Chapter 1

Introduction

1.1 Introduction

The field of Special Functions is enriched with hundreds of particular functions of Mathematical Physics, Chemistry, Astronomy, Statistics, Dynamics, Fiber optics, Systems and Control; including the classical orthogonal polynomials and their various generalizations (see [1], [40], [46], [47], [49], [53], [54], [62], [63], [64]). Besides this, Special functions also occur in certain Mathematics branches such as Lie algebra, Number theory, Combinatorics, Approximation theory etc. The Gauss function and its straight generalization which is well known as the generalized hypergeometric function encompass a vast number of particular functions as well as a number of polynomials that arise from a physical phenomenon or a natural phenomenon. For example, the Bessel function arise from the vibrating string or vibration of elastic rubber membrane. The Laguerre polynomials occur in Hydrogen atom study, the Legendre polynomials are associated with the Gravitational/electric potential theory. In geophysical and astrophysical applications, the Legendre polynomial of odd degree $(2k-1)$ given by [39, Eq. (2.2), p. 2205]

$$P_{2k-1}(\cos\theta) = \sum_{j=1}^{k} \frac{(-1)^{k-j}[2(k+j)-3]!!}{2^{k-1}(k-j)!(j-1)!(2j-1)!!}(\cos\theta)^{2j-1}$$

is usually employed to represent an axisymmetric flow on a spherical surface. This polynomial plays vital role in the construction of the geostrophic polynomial (see for more detail [39]) which helps in studying the fluid motion in the interiors of rapidly rotating planets or stars. Amongst the eminent mathematicians who contributed to a great extent in the development, the names of Carl F. Gauss, Leonhard Euler, Weierstrass, E. E. Kummer, John Wallis, G. Szego, H. Bateman, P. E. Bedient, L. Carlitz, R. P. Boas, W. N. Bailey, A. Erdelyi, E. D. Rainville, P. Humbert, D. Dikinson, H. W. Gould, H. L. Krall, O. Frink, R. Askey, W. A. Al-Salam, H. Exton, J. L. Burchnall, T. W. Chaundy, G. Gasper, H. M. Srivastava, T. S. Chihara, M. Rahman, M. E. H. Ismail, Tom Koorwinder, A. Verma, R. P. Agarwal are worth mentioning.

1.2 Definitions and Formulae

We shall use the following definitions and formulas in the work.

Definition 1.2.1. *The gamma function is defined as [53]:*

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, \mathrm{dt}; \quad \Re(\mathrm{z}) > 0. \tag{1.2.1}$$

Definition 1.2.2. *The Pochhammer symbol is defined as [53, 62]*

$$(\lambda)_n = \begin{cases} (\lambda)(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & \text{if } \ n \in \mathbb{N}, \\ \dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \ \text{if } \ n \in \mathbb{C}. \end{cases} \tag{1.2.2}$$

Here, $(\lambda)_n$ is also called the factorial function.
If $\lambda = 1$ then it reduces to $n!$, that is $(1)_n = n!$.

Definition 1.2.3. *The binomial coefficient.*

$$\binom{\lambda}{n} = \frac{(\lambda)(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n\,(-\lambda)_n}{n!},$$

or equivalently [53],

$$\binom{\lambda}{n} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)\; n!}.$$

Remark 1.2.1. *For $0 \le k \le n$,*

$$(\lambda)_{n-k} = \frac{(-1)^k\,(\lambda)_n}{(1-\lambda-n)_k}.$$

For $\lambda = 1$, it gives

$$(-n)_k = \frac{(-1)^k\, n!}{(n-k)!}, \quad 0 \le k \le n.$$

Definition 1.2.4. *The generalized hypergeometric function is denoted and defined by [53]:*

$${}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r; & z \\ b_1, b_2, \ldots, b_s; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{z^n}{n!},$$

where $b_1, b_2, \ldots, b_s$ are neither zero nor negative integers.

The series converges absolutely for $|z| < \infty$ if $r \leq s$ and if $r = s + 1$, then series converges absolutely for $|z| < 1$. On the boundary of the circle $|z| = 1$, the series converges provided that the $\Re(\sum b_j - \sum a_i) > 0$.
The operator differential equation satisfied by $w = {}_rF_s[z]$ is given by [53]

$$\left[\theta \prod_{j=1}^{s} (\theta + b_j - 1) - z \prod_{i=1}^{r} (\theta + a_i)\right] w = 0,$$

where $\theta = z\dfrac{d}{dz}$ and $r \leq s + 1$.
The following are the useful double series identities [62].

$$\begin{aligned} \sum_{k=0}^{nm} \sum_{j=0}^{\lfloor k/m \rfloor} A(k,j) &= \sum_{j=0}^{n} \sum_{k=0}^{mn-mj} A(k+mj,j) \\ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} A(k,n-mk) \end{aligned}$$

The binomial series is [53]

$$\sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} = (1-z)^{-a}, \quad |z| < 1. \tag{1.2.3}$$

