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A SYSTEM OF p-POLYNOMIALS AND ITS q-ANALOGUE

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SAVALIA RAJESH VASANTBHAI

UNDER THE GUIDANCE OF

DR. B. I. DAVE

PROFESSOR
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
THE MAHARAJA SAYAJIRAO UNIVERSITY OF BARODA
VADODARA 390-002

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The thesis entitled "A SYSTEM OF p-POLYNOMIALS AND ITS q-ANALOGUE" carries out the generalization of well known polynomials and their q-versions along with their inverse series relation in p-deformed sense.

In fact, the p-Gamma function and the Pochhammer p-symbol were introduced by Diaz and Pariguan [7] as follows.

For $z \in \mathbb{C},\ \mathrm{Re}(z) > 0$ and $p > 0$,

$$\Gamma_p(z) = \int_0^\infty t^{z-1} e^{-\frac{t^p}{p}}\ dt. \tag{1}$$

For $z \in \mathbb{C},\ p \in \mathbb{R}$ and $n \in \mathbb{N}$ is given by

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p). \tag{2}$$

The following properties follow from (2).

$$\Gamma_p(z+p) = z\Gamma_p(z), \tag{3}$$
$$\Gamma_p(p) = 1, \tag{4}$$
$$(z)_{k,p} = \frac{\Gamma_p(z+kp)}{\Gamma_p(z)}. \tag{5}$$

Diaz at el. [7] also proposed the following generalization of the hypergeometric series in the form of Pochhammer p-symbol (cf. [17] with $p = 1$), given by

$${}_rF_s(a,k,b,l)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s} n!} x^n, \tag{6}$$

where $a = (a_1, a_2, \cdots, a_r) \in \mathbb{C}^r,\ k = (k_1, k_2, \cdots, k_r) \in (\mathbb{R}^+)^r,\ b = (b_1, b_2, \cdots, b_s) \in \mathbb{C}^s \setminus (k\mathbb{Z}^-)^s$ and $l = (l_1, l_2, \cdots, l_s) \in (\mathbb{R}^+)^s$. This series converges for all x if $r \leq s$, and diverges if $r > s+1,\ x \neq 0$. If $r = s+1$, then the series converges for $|x| < \frac{l_1 l_2 \cdots l_s}{k_1 k_2 \cdots k_r}$. It also satisfies the differential equation [7]:

$$\begin{aligned}&[D\,(l_1 D + b_1 - l_1)\,(l_2 D + b_2 - l_2)\ \cdots\ (l_s D + b_s - l_s)\\ &\quad - x\,(k_1 D + a_1)\,(k_2 D + a_2)\ \cdots\ (k_r D + a_r)]y = 0,\end{aligned} \tag{7}$$

where $D = x\dfrac{d}{dx}$. For $p > 0,\ a \in \mathbb{C}$ and $|x| < \dfrac{1}{p}$, Diaz at el. [7] showed that

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!} x^n = (1 - px)^{-\frac{a}{p}}. \tag{8}$$

This may be regarded as the p-deformed binomial series.

In addition to this, Rafael Díaz and Carolina Teruel [8] defined a q-analogues for $p > 0$ in the form:

$$\Gamma_{q,p}(z) = \frac{(q^p;q)_{\infty,p}(1-q)^{1-z/p}}{(q^z;q)_{\infty,p}}, \quad z > 0, \tag{9}$$
$$(z;q)_{n,p} = (1-q^z)(1-q^{z+p})(1-q^{z+2p})\cdots(1-q^{z+(n-1)p}), z \in \mathbb{R}, n \in \mathbb{Z}^+, \tag{10}$$

where

$$(q^\alpha;q)_{\infty,p} = \prod_{n=0}^{\infty}(1-q^{\alpha+np}),\ |q| < 1$$

and

$$(a;q)_{n,p} = \begin{cases} 1, & if \ \ n=0 \\ (1-a)(1-aq^p)\cdots(1-aq^{p(n-1)}), & if \ n \in \mathbb{Z}_{>0} \\ [(1-aq^{-p})(1-aq^{-2p})\cdots(1-aq^{np})]^{-1}, & if \ n \in \mathbb{Z}_{<0} \\ (a;q)_{\infty,p}/(aq^{np};q)_{\infty,p} & if \ n \in \mathbb{C}. \end{cases}.$$

From this, it follows that

$$\begin{aligned} \Gamma_{q,p}(p) &= 1, & (11)\\ (a;q)_{n,p} &= \frac{\Gamma_{q,p}(a+np)}{\Gamma_{q,p}(a)}, \ n \in \mathbb{N}, & (12)\\ (a;q)_{n,p} &= \frac{(a;q)_{\infty,p}}{(aq^{pn};q)_{\infty,p}}, \ n \in \mathbb{C}. & (13) \end{aligned}$$

Having motivated by this introduction, certain polynomials systems are provided p-deformation in this work, in particular, the p-deformed Gould's generalized Humbert polynomials class : $\{P_n(m,x,y,p,C); n = 0,1,2,\ldots\}$ [11, Eq.5.11, p.707] is defined by

$$P_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\gamma^k c^{s-n+mk-k}}{(s+p)_{-n+mk-k,p}(n-mk)! \ k!}(-mx)^{n-mk}, \qquad (14)$$

in which the floor function $\lfloor r \rfloor$ = floor r, represents the greatest integer $\leq r$ and $\gamma, \ s, \ c \in \mathbb{C}, \ m \in \mathbb{N}, \ x \in \mathbb{R}, \ n \in \mathbb{N} \cup \{0\}$ and $p > 0$.

The p-deformed Wilson polynomials [9] is defined as

$$\begin{aligned} & W_{n,p}(x^2;a,b,c,d) = (a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} \\ & \times \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+np-p)_{k,p}(a+ix)_{k,p}(a-ix)_{k,p}}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}k!} \end{aligned} \qquad (15)$$

and the p-Racah polynomials [9] in the form:

$$\begin{aligned} & R_{n,p}\left(x(x+c+d+p);a,b,c,d\right) \\ = & \sum_{k=0}^{n} \frac{(-n)_k(a+b+np+p)_{k,p}(x+c+d+p)_{k,p}(-x)_{k,p}}{(p+a)_{k,p}(b+d+p)_{k,p}(c+p)_{k,p}k!}. \end{aligned} \qquad (16)$$

These polynomials contain as their special cases, a number of particular polynomials; in fact, the polynomial (14) includes p-deformed Humbert polynomials (cf. [12] with $p = 1$), the p-deformed Kinney polynomial, p-deformed Pincherle polynomial, p-deformed Gegenbauer polynomial and the p-deformed Legendre polynomial (cf. [11] with $p = 1$). On the other hand, (15) and (16) provide p-extension to the polynomials of Hahn, continuous Hahn, Dual Hahn, continuous dual Hahn, Meixner-Pollaczek, Krawtchouk, Jacobi etc.

The work incorporates the inverse series relation of these general classes of polynomials and towards the application of the inverse series relation, the generating function relations and the summation formulas are derived. Besides this, the differential equation, companion matrix representations are also obtained.

Definition 1. *If a polynomial $f(x) \in \mathbb{C}[X]$ and $f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_{k-1}x^{k-1} + x^k$ then the $k \times k$ matrix, called the companion matrix of $f(x)$ is denoted and defined by [15, p. 39]*

$$C(f(x)) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \ldots & -\delta_{k-1} \end{bmatrix}.$$

Lemma 1. *If $f \in K[x]$ is non constant and $A = C(f(x))$ then $f(A) = 0$.*

The thesis also deals with the q-analogues of the aforementioned polynomials and their properties.

Chapter 1 introduces the subject matter and enlists certain definitions, notations, formulae and results. The Riordan's classification of inverse series relations [18, Chapter-2] are tabulated including their basic analogues [3, 5].

A q-analogue of the function (6) is defined by taking $k_1 = k_2 = \ldots = k_r = l_1 = l_2 = \ldots = l_s = p \in \mathbb{R}^+$, as follows:

Definition 2. *If (a) stands for the array of r parameters $a_1,\ a_2, \cdots,\ a_r \in C^r$, (b) stands for the array of s parameters $b_1,\ b_2, \cdots,\ b_s \in C^s \setminus (Z^-)^s$, $p, \alpha \in \mathbb{R}^+$ and $|q| < 1$ then*

$$\begin{aligned} & {}_r\phi_s((a);\ (b)\ ; q^p)(x|q, q^\alpha) \\ = & \sum_{n=0}^{\infty} \frac{(a_1; q)_{n,p}(a_2; q)_{n,p} \cdots (a_r; q)_{n,p}}{(b_1; q)_{n,p}(b_2; q)_{n,p} \cdots (b_s; q)_{n,p}(q^\alpha; q^\alpha)_n} \left((-1)^n q^{\alpha\binom{n}{2}}\right)^{1+s-r} x^n. \end{aligned} \tag{17}$$

From d' Alembert's ratio test, it follows that the series converges for all x if $r \leq s$, and it diverges when $r > s + 1$ and $x \neq 0$. If $r = s + 1$, then it converges for $|x| < 1$.

