p)(x/p).\end{aligned}\tag{6.1}$$

□

When $p=1$, this reduces to the well known Euler integral representation of ${}_2F_1[*]$.

Corollary 6.2. If $p>0,\ c\neq -p,-2p,\ldots$ and $\Re(c-a-b)>0$, then

$${}_2F_1((a,b),p,(c),p)(1/p) = \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-a)\Gamma_p(c-b)}.\tag{6.2}$$

Proof. Let $x\to 1^-$ in (6.1), then

$$\begin{aligned}{}_2F_1((a,b),p,(c),p)(1/p) &= \frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-a}{p}-1}dx\\ &= \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-a)\Gamma_p(c-b)}.\end{aligned}$$

□

Corollary 6.3. In the above notations,

$${}_2F_1((-np,b),p,(c),p)(1/p) = \frac{(c-b)_{n,p}}{(c)_{n,p}}.\tag{6.3}$$

Proof. By putting $a=-np$ in (6.2), we get the terminating series

$${}_2F_1((-np,b),p,(c),p)(1/p) = \frac{\Gamma_p(c)\Gamma_p(c-b+np)}{\Gamma_p(c+np)\Gamma_p(c-b)} = \frac{(c-b)_{n,p}}{(c)_{n,p}}.$$

□

This coincides with the Chu-Vandermonde identity [8, Ex. 4, p.69] when $p=1$. We now obtain the summation formulas involving the above considered polynomials. We begin with the inverse series (4.2) with the assumption that $\gamma_n \neq 0\ \forall n$, in the form:

$$\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\,k!}\,\mathscr{B}^a_{k,m,p}(x;l) \quad = \quad x^n \tag{6.4}$$

and multiply the both sides by $1/n!$ and take the summation n from 0 to ∞, then we find

$$\sum_{n=0}^{\infty}\frac{1}{n!\,\gamma_n}\sum_{k=0}^{mn}\frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\,k!}\,\mathscr{B}^a_{k,m,p}(x;l) \quad = \quad e^x. \tag{6.5}$$

Here x may be assigned particular values to get a number of particular sums. Next, by taking $\gamma_n = (\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}/((\beta_1)_{n,p}\cdots\,(\beta_d)_{n,p}\,n!)$ in (6.5), we get a summation formula involving the p-deformed extended Jacobi polynomials (4.3):

$$\sum_{n=0}^{\infty}\frac{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p}}{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}}\sum_{k=0}^{mn}\frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!k!}\,\mathscr{F}^{(a)}_{n,l,m}[(\alpha);(\beta):x]=e^x.$$

The p-Brafman polynomials case follows immediately when $l=0$.
Now, the summation formula involving the p-deformed extended Konhauser polynomials (4.5) can be obtained from (6.4) by taking $l=0, \gamma_n = 1/((p+\alpha)_{sn,p}n!)$ and replacing x by $x^s, s\in\mathbb{N}$. With this, $\mathscr{B}^a_{n,m,p}(x;0) = Z^{(\alpha)}_{n,m,p}(x;s)/(p+\alpha)_{sn,p}$ yields the sum:

$$\sum_{n=0}^{\infty}(p+\alpha)_{sn,p}\sum_{k=0}^{mn}\frac{(-mn)_k}{(p+\alpha)_{sk,p}(mn)!}Z^{(\alpha)}_{k,m,p}(x;s) \quad = \quad e^{x^s}.$$

When $s=1$, this readily yields the summation formula involving the extended p-deformed Laguerre polynomials (4.6).
Next assuming $|x|<1$, and taking summation n from 0 to ∞ in (6.4), we find

$$\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\sum_{k=0}^{mn}\frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\,k!}\mathscr{B}^a_{k,m,p}(x;l) \quad = \quad \frac{1}{1-x}.$$

By assigning different values of x from $(-1,1)$, a number of particular summation formulas can be derived. The summation formulas corresponding to those of p-deformed extended Jacobi polynomial and the extended p-deformed Konhauser polynomial follow by specializing the parameters suitably. We now derive certain summation formulas involving the *extended p-deformed* Racah polynomials and the *extended p-deformed* Wilson polynomials. From the inverse series (4.8), we obtain the following formula by multiplying both sides of by p^{-n}, take the summation from $n=0$ to ∞, then in view of the p-Gauss sum (6.2), we get

$$\sum_{n=0}^{\infty}\frac{(b+d+p)_{n,p}(c+p)_{n,p}}{p^n\,(mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\,k!}$$