1.3 p-Gamma function

This function was introduced by Rafael Díaz and Eddy Pariguan [17]. In fact, the occurrence of the product of the form $x(x+p)(x+2p)\cdots(x+(n-1)p)$ in combinatorics of creation and annihilation operators [16, 18] and the perturbative computation of Feynman integrals [15] led them to generalize the gamma function in the form involving the above factors.
Diaz at el.[17] defined the Pochhammer p-symbol for $z \in \mathbb{C}$, $p \in \mathbb{R}$ and $n \in \mathbb{N}$, which is given by

$$(z)_{n,p} = z(z+p)(z+2p)\cdots(z+(n-1)p). \tag{1.3.1}$$

The following are varied representations of p-gamma function [17]. It is given in Euler integral form as follows. For $z \in \mathbb{C}$, $\Re(z) > 0$ and $p > 0$,

$$\Gamma_p(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^p}{p}}\, dt. \tag{1.3.2}$$

Note 1.3.1. *For $p = 2$, the function*

$$\Gamma_2(x) = \int_0^\infty t^{x-1} e^{-\frac{t^2}{2}} dt$$

is the Gaussian integral.

In Pochhammer p-symbol, it is given by

$$\Gamma_p(x) = \lim_{n\to\infty} \frac{n! p^n (np)^{\frac{x}{p}-1}}{(x)_{n,p}}, \quad p > 0,\ x \in \mathbb{C}\backslash p\mathbb{Z}^-.$$

Just as for the gamma function, Γ_p also admits the infinite product representation:

$$\frac{1}{\Gamma_p(x)} = x p^{-\frac{x}{p}} e^{\frac{x}{p}\gamma} \prod_{n=1}^{\infty} \left[\left(1 + \frac{x}{np}\right) e^{\frac{x}{np}} \right].$$

The Stirling's formula has p-generalization:

$$\Gamma_p(x+1) = (2\pi)^{\frac{1}{2}} (px)^{-\frac{1}{2}} x^{\frac{x+1}{p}} e^{-\frac{x}{p}} + O\left(\frac{1}{x}\right).$$

When $p = 1$, this reduces to $\Gamma(z)$. The following properties follow from (1.3.1) and (1.3.2).

$$\Gamma_p(z+p) = z\Gamma_p(z), \tag{1.3.3}$$

$$\Gamma_p(p) = 1, \tag{1.3.4}$$

$$(z)_{k,p} = \frac{\Gamma_p(z+kp)}{\Gamma_p(z)}, \tag{1.3.5}$$

$$(z)_{n-k,p} = \frac{(-1)^k (z)_{n,p}}{(p-z-np)_{k,p}}, \tag{1.3.6}$$

$$(z)_{mn,p} = m^{mn} \prod_{j=1}^{m} \left(\frac{z+jp-p}{m}\right)_{n,p}, \tag{1.3.7}$$

$$(z)_{m+n,p} = (z)_{m,p}\ (z+mp)_{n,p}. \tag{1.3.8}$$

When $p = 1$, these identities get reduce to the corresponding properties of the function $\Gamma(z)$ and the Pochhammer symbol $(z)_n$ [53, 62]. We shall make use of the notation

$$\triangle_p(m;n) = \prod_{j=1}^{m} \left(\frac{n+jp-p}{m}\right). \tag{1.3.9}$$

For $p = 1$, this gives the usual notation $(\triangle_1(m;n) =)\triangle(m;n)$ which indicates the array of m parameters

$$\frac{n}{m}, \frac{n+p}{m}, \ldots, \frac{n+mp-p}{m}.$$

p-Version of the well known Bohr-Mollerup theorem is given by [17, Theorem 7]

Theorem 1.3.1. *Let $f(x)$ be a positive valued function defined on $(0, \infty)$. Assume that $f(p) = 1, f(x+p) = xf(x)$ and f is logarithmically convex, then $f(x) = \Gamma_p(x), \forall x \in (0, \infty)$.*

Diaz at el.[17] also proposed the following generalization of the hypergeometric series in the form of Pochhammer p-symbol (cf. [53] with $p = 1$), given by

$$_rF_s(a,k,b,l)(x) \;=\; \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_r)_{n,k_r}}{(b_1)_{n,l_1}(b_2)_{n,l_2}\cdots(b_s)_{n,l_s}n!}x^n, \tag{1.3.10}$$

where $a = (a_1, a_2, \cdots, a_r) \in \mathbb{C}^r,\ k = (k_1, k_2, \cdots, k_r) \in (\mathbb{R}^+)^r,\ b = (b_1, b_2, \cdots, b_s) \in \mathbb{C}^s \setminus (l\mathbb{Z}^-)^s$ and $l = (l_1, l_2, \cdots, l_s) \in (\mathbb{R}^+)^s$.

This series converges for all x if $r \leq s$, and diverges if $r > s+1,\ x \neq 0$. If $r = s+1$, then the series converges for $|x| < \frac{l_1 l_2 \cdots l_s}{k_1 k_2 \cdots k_r}$. It also satisfies the differential equation [17]:

$$\left[D\prod_{i=1}^{s}(l_iD + b_i - l_i) - \ x\ \prod_{j=1}^{r}(k_jD + a_j)\right] y \;=\; 0, \tag{1.3.11}$$

where $D = x\dfrac{d}{dx}$. For $p > 0,\ a \in \mathbb{C}$ and $|x| < \dfrac{1}{p}$, Diaz at el.[17] showed that

$$\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}x^n = (1 - px)^{-\frac{a}{p}}. \tag{1.3.12}$$

This may be regarded as the p-deformed binomial series. It is interesting to note that the radius of convergence of this series can be enlarged or diminished by choosing p smaller or larger; unlike in the classical theory of radius of convergence of the binomial series which is fixed and is unity. This attracted us to study p-deformation of certain Special functions, in particular the polynomials' systems.