In chapter 2, the general inversion theorems are proved. They are stated below.

Theorem 2. *If $a = n \in \mathbb{N} \cup \{0\}$ and $b = -m,\ m \in \mathbb{N}$, then $n^* = \lfloor n/m \rfloor$, and there hold the series relations*

$$u(n) = \sum_{k=0}^{n^*} \frac{\gamma^k}{\Gamma_p(p + \alpha - ar + mrk - kp)k!}\ v(n - mk) \tag{18}$$

$$\Leftrightarrow$$

$$v(n) = \sum_{k=0}^{n^*} \frac{(-\gamma)^k(\alpha - ar + mrk)\Gamma_p(\alpha - ar + kp)}{k!}.\ u(n - mk) \tag{19}$$

Theorem 3. *If $\{u(*)\}$ and $\{v(*)\}$ are bounded sequences, $a = n \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$, then there hold the series relations*

$$u(n) = \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma_p(p + \alpha - ar - brk - kp)k!}\ v(n + bk) \tag{20}$$

$$\Leftrightarrow$$
$$v(n) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k(\alpha - ar - brk)\Gamma_p(\alpha - ar + kp)}{k!} \; u(n+bk). \tag{21}$$

Theorem - 2 yields the following inverse of the p-polynomials (14), (15) and (16).

$$\begin{aligned}\frac{(-mx)^n}{n!}x^n &= \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k c^{n-k-s} \frac{(s - np + mkp)}{(s - np + kp)\Gamma_p(s+p)k!} \\ &\quad \times \Gamma_p(s - np + kp + p) P_{n-mk,p}(m, x, \gamma, s, c),\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{(a+ix)_{n,p}(a-ix)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}} &= \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}} \\ &\quad \times \frac{W_{k,p}(x^2; a, b, c, d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p} \; k!},\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{(x+c+d+p)_{n,p}(-x)_{n,p}}{(p+a)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}} &= \sum_{k=0}^{n} \frac{(-n)_k \; (a+b+2kp+p)}{(a+b+kp+p)_{n+1,p} k!} \\ &\quad \times R_{k,p}\left(x(x+c+d+p); a, b, c, d\right),\end{aligned} \tag{24}$$

respectively. Theorem - 3 yields the p-Bessel function [20, Eq. (3.1)]

$$J_{n,p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p + np + kp)k!} \left(\frac{x}{2}\right)^{n+2k} \tag{25}$$

and its inverse series as the deformed Neumann's expansion:

$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} \frac{(np + 2pk)\Gamma_p(np + kp)}{k!} J_{n+2k,p}(x).$$

The usual Neumann's expansion occurs if $p = 1$ [17, Ex.22, p. 122].
Theorem - 2 and Theorem - 3 on the other hand, provide p-deformation to Riordan's classes of inverse series relations.
Among the generating functions derived for these polynomials, the one is stated below.

$$\sum_{n=0}^{\infty} P_{n,p}(m, x, \gamma, s, c)t^n = c^{(1-1/p)s} \left(c + \gamma p t^m - pmxt\right)^{s/p}. \tag{26}$$

The summation formulas occurring from the inverse series are also derived. For instance, taking $x = 0$ in Wilson polynomials, one finds the sum:

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n \; n!} \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p} \; k!} \\ &\times {}_4F_3((-k, a+b+c+d+k-1, a, a), p, (a+b, a+c, a+d), p)(1) \\ &= \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{[\Gamma_p(b)]^2}.\end{aligned}$$

The following is the differential equation satisfied by polynomial $y = P_{n,p}(m, x, \gamma, s, c)$.

Theorem 4. *Let $s \in \mathbb{C},\ p > 0$ and $m \in \mathbb{N}$. Then the polynomials $y = P_{n,p}(m, x, \gamma, s, c)$*

are a particular solution of the m^{th} order differential equation in the form

$$\gamma c^{m-1} y^{(m)} + \sum_{r=0}^{m} a_r x^r y^{(r)} \quad = \quad 0, \tag{27}$$

where $a_r = \dfrac{m^{m-1}\Delta^r f_0}{r!}$.

In chapter 3, the q-analogues of the aforementioned polynomials are defined. the basic analogue of (14) as

Definition 3. *For $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}, r, s, c \in \mathbb{C}, p > 0,$*

$$\begin{aligned} P_{n,p,r}(m, x, \gamma, s, c|q) \quad &= \quad \sum_{k=0}^{\lfloor n/m \rfloor} \gamma^k \frac{(q^{s-nr+mrk-kp+p}; q)_{\infty,p}(1-q^c)^{s-n+mk-k}}{(q^{s-nr+mkr+p}; q)_{\infty,p}(q;q)_{n-mk}} \\ &\quad \times \frac{(1-q)^{k-s}}{(q^{mr-p}; q^{mr-p})_k} \left((q^m-1)x\right)^{n-mk}. \end{aligned} \tag{28}$$

We shall call this polynomials as the p-deformed generalized q-Humbert polynomials. It extends the p-version of the q-Humbert polynomials, the q-Kinney polynomial, q-Pincherle polynomial, q-Gegenbauer polynomial and q-Legendre polynomial together with their inverse series relation. The inverse series of these polynomials (28) are then obtained from

Theorem 5. *If $p > 0,\ 0 < q < 1,\ \alpha,\ r \in \mathbb{C}$ and $a \in \mathbb{N}$, then*

$$F(a) \quad = \quad \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-ar-brk-kp}; q)_{\infty,p}}{(q_3; q_3)_k} G(a+bk) \tag{29}$$

$$\Leftrightarrow$$

$$G(a) \quad = \quad \sum_{k=0}^{N} (-\gamma)^k \ q_3^{k(k-1)/2} \frac{(1-q^{\alpha-ar-brk})}{(q^{\alpha-ar+kp}; q)_{\infty,p}(q_3;q_3)_k} F(a+bk), \tag{30}$$

where $q_3 = q^{-br-p},\ br \neq -p$.

For $br = -p$, the following theorem holds.

Theorem 6. *if $p > 0,\ 0 < q < 1,\ \alpha,\ r \in \mathbb{C},\ a \in \mathbb{N}$ and then*

$$F(a) \quad = \quad \sum_{k=0}^{N} \gamma^k \frac{(q^{p+\alpha-ar}; q)_{\infty,p}}{(q;q)_k} \ G(a+bk) \tag{31}$$

$$\Leftrightarrow$$

$$G(a) \quad = \quad \sum_{k=0}^{N} (-\gamma)^k q^{k(k-1)/2} \frac{F(a+bk)}{(q^{p+\alpha-ar+kp}; q)_{\infty,p}(q;q)_k}, \tag{32}$$

where N in both theorems, may be a positive integer or infinity; depending on whether b is a negative integer or a positive integer. For the polynomials' inverse series, N must be a positive integer; for that b must be a negative integer and a a non negative integer.

The inverse series of (28) occurs from this theorem in the form:

$$\frac{(q^m-1)^n}{(1-q^c)^n (q;q)_n} x^n = \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k \left(\frac{1-q}{1-q^c}\right)^{s+k} \quad q^{(mr-p)k(k-1)/2}$$

$$\times\frac{(1-q^{s-nr+mrk})}{(q^{s-nr+kp};q)_{\infty,p}}\frac{(q^{s-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}\ P_{n-mk,p,r}(m,x,\gamma,s,c|q). \quad (33)$$

Besides this, the q-analogues of Wilson polynomials and Racah polynomials are also obtained from this theorem, along with their inverse series relations as given below.

