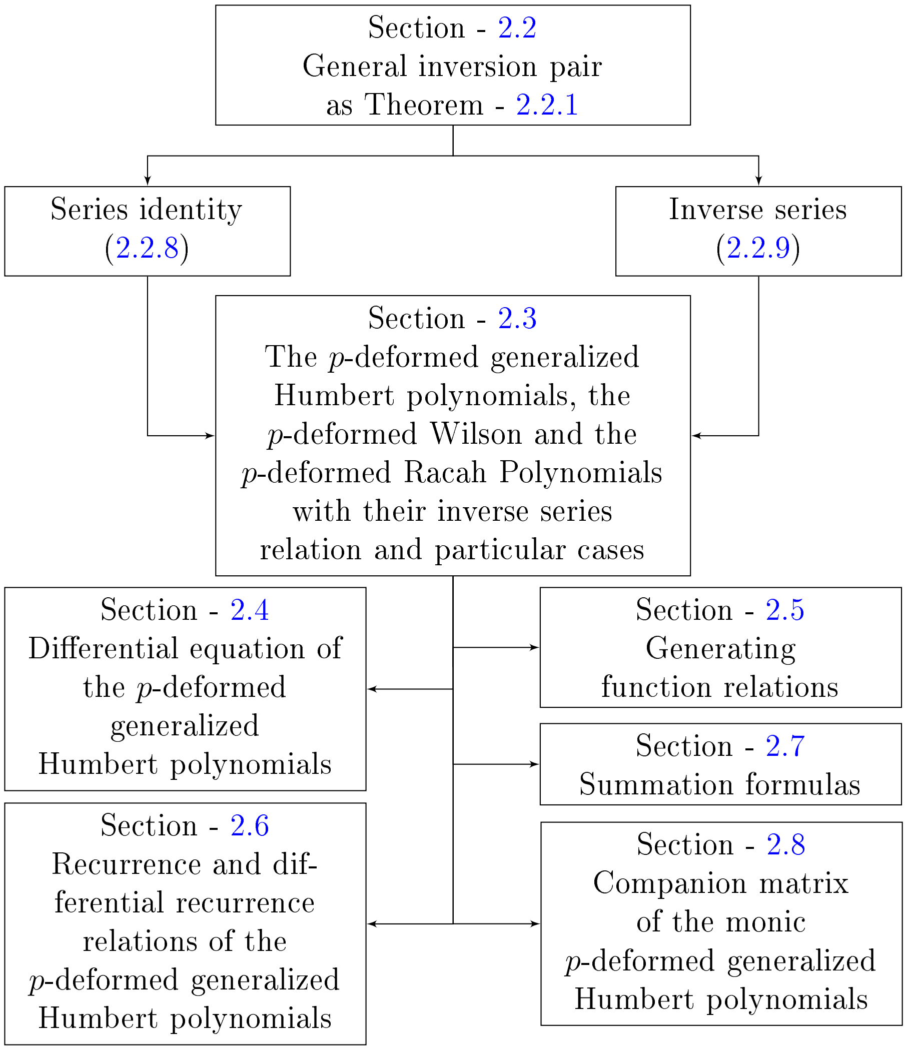
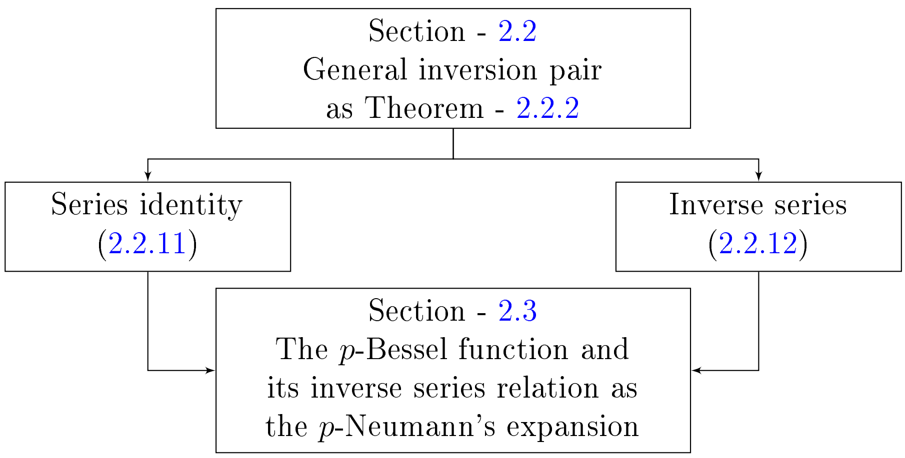
Chapter 2