$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \frac{\Gamma_p(p+a)\Gamma_p(a-c-d)}{\Gamma_p(a-x-c-d)\Gamma_p(p+a+x)},$$

If we multiply both side of (4.8) by $(-jp)_{n,p}\ p^{-n}$ and then take the summation from $n=0$ to j, then we get

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!}$$
$$\times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d) = \frac{(x+a+p)_{j,p}}{(a+p)_{j,p}},$$

using the p-Chu-Vandermonde identity (6.4). A worth mentioning sum occurs when $x=0$. In this case, $R_{k,p}(0;a,b,c,d)$ is unity; hence, this summation formula reduces to

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!} = 1.$$

In a similar manner, we may find the summation formulas involving the *extended p-deformed* Wilson polynomials. For that we first multiply both sides of (4.10) by p^{-n}, put $a+b+c+d=h$, take the summation from $n=0$ to ∞ then using the p-Gauss sum (6.2), we get

$$\sum_{n=0}^{\infty}\frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ (mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k\ (h-p+kp+lkp/m)}{(h+kp-p)_{ln+1,p}}$$
$$\times\frac{W_{k,l,m,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!} = \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{\Gamma_p(b-ix)\Gamma_p(b+ix)}.$$

Here the substitution $x=0$ leads us to

$$W_{k,l,m,p}(0;a,b,c,d) = (a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}$$

$$\times_{r+2}F_3((\Delta_p(m;-k),\Delta_p(l;h+kp-p+lkp/m),a,a),p,(a+b,a+c,a+d),p)(m^m l^l),$$

where $r=m+l$ and $h=a+b+c+d$. Hence, the above sum particularizes to

$$\frac{\Gamma_p(a+b)\Gamma_p(b-a)}{[\Gamma_p(b)]^2} = \sum_{n=0}^{\infty}\frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ (mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k\ (h-p+kp+lkp/m)}{(h+kp-p)_{ln+1,p}\ k!}$$
$$\times_{r+2}F_3\Big((\Delta_p(m;-k),\Delta_p(l;h+kp-p+lkp/m),a,a),p,(a+b,a+c,a+d),p\Big)(m^m l^l)$$

Now, if both sides of (4.10) are multiplied by by $(-jp)_{n,p}\ p^{-n}$, take the summation from $n=0$ to j and use the terminating p-Gauss sum (6.4) to derive

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}\ p^n(mn)!}\sum_{k=0}^{mn}\frac{(-mn)_k\ (h-p+kp+lkp/m)}{(h+kp-p)_{ln+1,p}}$$
$$\times\frac{W_{k,l,m,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!} = \frac{(b-ix)_{j,p}}{(a+b)_{j,p}}.$$