$$\frac{a^n\ p_{n,p,r}(cos\theta;a,b,c,d|q)}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}$$
$$=\sum_{k=0}^{n}q^{kr-kp}\frac{(q^{-n(r-p)};q^{r-p})_k(abcdq^{np-p};q)_{\frac{kr}{p}-k,p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{r-p};q^{r-p})_k} \quad (34)$$
$$\Leftrightarrow$$
$$\frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}}=\sum_{k=0}^{n}q^{nkr-nkp}\ \frac{(1-abcdq^{-p+kr})}{(abcdq^{kp-p};q)_{\frac{nr}{p}-n+1,p}}$$
$$\times\frac{(q^{-n(r-p)};q^{r-p})_k}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{r-p};q^{r-p})_k\ a^k}\ p_{k,p,r}(cos\theta;a,b,c,d|q), \quad (35)$$

and with $q^{-x}+cdq^{x+p}=\mu(x)$,

$$\begin{aligned}R_{n,p,r}(\mu(x);a,b,c,N|q) &= \sum_{k=0}^{n}q^{kr-kp}\frac{(q^{-n(r-p)};q^{r-p})_k(abq^{np+p};q)_{\frac{kr}{p}-k,p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}}\\ &\quad\times\frac{(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}}{(q^{r-p};q^{r-p})_k} \quad (36)\\ &\Leftrightarrow\\ \frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}} &= \sum_{k=0}^{n}\frac{q^{nkr-nkp}(1-abq^{p+kr})(q^{-n(r-p)};q^{r-p})_k}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}(q^{r-p};q^{r-p})_k}\\ &\quad\times R_{k,p,r}(\mu(x);a,b,c,N|q). \quad (37)\end{aligned}$$

The particular polynomials belonging to the above polynomials are the p-deformed q-Hahn polynomial, p-deformed little as well as big q-Jacobi polynomials, p-deformed q-Gegenbauer polynomial, p-deformed q-Legendre polynomial, p-deformed q-Chebyshev polynomial together their inverse series relation.

The above theorem also provides extension to the generalized Bessel function due to Mansour Mahmoud [14] to p-deformed extended q-Bessel function together with its q,p-Neumann's expansion as its inverse, in the form:

$$\begin{aligned}J_{n,p,r}(x;a,q) &= \frac{1}{(q^p;q)_{n,p}}\sum_{k=0}^{\infty}\frac{(-1)^{(a+1)k}q^{ak(n+k)/2}}{(q^{p-nr};q)_{-\frac{2rk}{p}-k,p}(q^{-2r-p};q^{-2r-p})_k}\left(\frac{x}{2}\right)^{n+2k}\\ &\Leftrightarrow\\ \left(\frac{x}{2}\right)^n &= \sum_{k=0}^{\infty}(-1)^{ak}q^{(-2r-p)\binom{k}{2}}q^{ak(n+k)/2}\frac{(1-q^{-nr-2rk})(q^p;q)_{\infty,p}}{(q^{-nr+kp};q)_{\infty,p}(q^{-2r-p};q^{-2r-p})_k}\\ &\quad\times J_{n+2k,p,r}(x;a,q).\end{aligned}$$

The inverse series may be used to derive certain summation formulas. For example, there follows the sum:

$$\sum_{n=0}^{\infty}\frac{(1-q^c)^n}{(q^m-1)^n}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k\left(\frac{1-q}{1-q^c}\right)^{s+k}\frac{q^{(mr-p)k(k-1)/2}}{(q^{s-np+kp};q)_{\infty,p}}$$

$$\times\frac{(1-q^{s-nr+mrk})(q^{s-nr+p};q)_{\infty,p}}{(q^{mr-p};q^{mr-p})_k}P_{n-mk,p,r}(m,x,\gamma,s,c|q) \;=\; e_q(x),$$

and the sum:

$$\sum_{n=0}^{j}\frac{(q^{-jp};q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}}{(cdq^{x+p};q)_{n,p}(q^p;q)_{n,p}}(axq^{jp+p})^n\sum_{k=0}^{n}\frac{q^{nk(r-p)}(1-abq^{p+kr})}{(abq^{kp+p};q)_{\frac{nr}{p}-n+1,p}}$$
$$\times\frac{(q^{-n(r-p)};q^{r-p})_k}{(q^{r-p};q^{r-p})_k}R_{k,p,r}(q^{-x}+cdq^{x+p};a,b,c,N|q) \;=\; \frac{(aq^{x+p};q)_{j,p}}{(aq^p;q)_{j,p}}.$$

Finally, the companion matrix of the generalized polynomial is obtained by taking $\lfloor n/m\rfloor = N$ in (28) and converting it to the monic form:

$$\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)=\sum_{k=0}^{N}\delta_k\; x^{n-mk},$$

where

$$\delta_k \;=\; \gamma^k\frac{(q^{mr-p};q^{mr-p})_n(q^{s-nr+mrk-kp+p};q)_{\infty,p}(1-q^c)^{mk-k}(1-q)^k}{(q^{s-nr+mrk+p};q)_{\infty,p}(q^{mr-p};q^{mr-p})_{n-mk}(q^{mr-p};q^{mr-p})_k}\,(q^m-1)^{-mk}.$$

With this δ_k, $C\left(\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)\right)$ assumes the form as stated in definition of companion matrix. The eigen values of this matrix will be the zeros of $\widetilde{P}_{n,p}(m,x,\gamma,s,c|q)$, (see [15, p. 39]).

The subject matter of Chapter 4 is to provide extension to the p-deformed Wilson polynomials (15) and the p-deformed Racah polynomials (16) of chapter 2 by considering the degree $\lfloor n/m\rfloor$ in stead of n. For that the following inversion theorem [21] is proved.

Theorem 7. *For $\lambda\in\mathbb{C},\ \alpha\in\mathbb{C},\ n\in\mathbb{N}\cup\{0\},\ m\in\mathbb{N}$ and $p>0$,*

$$F(n) \;=\; \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^{mk}\frac{1}{\Gamma_p(\alpha+mk\lambda+p-np)(n-mk)!}G(k) \tag{38}$$
$$\Rightarrow$$
$$G(n) \;=\; \sum_{k=0}^{mn}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+mn\lambda-kp)}{(mn-k)!}F(k) \tag{39}$$

and conversely, the series in (39) implies the series (38) if for $n\neq mr,\ r\in\mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k\frac{(\alpha+k\lambda-kp)\Gamma_p(\alpha+n\lambda-kp)}{(n-k)!}F(k) \;=\; 0. \tag{40}$$

Using this theorem, the Wilson polynomial and the Racah polynomial assume extension in the form along with their inverse series relations, as given below.

$$\frac{W_{n,l,m,p}(x^2;a,b,c,d)}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}} \;=\; \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-n)_{mk}(a+b+c+d+np-p)_{lk,p}}{(a+b)_{k,p}(a+c)_{k,p}}$$
$$\times\frac{(a+ix)_{k,p}(a-ix)_{k,p}}{(a+d)_{k,p}k!} \tag{41}$$
$$\Leftrightarrow$$
$$\frac{(a+ix)_{n,p}(a-ix)_{n,p}(mn)!}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}n!} \;=\; \sum_{k=0}^{mn}\frac{(-mn)_k(a+b+c+d-p+kp+lkp/m)}{(a+b+c+d-p+kp)_{ln+1,p}}$$

$$\times \frac{W_{k,l,m,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!}. \tag{42}$$

$$R_{n,l,m,p}(x(x+c+d+p);a,b,c,d) = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}\ (a+b+np+p)_{lk,p}\ (-x)_{k,p}}{(a+p)_{k,p}\ (b+d+p)_{k,p}}, \times \frac{(x+c+d+p)_{k,p}}{(c+p)_{k,p}\ k!} \tag{43}$$

$$\Leftrightarrow$$

$$\frac{(-x)_{n,p}(x+c+d+p)_{n,p}(mn)!}{(a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}\ n!} = \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}k!} \times R_{k,l,m,p}(x(x+c+d+p);a,b,c,d). \tag{44}$$

In addition to this, the above theorem is used to define a general class of polynomials

Definition 4. *For $a,\ l \in \mathbb{C},\ m \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}$ and $p>0$,*

$$\mathcal{B}^a_{n,m,p}(x;l) = \sum_{k=0}^{\lfloor n/m\rfloor} (-n)_{mk}(a+np)_{lk,p}\gamma_k x^k \tag{45}$$

in which the floor function $\lfloor r\rfloor = floor\ r$, represents the greatest integer $\le r$.