The p-deformed polynomials' system - I





2.1 Introduction

The aim of this chapter is to extend the general class $\{P_n(m,x,\gamma,s,c); n = 0,1,2,\ldots\}$ defined explicitly by

$$P_n(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor n/m \rfloor} \binom{s-n+mk}{k}\binom{s}{n-mk}\gamma^k c^{s-n+mk-k}(-mx)^{n-mk}, \tag{2.1.1}$$

due to H. W. Gould[26, Eq.5.11, p.707] in p-deformed version and obtain its properties such as the inverse series relation, differential equation, generating function relations, differential recurrence relations and summation formulas. This polynomials occur as the coefficients of t^n in a series expansion of $(c-mxt+\gamma t^m)^s$ as follows

$$(c-mxt+\gamma t^m)^s = \sum_{n=0}^{\infty} P_n(m,x,\gamma,s,c)t^n. \tag{2.1.2}$$

H. W. Gould obtained its inverse series relation given by [26, Eq.5.12, p.707]

$$\frac{(-m)^n}{(c)^{n-s}}\binom{s}{n}x^n = \sum_{k=0}^{\lfloor n/m \rfloor} \binom{s-n+k}{k}\frac{(-\gamma)^k(s-n+mk)}{c^k(s-n+k)}P_{n-mk}(m,x,\gamma,s,c). \tag{2.1.3}$$

It appears from the history [51] that as long ago as in 1722, Liouville discussed a paradox arising from the theories due to Galileo and Huygens related to isochronal property of the cycloid curve. Liouville obtained the power series expansion of $(p^2-2qx-x^2)^{-1/2}$ in powers of x. Nielsen[51] showed that the coefficients $f_n(p,q)$ of this expansion are connected with the Legendre polynomial $P_n(x)$ by the relation:

$$f_n(p,q) = i^{-n}p^{-n-1}P_n(iq/p),\ i^2 = -1.$$

Here, taking $p = 1$, and replacing x by t, we get the expression $(1-2qt-t^2)^{-1/2}$ which occurs in the potential theory corresponding to the Legendre polynomial. In fact, in potential theory, a Newtonian potential function [33] may be written as

$$U = \iiint_V \frac{\lambda}{r}\, dV,$$

where $r = (1-2qt-t^2)^{-1/2}$. It is noteworthy that the potential function U is associated with Laplace's equation [66]:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0.$$

The Humbert polynomials: $\Pi^{\nu}_{n,m}(x)$ occurred in his study(see [30–32]) of more generalized potential problems associated with the extended Laplace equation [66]

$$\frac{\partial^3 U}{\partial x^3}+\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial z^3}-3\frac{\partial^3 U}{\partial x\partial y\partial z}=0.$$

This polynomial occurs as the coefficient of the power series expansion of the function $(1-mxt+t^m)^{-\nu}$ in powers of t (cf. Liouville's function $(p^2-2qx-x^2)^{-1/2}$). These coefficients are explicitly represented[2, p.360] by

$$\Pi^{\nu}_{n,m}(x) = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(mx)^{n-mk}}{\Gamma(1-\nu-n+mk-k)(n-mk)!k!} \tag{2.1.4}$$

which is known as the Humbert polynomial due to P. Humbert[30]. This polynomial is seen to be included in the class $\{P_n(m,x,\gamma,s,c); n=0,1,2,\ldots\}$ when $s=-\nu$ and $c=1$. Its inverse series is given by

$$\frac{(-mx)^n}{n!}=\sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k(-\nu-n+mk)\Gamma(-\nu-n+k+1)}{(-\nu-n+k)k!}\;\Pi^{\nu}_{n-mk,m}(x). \tag{2.1.5}$$

The other special cases of the polynomials (2.1.1) are the Kinney polynomial, the Pincherle polynomial, the Gegenbauer polynomial and the Legendre polynomial (see [26]). Including the polynomial (2.1.4), we regard all these polynomials to constitute a family of (2.1.1). They are listed below along with their inverse series relations [9].