7. EXTENSION OF CERTAIN RIORDAN'S INVERSE PAIRS

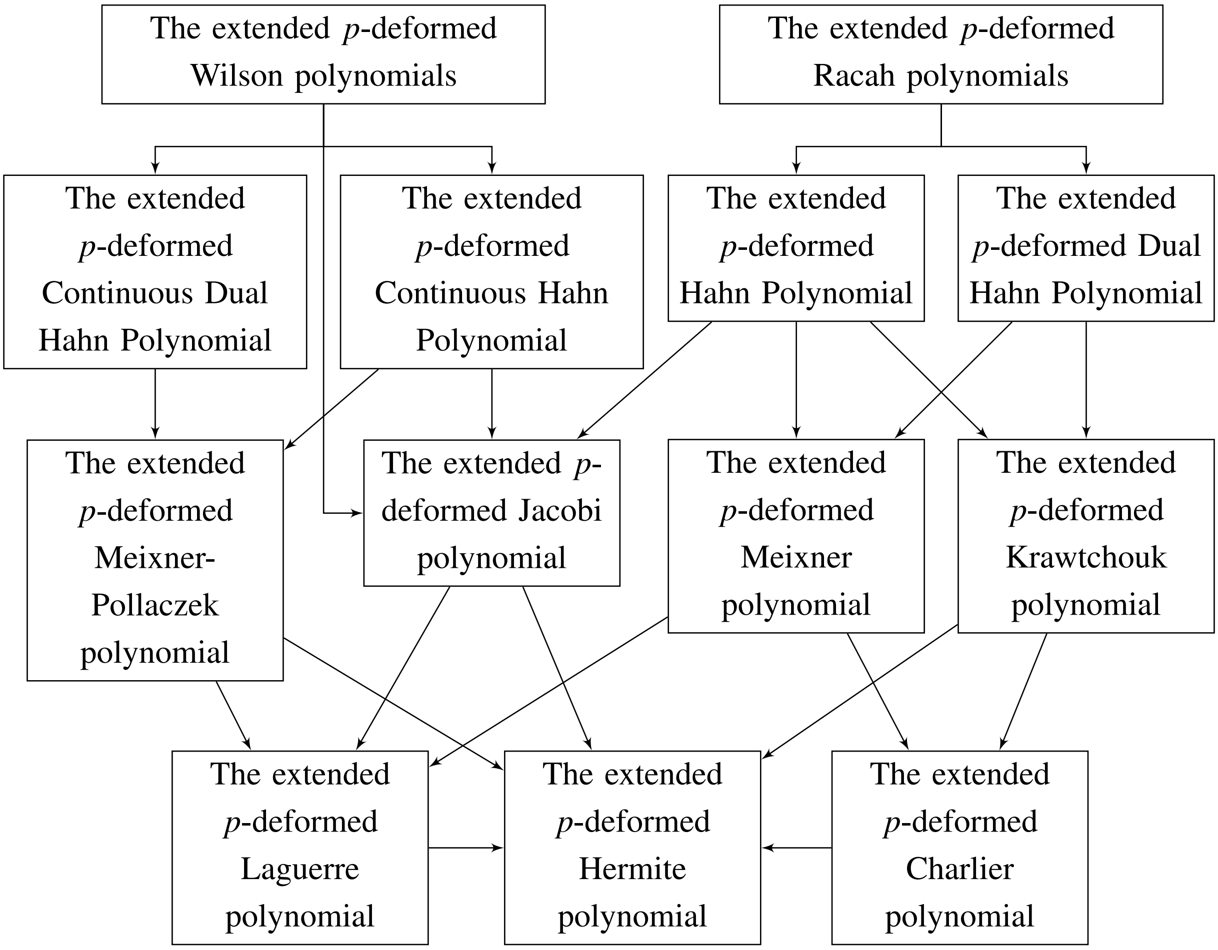
In this section, we illustrate the reducibility of the inverse pairs obtained in section 3 to some of the Riordan's inverse pairs [9, Ch. 2] in p-extended form. They are tabulated below.

Table 1 p-Deformed extension of Riordan's inverse series

$$F(n)=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{A_{n,k,p}}{(n-mk)!}G(k)\ ;\ G(n)=\sum_{k=0}^{mn}(-1)^{mn-k}\frac{B_{n,k,p}}{(mn-k)!}F(k)$$

Inverse pair -	λ	$A_{n,k,p}$	$B_{n,k,p}$	p-extension of the class No., Table No. in [9, Ch.2]
Th 1	l	$\frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{a+lkp-kp}{a+lmn-kp}\times\frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	Gould class 1, Table 2.2
2	l	$\frac{a+ln-np+p}{a+lmk-np+p}\times\frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	Gould class 2, Table 2.2
3	$p-2$	$\frac{\Gamma_p(a+np+2mk-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{a+2k+p}{a+2mn-mnp+kp+p}\times\frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	Simpler Legendre, class 1, Table 2.5
4	$p-2$	$\frac{a+2n}{a+2mk+np-mkp}\times\frac{\Gamma_p(a+2mk+np-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	Simpler Legendre, class 2, Table 2.5
4	$p-c$	$\frac{a+cn}{a+cmk+np-mkp}\times\frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	Legendre-Chebyshev, class 1, Table 2.6
Th 1	$p+c$	$\frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{a+ck}{a+cmn+mnp-kp}\times\frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	Legendre-Chebyshev, class 3, Table 2.6
3	$p-c$	$\frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{a+ck+p}{a+cmn-mnp+kp+p}\times\frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	Legendre-Chebyshev, class 5, Table 2.6
2	$p+c$	$\frac{a+cn+p}{a+cmk+mkp-np+p}\times\frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	Legendre-Chebyshev, class 7, Table 2.6

Polynomials' reducibility chart



8. ACKNOWLEDGEMENT

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