The theorem also yields its inverse series:

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{(-mn)_k\ (a+kp+lkp/m)}{(a+kp)_{ln+1,p}\ (mn)!\ k!}\ \mathcal{B}^a_{k,m,p}(x;l) \tag{46}$$

along with its particular cases like the p-deformed extended Jacobi polynomial and its inverse series as stated by

$$\mathcal{F}^{(a)}_{n,l,m,p}[(\alpha);(\beta):x] = \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(a+np)_{lk,p}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\ k!}\ x^k \tag{47}$$

$$\Leftrightarrow$$

$$\frac{(\alpha_1)_{n,p}\cdots(\alpha_c)_{n,p}\ (mn)!}{(\beta_1)_{n,p}\cdots(\beta_d)_{n,p}\ n!} x^n = \sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}k!}\ \mathcal{F}^{(a)}_{k,l,m,p}[(\alpha);(\beta):x], \tag{48}$$

in which the substitution $l=0$ yields the p-deformed Brafman polynomial and its inverse series relation. Also, the generalized p-deformed extended Jacobi polynomial and its inverse series may be deduced in the form:

$$P^{(\alpha,\beta)}_{n,l,m}(x) = \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}(\alpha+\beta+n+1)_{lk}}{(1+\alpha)_k\ k!} \left(\frac{1-x}{2}\right)^k \tag{49}$$

$$\Leftrightarrow$$

$$\frac{(1-x)^n}{(1+\alpha)_n 2^n n!} = \sum_{k=0}^{mn} \frac{(-mn)_k(\alpha+\beta+kp+lkp/m+1)}{(\alpha+\beta+kp+1)_{ln+1,p}(mn)!k!} P^{(\alpha,\beta)}_{k,l,m}(x), \tag{50}$$

the extended p-deformed Konhauser polynomial and its inverse series relation as stated by

$$Z^{(\alpha)}_{n,m,p}(x;s) = \frac{(p+\alpha)_{sn,p}}{n!} \sum_{k=0}^{\lfloor n/m\rfloor} \frac{(-n)_{mk}}{(p+\alpha)_{sk,p}\ k!}\ x^{sk}, \tag{51}$$

$$\Leftrightarrow$$

$$x^{sn} = \sum_{k=0}^{mn} \frac{(-1)^k (p+\alpha)_{sn,p}\ n!}{(p+\alpha)_{sk,p}\ (mn-k)!}\ Z^{(\alpha)}_{k,m,p}(x;s). \tag{52}$$

In this last inverse pair, taking $l=1$, readily yields the extended p-deformed Laguerre polynomial(cf. [17, p. 201, 207]) and its inverse series.

Next, we show that, for $l \in \mathbb{N}$, p-polynomial (47) which satisfies differential equation

$$\Bigg[D \prod_{g=1}^{d} (pD+\beta_g-p) - x \prod_{i=1}^{m}\prod_{j=1}^{l}\prod_{h=1}^{c} \Bigg\{ (mD-n+i-1)\,(lpD+e+np+jp-p) \times (pD+\alpha_h) \Bigg\} \Bigg] y = 0. \tag{53}$$

In which $l=0$ yields the differential equation having one solution as the p-deformed Brafman polynomial. Additionally, the differential equation satisfied by (51) is derived in the form of

$$\left[D \left\{ \prod_{j=1}^{l} (lpD+\alpha+jp-lp) \right\} - x^l \prod_{i=1}^{m} (mD-n+i-1) \right] y = 0.$$

Taking $l=1$, this reduces to the differential equation satisfied by the extended p-deformed Laguerre polynomial.

Next, the generating function relation(GFR) of the *extended p-deformed* Wilson polynomials occurs from (41) which is given by

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+b+c+d-p)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} n!}\ W_{n,l,m,p}(x^2;a,b,c,d)\ t^n \\
&= (1-tp)^{\frac{-a-b-c-d+p}{p}} \frac{\Gamma_p(a+b)\Gamma_p(a+c)\Gamma_p(a+d)}{\Gamma_p(a+b+c+d-p)\Gamma_p(a+ix)\Gamma_p(a-ix)} \\
&\times {}_3\Psi_3^p \left[\begin{array}{c} (a+b+c+d-p, mp+lp), (a+ix,p), (a-ix,p); \\ (a+b,p), (a+c,p), (a+d,p); \end{array} \frac{(-t)^m}{(1-tp)^{m+l}} \right],
\end{aligned}$$

in which we have used the generalized p-Write function due to K. Gehlot et al. [10]. If $l \in \mathbb{N} \bigcup \{0\}$, then this reduces to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a+b+c+d-p)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} n!}\ W_{n,l,m,p}(x^2;a,b,c,d) t^n \\
&= (1-tp)^{\frac{-a-b-c-d+p}{p}} \\
&\quad \times {}_{m+l+2}F_3 \left(\left(\triangle_p (m+l, a+b+c+d-p), a+ix, a-ix \right), p, \right. \\
&\qquad \left. (a+b, a+c, a+d), p \right) \left(\frac{(m+l)^{m+l}(-t)^m}{(1-tp)^{m+l}} \right),
\end{aligned}$$

wherein $m+l=1,2$ for convergence.

In a similar manner, the GFR of the *extended p-deformed* Racah polynomials may be deduced. The following is the generating function relation of (45).

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}\ \mathcal{B}^{a}_{n,m,p}(x;l)\ t^n = (1-tp)^{\frac{-a}{p}} \sum_{k=0}^{\infty} (a)_{mk+lk,p} \gamma_k \left(\frac{x(-t)^m}{(1-tp)^{m+l}} \right)^k. \tag{54}$$

This gives the generating function relation of the p-deformed extended Jacobi polynomial, p-deformed Brafman polynomial, extended Konhauser polynomial and that of the Laguerre polynomial.

From the inverse series of the main theorem, certain summation formulas involving the extended p-deformed Wilson polynomials, Racah polynomials etc. are derived. One such sum is stated below. Taking $x = 0$ in the extended Racah polynomials, it is found that $R_{k,p}\left(0(c+d+p); a, b, c, d\right) = 1$ hence, the summation formula occurs in the form:

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ (mn)!} \sum_{k=0}^{mn} \frac{(-mn)_k(a+b+p+kp+lkp/m)}{(a+b+p+kp)_{ln+1,p}\ k!} \ = \ 1.$$

also, the summation formulas involving (45) and its particular cases may be deduced with the help of their inverse series relations. This is stated as

$$\sum_{n=0}^{\infty} \frac{1}{n!\ \gamma_n} \sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\ k!}\ \mathcal{B}^a_{k,m,p}(x;l) \ = \ e^x. \tag{55}$$

Further, assuming $|x| < 1$, it can be proved that

$$\sum_{n=0}^{\infty} \frac{1}{\gamma_n} \sum_{k=0}^{mn} \frac{(-mn)_k(a+kp+lkp/m)}{(a+kp)_{ln+1,p}(mn)!\ k!} \mathcal{B}^a_{k,m,p}(x;l) \ = \ \frac{1}{1-x}.$$

By assigning different values to x from $(-1, 1)$, a number of particular summation formulas can be derived.

The objective of chapter 5 is to provide q-analogue to the general class of polynomial (45) of Chapter 4 and its inverse series relation (46) by establishing a general q-inversion pair. The q-analogue of (45) [19] is defined as follows.

Definition 5. *For $a \in \mathbb{C},\ m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, 0 < q < 1$ and $p > 0$,*

$$\mathcal{B}^a_{n,m,p}(x|q;l) \ = \ \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl}(q^{-nl/m}; q^{l/m})_{mk}(q^{a+np}; q)_{\frac{kl}{p},p}\ \gamma_k x^k, \tag{56}$$

in which $l = r - m, r \in \mathbb{C} \setminus \{m\}$, and the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\leq u$.

This general class extends the q-extended Jacobi polynomials [4, Eq. (3.8)] and hence the q-Brafman polynomials and the little q-Jacobi polynomials [13, Eq.(3.12.1, p. 92)] (also [9, Ex. 1.32, p. 27]). As a limiting case, this general class also extends the q-Konhauser polynomials [1, Eq. (3.1), p. 3] and thereby the q-Laguerre polynomials [16].

The q-inversion theorem [19] is proved in the following form.