$$P_n(m,x) = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(mx)^{n-mk}}{\Gamma\left(1-\frac{1}{m}-n+mk-k\right)(n-mk)!k!} \tag{2.1.6}$$

$$\Leftrightarrow$$

$$\frac{(-mx)^n}{n!} = \sum_{k=0}^{\lfloor n/m\rfloor}\frac{(-1)^k\left(-\frac{1}{m}-n+mk\right)\Gamma\left(-\frac{1}{m}-n+k+1\right)}{\left(-\frac{1}{m}-n+kp\right)k!}P_{n-mk}(m,x), \tag{2.1.7}$$

$$\mathcal{P}_n(x) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(3x)^{n-3k}}{\Gamma\left(\frac{1}{2}-n+2k\right)(n-3k)!k!} \tag{2.1.8}$$

$$\Leftrightarrow$$

$$\frac{(-3x)^n}{n!} = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(-1)^k\left(-\frac{1}{2}-n+3k\right)\Gamma\left(\frac{1}{2}-n+k\right)}{\left(-\frac{1}{2}-n+k\right)k!}\mathcal{P}_{n-3k}(x), \tag{2.1.9}$$

$$C_n^{\nu}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma(1-\nu-n+k)(n-2k)!k!} \tag{2.1.10}$$

$$\Leftrightarrow$$

$$\frac{(-2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(-\nu-n+2k)\Gamma(-\nu-n+k+1)}{(-\nu-n+k)k!}C_{n-mk}^{\nu}(x), \tag{2.1.11}$$

and

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma\left(\frac{1}{2}-n+k\right)(n-2k)!k!} \tag{2.1.12}$$

$$\Leftrightarrow$$

$$\frac{(-2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\left(-\frac{1}{2}-n+2k\right)\Gamma\left(\frac{1}{2}-n+k\right)}{\left(-\frac{1}{2}-n+k\right)k!}P_{n-mk}(x). \tag{2.1.13}$$

We propose the extension of the polynomials $P_n(m, x, \gamma, s, c)$ as

Definition 2.1.1. *For* $\gamma,\ s,\ c \in \mathbb{C},\ m \in \mathbb{N},\ x \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\}$ *and* $p > 0$,

$$P_{n,p}(m, x, \gamma, s, c) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(s+p)}{\Gamma_p(s-np+mkp-kp+p)(n-mk)!\ k!} \times \gamma^k c^{s-n+mk-k}(-mx)^{n-mk}, \tag{2.1.14}$$

in which the floor function $\lfloor r \rfloor = floor\ r$*, represents the greatest integer* $\leq r$.

We call this polynomial as the p-deformed generalized Humbert polynomials. When $p = 1$, this coincides with the polynomial (2.1.1). The particular polynomials belonging to this general p-polynomial provide the extension to the polynomials (2.1.4), (2.1.6), (2.1.8), (2.1.10) and (2.1.12).
Further, one of the inversion theorems also provides us deformed Bessel function along with its inverse series in the form of deformed Neumann expansion [62]. This will extend the inverse pair [53]:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+1+k)}\left(\frac{x}{2}\right)^{n+2k} \tag{2.1.15}$$

$$\Leftrightarrow$$

$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} \frac{(n+2k)\Gamma(n+k)}{k!} J_{n+2k}(x), \tag{2.1.16}$$

wherein the inverse series is actually the Neumann's expansion [53].

We derive the general inversion pairs in section - 2.2. On particularizing the parameters involved suitably in the general inversion pair, we will deduce inverse series relations of (2.1.14) and its family. Besides these, by means of the alternative inverse pairs, we shall obtain the p-deformed Wilson polynomials as well as the p-deformed Racah polynomials together with their inverse series relations. Thereby, the particular polynomials namely, the polynomials of Hahn, continuous Hahn, Dual Hahn, Meixner-Pollaczek, Krawtchouk, Jacobi and others would also assume deformation along with their inverse series. Moreover, We shall obtain the inversion pair which would provide the p-version of the Bessel function (2.1.15) and its inverse series (2.1.16). This is done in section - 2.3. Next, the differential equation of the p-deformed generalized Humbert polynomials and its particular cases are derived in section - 2.4. The generating function relations, recurrence relations and differential recurrence relations are derived in section - 2.5 and section - 2.6. As an application of the inverse series, the summation formulas involving p-polynomials are derived in section - 2.7. The Companion matrix of p-deformed monic polynomial obtained from the p-deformed generalized Humbert polynomials(2.1.14) is derived in section - 2.8.