Diaz at el.[17] gave the p-Beta function

$$\beta_p(a,b) = \frac{1}{p}\int_0^1 t^{\frac{a}{p}-1}(1-t)^{\frac{b}{p}-1}dt = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}, \tag{1.3.13}$$

where $\Re(a,b) \neq 0, -p, -2p, \cdots$. Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

where z is non integral, has p-deformed version

$$\Gamma_p(z)\Gamma_p(p-z) = \frac{\pi}{p \sin \frac{\pi z}{p}},$$

if $z/p \notin \mathbb{Z}$. If $P(x)$ is a polynomial in x of degree less than n $(n \geq 1)$, then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(a+bk) \quad = \quad 0. \tag{1.3.14}$$

The Companion matrix of the monic polynomial is defined as follows.

Definition 1.3.1. *If a monic polynomial* $f(x) \in \mathbb{C}[X]$ *and* $f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_{k-1} x^{k-1} + x^k$, *then the* $k \times k$ *matrix, called the Companion matrix of* $f(x)$ *is denoted and defined by [48, p. 39]*

$$C(f(x)) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \ldots & -\delta_{k-1} \end{bmatrix}.$$

Lemma 1.3.1. *If* $f \in K[x]$ *is non constant and* $A = C(f(x))$ *then* $f(A) = 0$.

1.4 Inverse series relations

A series is said to be the inverse series of a given series if one of the series when substituted into the other, simplifies to the expression involving the Kronecker delta :

$$\delta_{nk} = \begin{cases} 0, & if \quad k \neq n \\ 1, & if \quad k = n \end{cases}.$$

To illustrate this, consider the inverse pair

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \quad b_n = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} a_k.$$

Here, if second series is substituted into the first series then the inner sum simplifies to the form

$$\sum_{k=j}^{n}(-1)^{k+j}\binom{n}{k}\binom{k}{j} = \delta_{nj},$$

thus proving one side of inverse relation. The proof of the converse part is similar. The prior works of Gould[21–24] on inverse series evoked an influx of intrigue that was reflected in works of Jhon Riordan [55, Chapter-2] who studied the inverse series relations and classified them into several classes namely the simplest inverse pairs, the Gould classes, the simpler Chebyshev classes, the Chebyshev classes, the simpler Legendre classes and the Legendre - Chebyshev classes of inverse series. All these classes are recorded in the following tables.

TABLE 1.1: The Simplest inverse relations

(1)	$a_n = \sum_{k=0}^{n}\binom{n}{k}b_k$	$b_n = \sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}a_k$
(2)	$a_n = \sum_{k=n}\binom{k}{n}b_k$	$b_n = \sum_{k=n}(-1)^{k+n}\binom{k}{n}a_k$
(3)	$a_n = \sum_{k=0}\binom{p-k}{p-n}b_k$	$b_n = \sum_{k=0}(-1)^{k+n}\binom{p-k}{p-n}a_k$
(4)	$a_n = \sum_{k=0}^{n}\binom{n+p}{k+p}b_k$	$b_n = \sum_{k=0}^{n}(-1)^{k+n}\binom{n+p}{k+p}a_k$
(5)	$a_n = \sum_{k=0}^{n}\binom{k+p}{n+p}b_k$	$b_n = \sum_{k=n}(-1)^{k+n}\binom{k+p}{n+p}a_k$
(6)	$a_n = \sum_{k=1}\frac{n!}{k!}\binom{n-1}{k-1}b_k$	$b_n = \sum_{k=1}(-1)^{k+n}\frac{n!}{k!}\binom{n-1}{k-1}a_k$

([55, Tabel 2.1, p.49])

TABLE 1.2: The Gould classes of inverse relations

$a_n = \sum A_{n,k}b_k;\ b_n = \sum(-1)^{k+n}B_{n,k}a_k$

Sr. No.	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{p+qk-k}{n-k}$	$\frac{p+qk-k}{p+qn-k}\binom{p+qn-k}{n-k}$
(2)	$\frac{p+qn-n+1}{p+qk-n+1}\binom{p+qk-k}{n-k}$	$\binom{p+qn-k}{n-k}$
(3)	$\binom{p+qn-n}{k-n}$	$\frac{p+qn-n}{p+qk-n}\binom{p+qk-n}{k-n}$
(4)	$\frac{p+qk-k+1}{p+qn-k+1}\binom{p+qn-n}{k-n}$	$\binom{p+qk-n}{k-n}$

([55, Tabel 2.2, p.52])