Theorem 8. $0 < q < 1,\ \lambda \in \mathbb{C} \setminus \{0\},\ \alpha \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N}$ *and* $p > 0$*,*

$$F(n) \ = \ \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mk\lambda(mk-2n+1)/2}\ \frac{(q^{\alpha+mk\lambda+p-np}; q)_{\infty,p}}{(q^\lambda; q^\lambda)_{n-mk}}\ G(k) \tag{57}$$

$$\Rightarrow$$

$$G(n) = \sum_{k=0}^{mn} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\lambda};q^{\lambda})_{mn-k}(q^{\alpha+mn\lambda-kp};q)_{\infty,p}} F(k) \tag{58}$$

and conversely, the series in (58) implies the series (57) if for $n \neq mr,\ r \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \frac{(1-q^{\alpha+k\lambda-kp})}{(q^{\lambda};q^{\lambda})_{n-k}(q^{\alpha+n\lambda-kp};q)_{\infty,p}} F(k) = 0. \tag{59}$$

This theorem besides the aforementioned polynomials, also invert the Askey-Wilson polynomials [13, Eq.(3.1.1), p. 63] (also [9, Ex. 2.11, p.51]) and the q-Racah polynomials [13, Eq.(3.2.1), p. 66] (also [9, Ex. 2.10, p. 51]).
The inverse series of these polynomials are deduced from this theorem which are stated below, in the same order.

$$\gamma_n x^n = \sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)};q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp};q)_{\frac{ln}{p}+1,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k} \mathcal{B}^a_{k,m,p}(x|q;l), \tag{60}$$

$$\begin{aligned} \frac{p_{n,l,m,p}(cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}} &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k} \\ &\times \frac{(abcdq^{np-p};q)_{kl/p,p}\,(ae^{i\theta};q)_{k,p}\,(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}} \end{aligned} \tag{61}$$

$$\Leftrightarrow$$

$$\begin{aligned} \frac{(ae^{i\theta};q)_{n,p}(ae^{-i\theta};q)_{n,p}}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n} &= \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k} \\ \times \frac{(1-abcdq^{kL+kp-p})\ a^k\ p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}}, \end{aligned} \tag{62}$$

$$\begin{aligned} R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q) &= \sum_{k=0}^{\lfloor n/m \rfloor} q^{kl} \frac{(q^{-n(l/m)};q^{l/m})_{mk}}{(q^{l/m};q^{l/m})_k} \\ &\times \frac{(abq^{np+p};q)_{\frac{kl}{p},p}(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}} \end{aligned} \tag{63}$$

$$\Leftrightarrow$$

$$\begin{aligned} \frac{(q^{-x};q)_{n,p}(cdq^{x+p};q)_{n,p}}{(aq^p;q)_{n,p}(bdq^p;q)_{n,p}(cq^p;q)_{n,p}(q^{l/m};q^{l/m})_n} &= \sum_{k=0}^{mn} q^{nkl} \frac{(q^{-mn(l/m)};q^{l/m})_k}{(q^{l/m};q^{l/m})_k} \\ \times \frac{(1-abq^{kL+kp+p})}{(abq^{kp+p};q)_{ln/p+1,p}(q^{l/m};q^{l/m})_{mn}} R_{k,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q). \end{aligned} \tag{64}$$

These p-deformed q-polynomials provide p-extension to a number of particular q-polynomials (see [13, p. 61, 62] for complete reducibility chart and [13, Ch. 3]). They include among several polynomials the q-Hahn, dual q-Hahn, continuous q-Hahn, continuous dual q-Hahn, q-Meixner-Pollaczek, q-Meixner, q-Krawtchouk and q-Charlier polynomials together with their inverse series relations.
Now for $p > 0$, define the difference operator $\theta_{q,p}f(x) = f(x) - f(xq^p)$, such that

$$\frac{\theta_{q,p}f(x)}{(1-q)x} = D_{q,p}f(x), \quad (|q| < 1) \tag{65}$$

would give a p-deformed q-derivative of $f(x)$ in which $p = 1$ yields the θ-form q-

derivative of $f(x)$(cf. [9, Ex.1.12, p.22] with $p=1$). In this notations, the q,p-differential equation of the extended q,p-Jacobi polynomial is given by

$$\Bigg[(1-r)xD_r\left\{\prod_{v=1}^{d}((1-q)xD_{q,p}+q^{p-\beta_v}-1)\right\}-xq^{pl}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{l}\prod_{t=1}^{l}\prod_{u=1}^{c}\prod_{v=1}^{d}\Big\{r^{(i-1-n)/m}$$
$$\times q^{\alpha_u-\beta_v+p+(e+p(n+s-1))/l}w^{j-1}\nu^{t-1}((1-r)xD_r+r^{-(i-1-n)/m}w^{1-j}-1)$$
$$\times(x(1-q)D_{q,p}+q^{-(e+p(n+s-1))/l}\nu^{1-t}-1)((1-q)xD_{q,p}+q^{-\alpha_u}-1)\Big\}\Bigg]y=0, \quad (66)$$

where $q^{pl/m}=r$, w is m^{th} root of unity and ν is l^{th} root of unity. From this, the differential equations for p-deformed q-Brafman polynomials, p-deformed q-Konhauser polynomial and p-deformed q-Laguerre polynomial can be obtained.

Now, using the first series in general inversion theorem, GFR for the general class of q-polynomials (56) is obtained in the form:

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(a;q)_{n,p}\frac{F(n)}{(q^{l/m};q^{l/m})_n}\ t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{mk}q^{ln(n-1)/2m}\frac{(a;q)_{n+mk+\frac{kl}{p},p}}{(q^{l/m};q^{l/m})_n}\ G(k)\ t^{n+mk}.\end{aligned} \quad (67)$$

From this, the GFR of $\mathcal{B}^{a}_{n,m,p}(x|q;l)$ occurs as follows.

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}\frac{(a;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\mathcal{B}^{a}_{n,m,p}(x|q;l)\ t^n\\ =\ &\sum_{k=0}^{\infty}(-1)^{mk}(a;q)_{mk+\frac{kl}{p},p}\ {}_1\phi_1\left(aq^{mkp+kl};0;p\right)\left(t\middle|q,q^{l/m}\right)\ \gamma_k\ \left((-t)^m x\right)^k.\end{aligned} \quad (68)$$

This contains the GFR of $\mathcal{F}^{(e)}_{n,m,p,l}[(\alpha);(\beta):x|q]$, $p_{n,m,p,l}(x;a,b;q)$, etc. The GFRs of the polynomials (61) and (63) from the general GFR (67) are obtained as follows.

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}(abcdq^{-p};q)_{n,p}\frac{p_{n,l,m,p}(cos\theta;a,b,c,d|q)a^n}{(ab;q)_{n,p}(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}\ t^n\\ =\ &\sum_{k=0}^{\infty}\frac{(abcdq^{-p};q)_{mk+\frac{kl}{p},p}(ae^{i\theta};q)_{k,p}(ae^{-i\theta};q)_{k,p}}{(ab;q)_{k,p}(ac;q)_{k,p}(ad;q)_{k,p}(q^{l/m};q^{l/m})_k}\\ &\times\ {}_1\phi_1\left(abcdq^{mkp+kl-p};0;p\right)(t|q,q^{l/m})(-t)^{mk},\end{aligned}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}q^{ln(n-1)/2m}\frac{(abq^p;q)_{n,p}}{(q^{l/m};q^{l/m})_n}\ R_{n,m,p,l}(q^{-x}+cdq^{x+1};a,b,c,d|q)\ t^n\\ =\ &\sum_{k=0}^{\infty}\frac{(q^{-x};q)_{k,p}(cdq^{x+p};q)_{k,p}(abq^p;q)_{mk+\frac{kl}{p},p}}{(aq^p;q)_{k,p}(bdq^p;q)_{k,p}(cq^p;q)_{k,p}(q^{l/m};q^{l/m})_k}\\ &\times\ {}_1\phi_1\left(abq^{p+mkp+kl};0;p\right)(t|q,q^{l/m})\ (-t)^{mk}.\end{aligned}$$

Next, the use of the inverse series of the theorem and in particular, the inverse series (60) of the general class (56), is used to deduce certain summation formulas with the assumption that $\gamma_n \neq 0, \forall n = 0, 1, 2, \ldots$. Thus, from

$$\frac{1}{\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)}; q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp}; q)_{\frac{ln}{p}+1,p}(q^{l/m}; q^{l/m})_{mn}(q^{l/m}; q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) \;=\; x^n, \qquad (69)$$

on multiplying both sides by $(a;q)_n/(q;q)_n$ and taking the summation from $n=0$ to ∞ and then using the q-Binomial theorem [9] with $|x|<1$, one finds the sum:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n\gamma_n}\sum_{k=0}^{mn} \frac{q^{nkl}(q^{-mn(l/m)}; q^{l/m})_k(1-q^{a+k(l/m)+kp})}{(q^{a+kp}; q)_{\frac{ln}{p}+1,p}(q^{l/m}; q^{l/m})_{mn}(q^{l/m}; q^{l/m})_k}\mathcal{B}^a_{k,m,p}(x|q;l) \;=\; \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

Next, in the inverse series (62), using the q-sum

$${}_2\phi_1\left(a,b;c;q^p,\frac{c}{ab}\right) \;=\; \frac{(c/a;q)_{\infty,p}\,(c/b;q)_{\infty,p}}{(c;q)_{\infty,p}\,(c/ab;q)_{\infty,p}}, \qquad (70)$$

one arrives at the summation formula:

$$\sum_{n=0}^{\infty}\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$

$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}q^{n(b-a-2cos\theta)} \;=\; \frac{\left(be^{-i\theta};q\right)_{\infty,p}\left(be^{i\theta};q\right)_{\infty,p}}{(ab;q)_{\infty,p}\left(q^{b-a-2cos\theta};q\right)_{\infty,p}}.$$

In a similar manner, the summation formula from the inverse series inverse series (64) can be obtained. We multiply both sides of (62) by

$$\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}},$$

taking the sum from $n=0$ to j and then using the sum:

$${}_2\phi_1\left(q^{-np},b;c;q^p,\frac{cq^{np}}{b}\right) \;=\; \frac{(c/b;q)_{n,p}}{(c;q)_{n,p}}. \qquad (71)$$

on the right hand side, we obtain

$$\sum_{n=0}^{j}\frac{(ac;q)_{n,p}(ad;q)_{n,p}(q^{l/m};q^{l/m})_n}{(ae^{-i\theta};q)_{n,p}(q^p;q)_{n,p}}\sum_{k=0}^{mn}\frac{q^{nkl}(q^{-ln};q^{l/m})_k(1-abcdq^{kL+kp-p})}{(abcdq^{kp-p};q)_{\frac{ln}{p}+1,p}(ab;q)_{k,p}(ac;q)_{k,p}}$$

$$\times\frac{a^k p_{k,l,m,p}(cos\theta;a,b,c,d|q)}{(ad;q)_{k,p}(q^{l/m};q^{l/m})_{mn}(q^{l/m};q^{l/m})_k}(q^{-jp};q)_{n,p}(q^{jp}be^{-i\theta})^n \;=\; \frac{\left(be^{-i\theta};q\right)_{j,p}}{(ab;q)_{j,p}}.$$

In chapter 6, general class of polynomials

$$S_n(l,m,\alpha,\beta:x) \;=\; \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^{mk}\;\lambda_k}{\Gamma(1+\beta-n\alpha+lk)(n-mk)!}\;x^k, \qquad (72)$$

due Dalbhide and Dave [2], is extended in the light of p-Gamma function and p-Pochhammer symbol and derive its inverse series relation along with particular cases with the help of general inversion pair. The p-deformation of polynomial (72) [20] is define as follows

Definition 6. *For* $0 \leq \alpha \leq 1, \beta \in \mathbb{C},\ m \in \mathbb{N},\ n, l = m\alpha \in \{0\} \cup \mathbb{N}$ *and* $p > 0$*,*

$$S_{n,p}(l,m,\alpha,\beta : x) \;=\; \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^{mk}\ \lambda_k\ x^k}{\Gamma_p(p+\beta-pn\alpha+plk)(n-mk)!}, \tag{73}$$

in which the floor function $\lfloor r \rfloor = floor\ r$*, represents the greatest integer* $\leq r$*.*

This polynomial contains the p-deformed extended Jacobi polynomial:

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m,p}[(\alpha);(\beta):x] \;=\; \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)_{mk}(\alpha_1)_{k,p}\cdots(\alpha_c)_{k,p}}{(\beta+p-pn\alpha)_{lk,p}(\beta_1)_{k,p}\cdots(\beta_d)_{k,p}\ k!}\ x^k. \tag{74}$$

When $l = 0$, then it reduces to the p-deformed Brafman polynomial. Also, the extended p-deformed Konhauser polynomial: $Z^{(\alpha)}_n(x;l)$ and its evident case $l = 1$ yields the extended p-deformed Laguerre polynomial.
A general inversion pair is proved as

Theorem 9. *Let* $0 \leq \alpha \leq 1,\ n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N}$ *such that* αm *is a non negative integer and* $\beta \in \mathbb{C} \setminus \{0\},$ *then*

$$G(n) \;=\; \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{\Gamma_p(\beta+pmk\alpha-pn\alpha+p)(n-mk)!}\ F(k) \tag{75}$$

$$\Rightarrow$$

$$F(n) \;=\; \sum_{k=0}^{nm} \frac{(-1)^{mn-k}\ \beta\ \Gamma_p(\beta+pmn\alpha-pk\alpha)}{(mn-k)!}\ G(k), \tag{76}$$

and conversely, the series in (76) implies the series (75) if for $n \neq mr,\ r \in \mathbb{N}$*,*

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}\ \beta\ \Gamma_p(\beta+pn\alpha-pk\alpha)}{(n-k)!}\ G(k) \;=\; 0. \tag{77}$$

From this, the inverse series of (73) is obtained in the form:

$$\lambda_n x^n = \sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta : x). \tag{78}$$

From this, the inverse series of the extended Jacobi polynomial (74), and hence that of the p-deformed Brafman polynomial, and also, the inverse of the extended p-deformed Konhauser polynomials (51) and the extended deformed Laguerre polynomial can be deduced.
The differential equation of (74) is derived as follows.

$$\left[\theta\left\{\prod_{j=1}^{l}\prod_{s=1}^{d}(lp\theta+\beta-pn\alpha+jp-lp)(p\theta+\beta_s-p)\right\}\right.$$
$$\left.-x\prod_{i=1}^{m}\prod_{r=1}^{c}\left\{(m\theta-n+i-1)(p\theta+\alpha_r)\right\}\right]\mathcal{H}^{(\alpha,\beta)}_{n,l,m,p}[(\alpha);(\beta):x] = 0, \tag{79}$$

where $\theta = x\dfrac{d}{dx}$.
The generating function relation for the polynomial (74) is derived from the first

series of the theorem. It is given as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\ \mathcal{H}_{n,m\alpha,m,p}^{(\alpha,\beta)}[(\alpha);(\beta):x]\ t^n \\
=\ &\sum_{n=0}^{\infty} \frac{(-1)^{n\alpha}(-\beta)_{n\alpha,p}}{n!}\ t^n\ {}_cF_d\left((\alpha),p,(\beta),p\right)(x(-t)^m)\,. \qquad (80)
\end{aligned}$$

By defining the p-deformed generalized Bessel function as

$$J_{\nu,p}^{\mu}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma_p(p+\nu p+kp\mu)k!}(-x)^k,$$

one more GFR is derived for the p-deformed generalized Konhauser polynomial in the form:

$$\sum_{n=0}^{\infty} \frac{Z_{n,m,p}^{\alpha}(x;l)}{(p+\alpha)_{nl,p}}t^n = \frac{e^t\Gamma_p(p+\alpha)}{\Gamma_p(1+p)}J_{\frac{\alpha}{p},p}^{l}\left((-1)^{m+1}x^l t^m\right). \qquad (81)$$

The particular cases $p=1,\ m=1$ provides the generating function relation obtained in [22, Ex. 65, p. 198].

The summation formulas implied by the inverse series are next considered here. The first sum is corresponding to the inverse series relation of (78):

$$\frac{1}{\lambda_n}\sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta:x)\ =\ x^n, \qquad (82)$$

assuming $\lambda_n \neq 0, \forall n \in \mathbb{N}$. Now multiplying both sides by $1/n!$ and taking summation from $n=0$ to ∞, this gives for all x,

$$\sum_{n=0}^{\infty}\frac{1}{n!\lambda_n}\sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta:x)\ =\ e^x, \qquad (83)$$

and with $|x|<1, \lambda_n \neq 0 \forall n$, the following sum is obtained.

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_n}\sum_{k=0}^{nm} \frac{(-1)^k\ \beta\ \Gamma_p(\beta+pnl-pk\alpha)}{(mn-k)!}\ S_{k,p}(l,m,\alpha,\beta:x)\ =\ \frac{1}{1-x}. \qquad (84)$$

From this the summation formulas involving the particular polynomials can be deduced.

Taking $\lfloor n/m \rfloor = N$ in (73) and converting it to the monic form $\widetilde{S}_{n,p}(l,m,\alpha,\beta:x)$, we get

$$\widetilde{S}_{n,p}(l,m,\alpha,\beta:x) = \sum_{k=0}^{N} \delta_k\ x^k,$$

where

$$\delta_k\ =\ \frac{(-1)^{(k-N)m}\ \Gamma_p(p+\beta-pn\alpha+plN)\lambda_k\ (n-mN)!\ x^k}{\Gamma_p(p+\beta-pn\alpha+plk)\lambda_N(n-mk)!}.$$

With this $\delta_k,\ C\left(\widetilde{S}_{n,p}(l,m,\alpha,\beta:x)\right)$ assumes the form as stated in definition of companion matrix. The eigen values of this matrix will be then precisely the zeros of $\widetilde{S}_{n,p}(l,m,\alpha,\beta:x)$ (see [15, p. 39]).

Chapter-7 incorporates the q-extension to the polynomial (73) and derive analogous properties. A p-deformed q-extension of the general class of p-polynomial (73) is defined as follows.