TABLE 1.3: The Simpler Chebyshev classes

(1)	$a_n = \sum \binom{n}{k} b_{n-2k}$	$b_n = \sum (-1)^k \frac{n}{n-k} \binom{n-k}{k} a_{n-2k}$
(2)	$a_n = \sum \frac{n-2k+1}{n-k+1} \binom{n}{k} b_{n-2k}$	$b_n = \sum (-1)^k \binom{n-k}{k} a_{n-2k}$
(3)	$a_n = \sum \binom{n+2k}{k} b_{n+2k}$	$b_n = \sum (-1)^k \frac{n+2k}{n+k} \binom{n+k}{k} a_{n+2k}$
(4)	$a_n = \sum \frac{n+1}{n+k+1} \binom{n+2k}{k} b_{n+2k}$	$b_n = \sum (-1)^k \binom{n+k}{k} a_{n+2k}$
(5)	$a_n = \sum \binom{n-k}{k} b_{n-k}$	$b_n = \sum (-1)^k \frac{n-k}{n+k} \binom{n+k}{k} a_{n-k}$
(6)	$a_n = \sum \frac{n+1}{n-2k+1} \binom{n-k}{k} b_{n-k}$	$b_n = \sum (-1)^k \binom{n+k}{k} a_{n-k}$

([55, Tabel 2.3, p.62])

TABLE 1.4: The Chebyshev classes of inverse series relations

$a_n = \sum A_{n,k} b_{n+ck};\ b_n = \sum B_{n,k} a_{n+ck}$

Sr. No.	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{n}{k}$	$\frac{n}{n+ck+k} \binom{n+ck+k}{k}$
(2)	$\frac{n+ck+1}{n-k+1} \binom{n}{k}$	$\binom{n+ck+k}{k}$
(3)	$\binom{n+ck}{k}$	$\frac{n+ck}{n+k} \binom{n+k}{k}$
(4)	$\frac{n+1}{n+ck-k+1} \binom{n+ck}{k}$	$\binom{n+k}{k}$

([55, Tabel 2.4, p.63])

TABLE 1.5: The Simpler Legendre inverse relations

(1)	$a_n = \sum \binom{p+n+k}{n-k} b_k$	$b_n = \sum (-1)^{n+k} \frac{p+2k+1}{p+n+k+1} \binom{p+2n}{n-k} a_k$
(2)	$a_n = \sum \binom{p+2n}{n-k} b_k$	$b_n = \sum (-1)^{n+k} \frac{p+2n}{p+n+k} \binom{p+n+k}{n-k} a_k$
(3)	$a_n = \sum \binom{p+n+k}{k-n} b_k$	$b_n = \sum (-1)^{n+k} \frac{p+2n+1}{p+n+k+1} \binom{p+2k}{k-n} a_k$
(4)	$a_n = \sum \binom{p+2k}{k-n} b_k$	$b_n = \sum (-1)^{n+k} \frac{p+2k}{p+n+k} \binom{p+n+k}{k-n} a_k$
(5)	$a_n = \sum \binom{p+2n}{k} b_{n-2k}$	$b_n = \sum (-1)^k \frac{p+2n}{p+2n-3k} \binom{p+2n-3k}{k} a_{n-2k}$
(6)	$a_n = \sum \frac{p+2n-4k+1}{p+2n-k+1} \binom{p+2n}{k} b_{n-2k}$	$b_n = \sum (-1)^k \binom{p+2n-3k}{k} a_{n-2k}$

([55, Tabel 2.5, p.68])

TABLE 1.6: The Legendre - Chebyshev classes of inverse relations
$a_n = \sum A_{n,k}\, b_k;\ b_n = \sum (-1)^{n+k} B_{n,k}\, a_k$

Sr. No.	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{p+cn}{n-k}$	$\frac{p+cn}{p+ck}\binom{p+n+ck-k-1}{n-k}$
(2)	$\binom{p+cn}{k-n}$	$\frac{p+cn}{p+ck}\binom{p+ck+k-n-1}{k-n}$
(3)	$\binom{p+ck}{n-k}$	$\frac{p+ck}{p+cn}\binom{p+cn+n-k-1}{n-k}$
(4)	$\binom{p+ck}{k-n}$	$\frac{p+ck}{p+cn}\binom{p+cn-n+k-1}{k-n}$
(5)	$\frac{p+ck+1}{p+cn-n+k+1}\binom{p+cn}{n-k}$	$\binom{p+n+ck-k}{n-k}$
(6)	$\frac{p+ck+1}{p+cn+n-k+1}\binom{p+cn}{k-n}$	$\binom{p+ck+k-n}{k-n}$
(7)	$\frac{p+cn+1}{p+ck-n+k+1}\binom{p+ck}{n-k}$	$\binom{p+cn+n-k}{n-k}$
(8)	$\frac{p+cn+1}{p+ck-n+k+1}\binom{p+ck}{n-k}$	$\binom{p+cn+n-k}{n-k}$

([55, Tabel 2.6, p.69])

1.5 q-Analogue

Recently, Rafael Díaz and Carolina Teruel [18] introduced two parameter deformation of the classical gamma function by means of the q,k-Pochhammer symbol[18, Def. 4, p.121] which is denoted and defined by

$$[t]_{n,k} = \prod_{j=0}^{n-1} [t+jk]_q, \quad t>0,\ k>0, \tag{1.5.1}$$

where $[a]_q = 1-q^a$. Using this, the q,k-generalized gamma function[18, Def. 6, p.122] was defined in the form:

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ k>0,$$

where $(1+x)_{q,k}^t = \dfrac{(1+x)_{q,k}^{\infty}}{(1+xq^{kt})_{q,k}^{\infty}}$ and $(x+y)_{q,k}^n = \prod_{j=0}^{n-1}(x+yq^{jk})$.
Alternatively [18, Lem-2, p.122],

$$\Gamma_{q,k}(t) = \frac{\left(1-q^k\right)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}\,(1-q)^{\frac{t}{k}-1}}, \quad t>0,\ k>0. \tag{1.5.2}$$

As $q \to 1^-$ from within the interval $(0, 1)$, the defining expressions in (1.5.1) and (1.5.2) yield the k-generalized Pochhammer symbol $(t)_{n,k}$ and the k-deformed classical gamma function $\Gamma_k(t)$([18, p.119] and [17]). We replace in the present work, k by p, and write $(q^t; q)_{n,p}$ in stead of $[t]_{n,k}$, where $t \in \mathbb{C}$. In the notations of (1.5.1) and (1.5.2), we have

$$(z;q)_{n,p} = (1-q^z)(1-q^{z+p})(1-q^{z+2p})\cdots(1-q^{z+(n-1)p}), \quad z \in \mathbb{R}, n \in \mathbb{Z}^+, \tag{1.5.3}$$

$$\Gamma_{q,p}(z) = \frac{(q^p;q)_{\infty,p}(1-q)^{1-z/p}}{(q^z;q)_{\infty,p}}, \quad |z| > 0, \tag{1.5.4}$$

where

$$(a;q)_{n,p} = \begin{cases} 1, & if\ n = 0, \\ (1-a)(1-aq^p)\cdots(1-aq^{p(n-1)}), & if\ n \in \mathbb{Z}_{>0}, \\ [(1-aq^{-p})(1-aq^{-2p})\cdots(1-aq^{np})]^{-1}, & if\ n \in \mathbb{Z}_{<0}, \\ (a;q)_{\infty,p}/(aq^{np};q)_{\infty,p}, & if\ n \in \mathbb{C}, \end{cases}$$

and

$$(q^\alpha;q)_{\infty,p} = \prod_{n=0}^{\infty}(1-q^{\alpha+np}), \ |q| < 1.$$

It may be mentioned here that for a parameter $\alpha \in \mathbb{C}$, $q^\alpha \equiv \alpha$.
In what follows, the following formulas will be used in the work for $0 < q < 1$

$$\Gamma_{q,p}(p) = 1, \tag{1.5.5}$$

$$(a;q)_{n,p} = \frac{\Gamma_{q,p}(a+np)}{\Gamma_{q,p}(a)}, \ n \in \mathbb{N}, \tag{1.5.6}$$

$$(a;q)_{n,p} = \frac{(a;q)_{\infty,p}}{(aq^{pn};q)_{\infty,p}}, \ n \in \mathbb{C}, \tag{1.5.7}$$

$$(a;q)_{m+n,p} = (a;q)_{m,p}(aq^{mp};q)_{n,p}, m, n \in \mathbb{N}, \tag{1.5.8}$$

$$(aq^{-np};q)_{n,p} = (-1)^n a^n q^{-pn(n+1)/2}\left(\frac{q^p}{a};q\right)_{n,p}, \tag{1.5.9}$$

$$(a;q)_{n-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2-nkp}\frac{(a;q)_{n,p}}{(q^{p-np}/a;q)_{k,p}}, \tag{1.5.10}$$

$$(a;q)_{-k,p} = \left(-\frac{1}{a}\right)^k q^{pk(k+1)/2}\frac{1}{(q^p/a;q)_{k,p}}, \tag{1.5.11}$$

$$(a;q)_{km,p} = (a, aq^p, aq^{2p}, \cdots, aq^{p(m-1)}; q^m)_{k,p}, \tag{1.5.12}$$

$$(a^m;q^m)_{k,p} = (a, a\omega, a\omega^2, \cdots, a\omega^{m-1}; q)_{k,p}, \text{ where } \omega = e^{2\pi i/m}. \tag{1.5.13}$$

When $p = 1$, these formulas get reduced to those listed in [19, Appendix I, p.233-234]. $q-$Binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

Next, We define q-analogue of (1.3.10) in the form of bibasic series with $k_1 = k_2 = \ldots = k_r = l_1 = l_2 = \ldots = l_s = p \in \mathbb{R}^+$ as follows:

Definition 1.5.1. *If (a) stands for the array of r parameters $a_1,\ a_2, \cdots,\ a_r \in C^r$, (b) stands for the array of s parameters $b_1,\ b_2, \cdots,\ b_s \in C^s \setminus (Z^-)^s$, $p,\ \alpha \in \mathbb{R}^+$ and $|q| < 1$ then*

$$\begin{aligned} &{}_r\phi_s((a);\ (b)\ ;q^p)(x|q,q^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_{n,p}(a_2;q)_{n,p}\cdots(a_r;q)_{n,p}}{(b_1;q)_{n,p}(b_2;q)_{n,p}\cdots(b_s;q)_{n,p}(q^\alpha;q^\alpha)_n} \left((-1)^n q^{\alpha\binom{n}{2}}\right)^{1+s-r} x^n. \end{aligned} \tag{1.5.14}$$