Definition 7. *For $\beta \in \mathbb{C}$, $r, \alpha \in \mathbb{C}/\{0\}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $|q| < 1$ and $p > 0$,*

$$S_{n,p,r}(l, m, \alpha, \beta : x|q^\alpha) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mkr\alpha(mk-2n+1)/2} \frac{(\beta q^{rlk+p-n\alpha p}; q^\alpha)_{\infty,p}}{(q^{r\alpha}; q^{r\alpha})_{n-mk}} \lambda_k x^k, \quad (85)$$

where $\lfloor n/m \rfloor = floor\ n/m$, represents the greatest integer $\leq n/m$.

When $q \to 1$ and $r = p$, this coincides with (73). Moreover, this general class of q, p-polynomials ((85) above) extends the general class of p-deformed q-polynomials (56) of chapter 5 by taking $\alpha = 1$ and $r = \lambda$.

The special case $\lambda_n = (\alpha_1; q)_{n,p}(\alpha_2; q)_{n,p} \cdots (\alpha_c; q)_{n,p}/((\beta_1; q)_{n,p}(\beta_2; q)_{n,p} \cdots (\beta_d; q)_{n,p}$ $(q^{r\alpha^2}; q^{r\alpha^2})_n)$ with r is replaced by $r\alpha$, yields the *tribasic* p-deformed extended q-Jacobi polynomial (cf. [6, Eq.(1.2), p. 77] with $p = 1$):

$$\begin{aligned} &\mathcal{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha); (\beta) : xq^{r\alpha l}|q^\alpha] \\ &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(q^{-nr\alpha^2}; q^{r\alpha^2})_{mk}(\alpha_1; q)_{k,p}(\alpha_2; q)_{k,p} \cdots (\alpha_c; q)_{k,p}}{(\beta q^{p-pn\alpha}; q^\alpha)_{\frac{rlk}{p},p}(\beta_1; q)_{k,p}(\beta_2; q)_{k,p} \cdots (\beta_d; q)_{k,p}(q^{r\alpha^2}; q^{r\alpha^2})_k} (xq^{r\alpha l})^k. \quad (86) \end{aligned}$$

When $\beta \to \infty$ in q^β, this polynomial reduces to the p-deformed q-Brafman polynomial. Further, replacing x by $x^l q^{nl}$, letting $q^\beta \to 0$, taking $r\alpha = l \in \mathbb{N}$ and $\lambda_n = q^{ln(\alpha+1)-lmn+ln(ln-1)/2}/((p\alpha; q)_{nl,p}(q^l; q^l)_{mn})$ in (85), we obtain the extended p-deformed q-Konhauser polynomial (cf. [1] with $p = 1$ and $m = 1$). The obvious specialization $l = 1$ is the extended p-deformed q-Laguerre polynomial. Here, a general inversion pair [20] is proved as

Theorem 10. *If $r, \alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $p > 0$ and $0 < q < 1$, then*

$$\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^{mk} q^{mkr\alpha(mk-2n+1)/2} \frac{(q^{mkr\alpha+\beta+p-n\alpha p}; q^\alpha)_{\infty,p}}{(q^{r\alpha}; q^{r\alpha})_{n-mk}}\ G(k) \quad (87) \\ &\Rightarrow \\ G(n) &= \sum_{k=0}^{mn} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1 - q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p}; q^\alpha)_{\infty,p}(q^{r\alpha}; q^{r\alpha})_{mn-k}}\ F(k), \quad (88) \end{aligned}$$

and conversely, the series in (88) implies the series (87) if for $n \neq mv$, $v \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1 - q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{nr\alpha+\beta+p-(k+1)\alpha p}; q^\alpha)_{\infty,p}(q^{r\alpha}; q^{r\alpha})_{n-k}}\ F(k) = 0. \quad (89)$$

The inverse series relation of (85) is obtained in the form:

$$\lambda_n\ x^n = \sum_{k=0}^{mn} (-1)^k q^{kr\alpha(k-1)/2} \frac{(1 - q^{kr\alpha+\beta+p-(k+1)\alpha p})\ S_{k,p,r}(l, m, \alpha, \beta : x|q^\alpha)}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p}; q^\alpha)_{\infty,p}(q^{r\alpha}; q^{r\alpha})_{mn-k}(q^{r\alpha}; q^{r\alpha})_k}. \quad (90)$$

From this the inverse series of $\mathcal{H}^{(\alpha,\beta)}_{k,m,l,p,r}[(\alpha); (\beta) : xq^{r\alpha l}|q^\alpha], Z^{(\alpha)}_{k,m,p}(x; l|q)$ can be deduced.

The p-deformed q-differential equations for the particular cases of the polynomial (85), are derived with the aid of the q-difference operator $\theta_{q,p}f(x) = f(x) - f(xq^p)$ for $p > 0$, and the p-deformed q-derivative (Eq.(65)) of $f(x)$ (cf. [9, Ex.1.12, p.22] with $p = 1$). The p-deformed tribasic q-differential equation of (86), with $rl/p = h \in \mathbb{N}$, w is m^{th} root of unity and ν is h^{th} root of unity, is derived as follows.

Corollary 11. *The polynomial* $y = \mathcal{H}^{(\alpha,\beta)}_{n,m,l,p,r}[(\alpha);(\beta) : xq^{r\alpha l}|q^{\alpha}]$ *satisfies the equation:*

$$\Bigg[(1-q^{r\alpha^2})xD_{q^{r\alpha^2}}\Bigg\{\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{v=1}^{d}((1-q^{\alpha})xD_{q^{\alpha},p} + q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}\nu^{1-t} - 1)$$
$$\times((1-q)xD_{q,p} + q^{p-\beta_v} - 1)\Bigg\} - xq^{r\alpha l}\prod_{i=1}^{m}\prod_{j=1}^{m}\prod_{s=1}^{h}\prod_{t=1}^{h}\prod_{u=1}^{c}\prod_{v=1}^{d}\Bigg\{q^{p\alpha-(\beta+p-pn\alpha+p\alpha(s-1))/h}$$
$$\times\, q^{p+((i-1)r\alpha^2-nr\alpha^2)/m-\beta_v+\alpha_u}w^{j-1}\nu^{1-t}((1-q)xD_{q,p} + q^{-\alpha_u} - 1)((1-q^{r\alpha^2})xD_{q^{r\alpha^2}}$$
$$+q^{-((i-1)r\alpha^2-nr\alpha^2)/m}w^{1-j} - 1)\Bigg\}\Bigg]y = 0. \quad (91)$$

From this, the p-deformed q-differential equations of the p-deformed q-Brafman polynomial, p-deformed q-Konhauser polynomial and the extended p-deformed q-Laguerre polynomial can be deduced.

The generating function relation for the polynomial (85) is derived in the form:

$$\sum_{n=0}^{\infty} q^{r\alpha n(n-1)/2}\,\frac{S_{n,p,r}(l,m,\alpha,\beta : x|q^{\alpha})}{(\beta q^{p-pn\alpha})_{\infty,p}}\,t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{mk}q^{r\alpha n(n-1)/2}}{(\beta q^{p-(n+mk)\alpha p};q^{\alpha})_{\frac{rlk}{p\alpha},p}(q^{r\alpha};q^{r\alpha})_n}\,\lambda_k\,x^k t^{n+mk}. \quad (92)$$

From this, the generating function relation of (74) and other polynomials can be obtained.

Next, writing the inverse series (90) for all $\lambda_n \neq 0$, in the form

$$\frac{1}{\lambda_n}\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}S_{k,p,r}(l,m,\alpha,\beta : x|q^{\alpha})$$
$$= x^n \quad (93)$$

and multiplying both sides by $1/(q^{r\alpha};q^{r\alpha})_n$, and taking summation from $n = 0$ to ∞, one obtains

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_n\,(q^{r\alpha};q^{r\alpha})_n}\,\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}(q^{r\alpha};q^{r\alpha})_k}$$
$$\times S_{k,p,r}(l,m,\alpha,\beta : x|q^{\alpha}) \;=\; e_{q^{r\alpha}}(x), \quad (94)$$

where $|x| < 1$. Next, using the summation formula of ${}_1\phi_1$ given by [9, Eq.(II.5), p.236], one finds the summation formula:

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(a;q)_n}{\lambda_n(c;q)_n(q;q)_n}\left(-\frac{c}{a}\right)^n\sum_{k=0}^{mn}\frac{(-1)^k q^{kr\alpha(k-1)/2}(1-q^{kr\alpha+\beta+p-(k+1)\alpha p})}{(q^{mnr\alpha+\beta+p-(k+1)\alpha p};q^{\alpha})_{\infty,p}(q^{r\alpha};q^{r\alpha})_{mn-k}}$$

$$\times \frac{1}{(q^{r\alpha};q^{r\alpha})_k} S_{k,p,r}\left(l,m,\alpha,\beta:\frac{c}{a}\middle|q^{\alpha}\right) = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \qquad (95)$$

The reducibility of summation formulas (94) and (95) corresponding to the particular cases may be obtained by the suitable substitutions of the parameters involved and the sequence $\{\lambda_n\}$ as stated above sections.

Taking $\lfloor n/m \rfloor = N$ in (85) and converting it to the monic form as denoted by $\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})$, we get

$$\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha}) = \sum_{j=0}^{N} \delta_j \; x^j,$$

where

$$\delta_j = (-1)^{\sigma m} \; q^{r\alpha m^2\sigma(j+N)/2+m\sigma/2-mn\sigma} \; \frac{(\beta q^{lj-pn\alpha+p};q^{r\alpha})_{\infty,p} \; (q^{r\alpha};q^{r\alpha})_{n-mN}\lambda_j}{(\beta q^{lN-pn\alpha+p};q^{r\alpha})_{\infty,p} \; (q^{r\alpha};q^{r\alpha})_{n-mj}\lambda_N},$$

and $\sigma = j-N$. With this δ_j, $C\left(\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})\right)$ assumes the form as stated in Definition of Companion matrix. The eigen values of this matrix will be then precisely the zeros of $\widetilde{S}_{n,p,r}(l,m,\alpha,\beta:x|q^{\alpha})$ (see [15, p. 39]).

Chapters 8 and 9 incorporate the p-deformation of Riordan's classified inverse pairs and their q-analogues. For that the inversion theorems of the preceding chapters are used. In chapter 8, using the inversion theorems of chapters 2 and 4 and their alternative forms, the p-deformed extended pairs of Riordan's six classes are deduced. In chapter 9, their basic analogues are obtained with the help of the theorems of chapters 3, 5 and 7. In the following, certain p-deformed extended Gould's classes, Simpler Legendre classes and the Legendre-Chebyshev classes are tabulated in Table 1; whereas their basic analogues are tabulated in Table 2.

Table 1 : The p-deformed extension of Riordan's inverse series
$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{A_{n,k,p}}{(n-mk)!} G(k)$; $G(n) = \sum_{k=0}^{mn} (-1)^{mn-k} \frac{B_{n,k,p}}{(mn-k)!} F(k)$

$A_{n,k}$	$B_{n,k}$	Riordan's classes
$\frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{(a+lk-kp)}{(a+lmn-kp)} \times \frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	Gould Class
$\frac{(a+ln-np+p)}{(a+lmk-np+p)} \times \frac{\Gamma_p(a+lmk-mkp+p)}{\Gamma_p(a+lmk-np+p)}$	$\frac{\Gamma_p(a+lmn-kp+p)}{\Gamma_p(a+lmn-mnp+p)}$	Gould class
$\frac{\Gamma_p(a+np+2mk-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{(a+2k+p)}{(a+2mn-mnp+kp+p)} \times \frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	Simpler Legendre
$\frac{(a+2n)}{(a+2mk+np-mkp)} \times \frac{\Gamma_p(a+2mk+np-mkp+p)}{\Gamma_p(a+2mk+p)}$	$\frac{\Gamma_p(a+2mn+p)}{\Gamma_p(a+2mn-mnp+kp+p)}$	Simpler Legendre

Table 1 : *Continue*

$A_{n,k}$	$B_{n,k}$	Riordan's classes
$\frac{(a+cn)}{(a+cmk+np-mkp)} \times \frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	Legendre-Chebyshev
$\frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{(a+ck)}{(a+cmn+mnp-kp)} \times \frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	Legendre-Chebyshev
$\frac{\Gamma_p(a+cmk+np-mkp+p)}{\Gamma_p(a+cmk+p)}$	$\frac{(a+ck+p)}{(a+cmn-mnp+kp+p)} \times \frac{\Gamma_p(a+cmn+p)}{\Gamma_p(a+cmn-mnp+kp+p)}$	Legendre-Chebyshev
$\frac{(a+cn+p)}{(a+cmk+mkp-np+p)} \times \frac{\Gamma_p(a+cmk+p)}{\Gamma_p(a+cmk-np+mkp+p)}$	$\frac{\Gamma_p(a+cmn+mnp-kp+p)}{\Gamma_p(a+cmn+p)}$	Legendre-Chebyshev

Table 2 : p-Deformed extension of certain q-Riordan inverse's pairs

$$F(n) = \sum_{k=0}^{\lfloor n/m \rfloor} q^{\beta mk(mk-1)/2}\ A_{n,k}\ G(k);\ G(n) = \sum_{k=0}^{mn} (-1)^{mn+k} q^{\beta k(k-2mn+1)/2}\ B_{n,k}\ F(k)$$

β	$A_{n,k}$	$B_{n,k}$	q-Riordan Classes
$-l$	$\frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+lmk-np}{p},p}} \times \frac{1}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(1-q^{\alpha+lk-kp})}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+lmn-kp-p}{p},p}}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold class
$-l$	$\frac{(1-q^{\alpha+ln-np+p})}{(q^p;q)_{\frac{\alpha+lmk-np+p}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+lmk-mkp}{p},p}}{(q^{-l};q^{-l})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+lmn-kp}{p},p}}{(q^p;q)_{\frac{\alpha+lmn-mnp}{p},p}} \times \frac{1}{(q^{-l};q^{-l})_{mn-k}}$	q-Gold class
$p-2$	$\frac{(q^p;q)_{\frac{\alpha+np+2mk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times \frac{1}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(1-q^{\alpha+2k+p})}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp+p}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class
$p-2$	$\frac{(1-q^{\alpha+2n})}{(q^p;q)_{\frac{\alpha+2mk}{p},p}} \times \frac{(q^p;q)_{\frac{\alpha+np-mkp+2mk-p}{p},p}}{(q^{(p-2)};q^{(p-2)})_{n-mk}}$	$\frac{(q^p;q)_{\frac{\alpha+2mn}{p},p}}{(q^p;q)_{\frac{\alpha+2mn-mnp+kp}{p},p}} \times \frac{1}{(q^{(p-2)};q^{(p-2)})_{mn-k}}$	q-Simpler Legendre Class
$p-c$	$\frac{(1-q^{\alpha+cn})}{(q^p;q)_{\frac{\alpha+cmk}{p},p}}$	$\frac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp}{p},p}}$	q-Legendre-Chebyshev

Table 2 : *Continue*

β	$A_{n,k}$	$B_{n,k}$	q-Riordan Classes
	$\times\dfrac{(q^p;q)_{\frac{\alpha+np+cmk-mkp-p}{p},p}}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\times\dfrac{1}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	Class
$-p-c$	$\dfrac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^p;q)_{\frac{\alpha+cmk+mkp-np}{p},p}}$ $\times\dfrac{1}{(q^{-(c+p)};q^{-(c+p)})_{n-mk}}$	$\dfrac{(1-q^{\alpha+ck})}{(q^p;q)_{\frac{\alpha+cmn}{p},p}}$ $\times\dfrac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp-p}{p},p}}{(q^{-(c+p)};q^{-(c+p)})_{mn-k}}$	q-Legendre -Chebyshev Class
$p-c$	$\dfrac{(q^p;q)_{\frac{\alpha+np+cmk-mkp}{p},p}}{(q^p;q)_{\frac{\alpha+cmk}{p},p}}$ $\times\dfrac{1}{(q^{(p-c)};q^{(p-c)})_{n-mk}}$	$\dfrac{(1-q^{\alpha+ck+p})}{(q^p;q)_{\frac{\alpha+cmn-mnp+kp+p}{p},p}}$ $\times\dfrac{(q^p;q)_{\frac{\alpha+cmn}{p},p}}{(q^{(p-c)};q^{(p-c)})_{mn-k}}$	q-Legendre-Chebyshev Class
$-p-c$	$\dfrac{(1-q^{\alpha+cn+p})}{(q^p;q)_{\frac{\alpha+cmk+mkp-np+p}{p},p}}$ $\times\dfrac{(q^p;q)_{\frac{\alpha+cmk}{p},p}}{(q^{(c+p)};q^{(c+p)})_{n-mk}}$	$\dfrac{(q^p;q)_{\frac{\alpha+cmn+mnp-kp}{p},p}}{(q^p;q)_{\frac{\alpha+cmn}{p},p}}$ $\times\dfrac{1}{(q^{(c+p)};q^{(c+p)})_{mn-k}}$	q-Legendre-Chebyshev Class

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