Note 1.5.1. *The limiting case:*

$$\lim_{q\to 1^-} {}_r\phi_s((a);\ (b)\ ;q^p)\left((1-q)^{1+s-r}x\middle|\, q,q^\alpha\right) = {}_rF_s((a),p,(b),p)(x).$$

The series behaves similarly as the series (1.3.10). In fact, if

$${}_r\phi_s((a);\ (b)\ ;p)(x|q,q^\alpha) = \sum_{n=0}^{\infty} A_n x^n,$$

then by d' Alembert's ratio test,

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{(1-a_1q^{np})(1-a_2q^{np})\cdots(1-a_rq^{np})q^{\alpha n(s+1-r)}}{(1-b_1q^{np})(1-b_2q^{np})\cdots(1-b_sq^{np})(1-q^{\alpha(n+1)})}x\right|.$$

From this, it follows that the series converges for all x if $r \le s$, and it diverges when $r > s+1$ and $x \ne 0$. If $r = s+1$, then it converges for $|x| < 1$.

1.6 Basic inverse series relations

In order to illustrate a q-analogue of inverse pair, consider the series

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k.$$

Then its inverse series is given by

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} \; q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \; a_k$$

and vise versa. B. I. Dave[9, 11] derived basic analogues of Riordan's classes of inverse series relation and they are listed below

TABLE 1.7: Basic analogues of the simplest inverse pair

$$F(n) = \sum q^{k(k-1)/2} A_{n,k} G(k); \; G(n) = \sum (-1)^{k+n} q^{k(k-2n+1)/2} B_{n,k} F(k)$$

Sr. No.	$A_{n,k}$	$B_{n,k}$
(1)	$\begin{bmatrix} n \\ k \end{bmatrix}_q$	$\begin{bmatrix} n \\ k \end{bmatrix}_q$
(2)	$q^k \begin{bmatrix} k \\ n \end{bmatrix}_q$	$q^k \begin{bmatrix} k \\ n \end{bmatrix}_q$
(3)	$\begin{bmatrix} \alpha-k \\ \alpha-n \end{bmatrix}_q$	$\begin{bmatrix} \alpha-k \\ \alpha-n \end{bmatrix}_q$
(4)	$\begin{bmatrix} \alpha+n \\ \alpha+k \end{bmatrix}_q$	$\begin{bmatrix} \alpha+n \\ \alpha+k \end{bmatrix}_q$
(5)	$q^k \begin{bmatrix} \alpha+k \\ \alpha+n \end{bmatrix}_q$	$q^k \begin{bmatrix} \alpha+k \\ \alpha+n \end{bmatrix}_q$
(6)	$\frac{(q;q)_{n,p}}{(q;q)_{k,p}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$	$\frac{(q;q)_{n,p}}{(q;q)_{k,p}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$

([11, Table 1, p.17])

TABLE 1.8: Basic analogues of the Gould Classes

$$f(n) = \sum_{k=0}^{n} q^{\beta k(k-1)/2} A_{n,k} g(k); \; g(n) = \sum_{k=0}^{n} (-1)^{k+n} q^{\beta k(k-2n-1)/2} B_{n,k} f(k)$$

Sr. No.	β	$A_{n,k}$	$B_{n,k}$
(1)	$-m$	$\frac{(q;q)_{\alpha+mk-k}}{(q;q)_{\alpha+mk-n}(q^\beta;q^\beta)_{n-k}}$	$\frac{q^{-mk}(1-q^{\alpha+mk-k})(q;q)_{\alpha+nm-k-1}}{(q;q)_{\alpha+nm-n}(q^\beta;q^\beta)_{n-k}}$
(2)	$-m$	$\frac{(1-q^{\alpha+1+mn-n})(q;q)_{\alpha+mk-k}}{(q;q)_{\alpha+mk-n+1}(q^\beta;q^\beta)_{n-k}}$	$\frac{q^{-mk}(q;q)_{\alpha+mn-k}}{(q;q)_{\alpha+nm-n}(q^\beta;q^\beta)_{n-k}}$
(3)	m	$\frac{q^{mk}(1-q^{\alpha+mn-n})(q;q)_{\alpha+mk-n-1}}{(q;q)_{\alpha+mk-k}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q;q)_{\alpha+mn-n}}{(q;q)_{\alpha+mn-k}(q^\beta;q^\beta)_{k-n}}$
(4)	m	$\frac{q^{mk}(q;q)_{\alpha+mk-n}}{(q;q)_{\alpha+mk-k}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+1+mk-k})(q;q)_{\alpha+mn-n}}{(q;q)_{\alpha+mn-k+1}(q^\beta;q^\beta)_{k-n}}$

([11, Table 2, p.18])

TABLE 1.9: Basic analogues of the simpler Chebyshev Classes
$F(n) = \sum \gamma^k A_{n,k} G(n + bk);\ G(n) = \sum (-\gamma)^k q^{\beta k(k-1)/2} B_{n,k} F(n + bk)$

Sr. No.	b	β	γ	$A_{n,k}$	$B_{n,k}$
(1)	-2	-1	-1	$\frac{(1-q^n)(q;q)_{n-k-1}}{(q;q)_{n-2k}(q^\beta;q^\beta)_k}$	$\frac{(q;q)_n}{(q;q)_{n-k}(q^\beta;q^\beta)_k}$
(2)	-2	-1	-1	$\frac{(q;q)_{n-k}}{(q;q)_{n-2k}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{1+n-2k})(q;q)_n}{(q;q)_{n-k+1}(q^\beta;q^\beta)_k}$
(3)	2	1	1	$\frac{(q;q)_{n+2k}}{(q;q)_{n+k}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+2k})(q;q)_{n+k}}{(1-q^{n+k})(q;q)_n(q^\beta;q^\beta)_k}$
(4)	2	1	1	$\frac{(q;q)_{n+k}}{(q;q)_n(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n-1})(q;q)_{n+2k}}{(1-q^{n+k+1})(q;q)_{n+k}(q^\beta;q^\beta)_k}$
(5)	-1	-2	1	$\frac{(q;q)_{n-k}}{(q;q)_{n-2k}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n-k})(q;q)_{n+k-1}}{(q;q)_n(q^\beta;q^\beta)_k}$
(6)	-1	-2	1	$\frac{(1-q^{n-1})(q;q)_{n-k}}{(q;q)_{n-2k+1}(q^\beta;q^\beta)_k}$	$\frac{(q;q)_{n+k}}{(q;q)_n(q^\beta;q^\beta)_k}$

([11, Table 3, p.17])

TABLE 1.10: Basic analogues of the Chebyshev Classes

$$F(n) = \sum C_{n,k,p} \frac{(q;q)_k}{(q^\beta;q^\beta)_k} G(n+bk);$$

$$G(n) = \sum (-1)^k q^{\beta k(k-1)/2} B_{n,k} \frac{(q;q)_k}{(q^\beta;q^\beta)_k} F(n+bk)$$

Sr. No.	β	$A_{n,k}$	$B_{n,k}$
(1)	$c+1$	$\frac{(1-q^n)(q;q)_{n+ck+k}}{(1-q^{n+ck+k})(q;q)_{n+ck}(q^\beta;q^\beta)_k}$	$\frac{(q;q)_n}{(q;q)_{n-k}(q^\beta;q^\beta)_k}$
(2)	$c+1$	$\frac{(q;q)_{n+ck+k}}{(q;q)_{n+ck}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+1+ck})(q;q)_n}{(1-q^{n+1-k})(q;q)_{n-k}(q^\beta;q^\beta)_k}$
(3)	$c-1$	$\frac{(q;q)_{n+ck}}{(q;q)_{n+ck-k}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{n+ck})(q;q)_{n+k}}{(1-q^{n+k})(q;q)_n(q^\beta;q^\beta)_k}$
(4)	$c-1$	$\frac{(1-q^{n+1})(q;q)_{n+ck}}{(1-q^{n+1+ck-k})(q;q)_{n+ck-k}(q^\beta;q^\beta)_k}$	$\frac{(q;q)_{n+k}}{(q;q)_n(q^\beta;q^\beta)_k}$

([11, Table 4, p.19])

TABLE 1.11: Basic analogues of the simpler Legendre classes I

$$F(n) = \sum q^{(\beta k^2+k)/2} A_{n,k} G(k);$$
$$G(n) = \sum (-1)^{k+n} q^{\beta k(k-2n+1)/2} B_{n,k} F(k)$$

Sr. No.	β	$A_{n,k}$	$B_{n,k}$
(1)	-1	$\frac{(q;q)_{\alpha+k+n}}{(q;q)_{\alpha+2k}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+2k+1})(q;q)_{\alpha+2k+1}}{(q;q)_{\alpha+n+k+1}(q^\beta;q^\beta)_{n-k}}$
(2)	-1	$\frac{(1-q^{\alpha+2n})(q;q)_{\alpha+k+n-1}}{(q;q)_{\alpha+2k}(q^\beta;q^\beta)_{n-k}}$	$\frac{(q;q)_{\alpha+2n}}{(q;q)_{\alpha+n+k}(q^\beta;q^\beta)_{n-k}}$
(3)	1	$\frac{(1-q^{\alpha+2n+1})(q;q)_{\alpha+2k}}{(1-q^{\alpha+1+k+n})(q;q)_{\alpha+k+n}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q;q)_{\alpha+n+k}}{(q;q)_{\alpha+2n}(q^\beta;q^\beta)_{k-n}}$
(4)	1	$\frac{(q;q)_{\alpha+2k}}{(q;q)_{\alpha+k+n}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+2k})(q;q)_{\alpha+n+k}}{(1-q^{\alpha+n+k})(q;q)_{\alpha+2n}(q^\beta;q^\beta)_{k-n}}$

([11, Table 5, p.19])

TABLE 1.12: Basic analogues of the simpler Legendre classes II

$$F(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(q;q)_k A_{n,k}}{(q^{-3};q^{-3})_k} G(n-2k);$$
$$G(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{-3k(k-1)/2} \frac{(q;q)_k B_{n,k}}{(q^{-3};q^{-3})_k} F(n-2k)$$

Sr. No.	β	$C_{n,k,p}$	$D_{n,k,p}$
(1)	-3	$\frac{(1-q^{\alpha+2n})(q;q)_{\alpha+2n-3k}}{(1-q^{\alpha+2n-3k})(q;q)_{\alpha+2n-4k}(q^\beta;q^\beta)_k}$	$\frac{(q;q)_{\alpha+2n}}{(q;q)_{\alpha+2n-k}(q^\beta;q^\beta)_k}$
(2)	-3	$\frac{(q;q)_{\alpha+2n-3k}}{(q;q)_{\alpha+2n-4k}(q^\beta;q^\beta)_k}$	$\frac{(1-q^{\alpha+2n-4k+1})(q;q)_{\alpha+2n}}{(1-q^{\alpha+2n-k+1})(q;q)_{\alpha+2n-k}(q^\beta;q^\beta)_k}$

([11, Table 6, p.19])

TABLE 1.13: Basic analogues of the Legendre-Chebyshev classes

$$F(n) = \sum \gamma^{n+k} q^{(\alpha k+\beta)/2} A_{n,k} G(k);\; G(n) = \sum (-\gamma)^{n+k} q^{k(\alpha k-2\alpha n-\beta)/2} B_{n,k} F(k)$$

Sr. No.	b	α	β	γ	$A_{n,k}$	$B_{n,k}$
(1)	-1	$1-c$	$c-1$	-1	$\frac{(1-q^{\alpha+cn})(q;q)_{\alpha+ck+n-k-1}}{(q;q)_{\alpha+ck}(q^\beta;q^\beta)_{n-k}}$	$\frac{(q;q)_{\alpha+cn}}{(q;q)_{\alpha+cn-n+k}(q^\beta;q^\beta)_{n-k}}$
(2)	1	$c+1$	$c+1$	-1	$\frac{(1-q^{\alpha+cn})(q;q)_{\alpha+ck+k-n-1}}{(q;q)_{\alpha+ck}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q;q)_{\alpha+cn}}{(q;q)_{\alpha+cn-k+n}(q^\beta;q^\beta)_{k-n}}$
(3)	-1	$-c-1$	$c+1$	1	$\frac{(q;q)_{\alpha+ck}}{(q;q)_{\alpha+ck-n+k}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+ck})(q;q)_{\alpha+cn+n-k-1}}{(q;q)_{\alpha+cn}(q^\beta;q^\beta)_{n-k}}$

Table 1.13: – *Continue*

Sr. No.	b	α	β	γ	$A_{n,k}$	$B_{n,k}$
(4)	1	$c-1$	$c-1$	1	$\frac{(q;q)_{\alpha+ck}}{(q;q)_{\alpha+ck-k+n}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+ck})(q;q)_{\alpha+cn+k-n-1}}{(q;q)_{\alpha+cn}(q^\beta;q^\beta)_{k-n}}$
(5)	-1	$1-c$	$c-1$	-1	$\frac{(q;q)_{\alpha+ck+n-k}}{(q;q)_{\alpha+kc}(q^\beta;q^\beta)_{n-k}}$	$\frac{(1-q^{\alpha+ck+1})(q;q)_{\alpha+cn}}{(q;q)_{\alpha+cn-n+k+1}(q^\beta;q^\beta)_{n-k}}$
(6)	1	$c+1$	$c+1$	-1	$\frac{(q;q)_{\alpha+ck+k-n}}{(q;q)_{\alpha+ck}(q^\beta;q^\beta)_{k-n}}$	$\frac{(1-q^{\alpha+ck+1})(q;q)_{\alpha+cn}}{(q;q)_{\alpha+cn+n-k+1}(q^\beta;q^\beta)_{k-n}}$
(7)	-1	$-c-1$	$c+1$	1	$\frac{(1-q^{\alpha+cn+1})(q;q)_{\alpha+ck}}{(q;q)_{\alpha+ck-n+k+1}(q^\beta;q^\beta)_{n-k}}$	$\frac{(q;q)_{\alpha+cn+n-k}}{(q;q)_{\alpha+cn}(q^\beta;q^\beta)_{n-k}}$
(8)	1	$c-1$	$c-1$	-1	$\frac{(1-q^{\alpha+cn+1})(q;q)_{\alpha+ck}}{(q;q)_{\alpha+ck+n-k+1}(q^\beta;q^\beta)_{k-n}}$	$\frac{(q;q)_{\alpha+cn+k-n}}{(q;q)_{\alpha+cn}(q^\beta;q^\beta)_{k-n}}$

([11, Table 7, p.20])

Having motivated by the works of R. Diaz and C. Teruel[18], and R. Diaz and E. Pariguan [17], we provide here the extension to certain classical polynomials along with their q-versions in the sense of p-deformation and derive their inverse series relations. Further, we obtain differential equation and the generating function relations of these polynomials; and using the inverse series, we deduce certain summation formulas involving the corresponding polynomials.