2.2 Inverse series relations

Let $\{f(n); n = 0, 1, 2, \ldots\}$ and $\{g(n); n = 0, 1, 2, \ldots\}$ be two sequences such that

$$f(n) = \sum_{k=0}^{n} \alpha_{n,k}\, g(k); \quad g(n) = \sum_{k=0}^{n} \beta_{n,k}\, f(k). \tag{2.2.1}$$

If $\alpha_{r,r}$ and $\beta_{r,r}$ are non zero for all $r = 0, 1, 2, \ldots, n$, then these two series are said to form a pair of inverse series relations. In fact, the coefficient $\alpha_{r,r}$ and $\beta_{r,r}$ are the diagonal elements of the matrices corresponding to these series; and if they are all non zero then their inverses do exist (which are unique !); thus providing the inverse series of those stated in (2.2.1). This eventually leads us to the identity:

$$\sum_{k=j}^{n} \alpha_{n,k}\, \beta_{k,j} = \delta_{nj}$$

involving the Kronecker δ_{nj}. A number of particular and general inverse series pairs of this form occur in the literature [4, 37, 60, 65]. Now replace $\alpha_{n,k}$ by $A(n,k;-b)$ and $\beta_{n,k}$ by $B(n,k;-b)$ and consider the series

$$f(a) = \sum_{k=0}^{N} A(a,b;k)\ g(a+bk); \qquad g(a) = \sum_{k=0}^{N} B(a,b;k)\ f(a+bk).$$

Here, if a is a non negative integer n and b is a negative integer $-m$, then $a+bk = n-mk$. In order that $n-mk \geq 0, k$ will not exceed (the floor function) $\lfloor n/m \rfloor$, representing the greatest integer part of n/m. Hence $N = \lfloor n/m \rfloor$.
If a is a non negative integer n and b is a positive integer m, then $a+bk = n+mk \geq 0$ always and in this case, $N = \infty$.
We prove here a general inversion pair (GIP) involving the p-generalized gamma function: $\Gamma_p(z)$ and the Pochhammer p-symbol $(z)_{n,p}$ due to Díaz at el. [17]. The GIP will now onward be referred to as p-deformed GIP or simply GIP.
As main results, the p-deformed GIP:

$$u(a) = \sum_{k=0}^{N} \frac{\gamma^k}{\Gamma_p(p+\alpha-ar-brk-kp)k!}\ v(a+bk) \tag{2.2.2}$$
$$\Leftrightarrow$$
$$v(a) = \sum_{k=0}^{N} \frac{(-\gamma)^k(\alpha-ar-brk)\Gamma_p(\alpha-ar+kp)}{k!}\ u(a+bk), \tag{2.2.3}$$

will be proved here by choosing a to be a non negative integer n, b to be (i) negative integer $-m$ in Theorem - 2.2.1 and (ii) positive integer in Theorem - 2.2.2. In case (i), $N = \lfloor n/m \rfloor$ whereas in (ii) $N = \infty$. These theorems will be proved with the aid of series orthogonality relation of the form

$$\sum_{r=0}^{s} \binom{s}{r}\ \zeta(r,s) = \binom{0}{s}$$

for appropriate $\zeta(r,s)$. For that we need the following particular inverse pair which we prove as

Lemma 2.2.1. *For $p > 0$, and α, r arbitrary,*

$$f(j) = \sum_{k=0}^{j} (-1)^k \binom{j}{k}\ \Gamma_p(\alpha - nr + mrk - kp + jp)\ g(k) \tag{2.2.4}$$
$$\Leftrightarrow$$
$$g(j) = \sum_{k=0}^{j} (-1)^k \binom{j}{k}\ \frac{(\alpha - nr + mrk)}{\Gamma_p(p + \alpha - nr + j(mr-p) + kp)}\ f(k). \tag{2.2.5}$$

Proof. We observe that the diagonal elements of the coefficient matrix of first series are $(-1)^i\ \Gamma_p(\alpha - nr + mri)$ and the diagonal elements of the coefficient matrix of second series are $(-1)^i/\Gamma_p(\alpha - nr + mri)$ which are all non zero, implying that these matrices have unique inverse. Hence, it suffice to prove that one of these series implies the other. We shall show that (2.2.5) implies (2.2.4).
We denote the right hand side of (2.2.4) by $\Phi(j)$ and then substitute for $g(k)$ from (2.2.5) to get

$$\begin{aligned}\Phi(j) &= \sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma_p(\alpha - nr + mrk - kp + jp)\\ &\quad\times\sum_{i=0}^{k}(-1)^i\binom{k}{i}\ \frac{(\alpha - nr + mri)}{\Gamma_p(p+\alpha - nr + k(mr-p)+ip)}\ f(i).\end{aligned}$$

Here applying double series relation

$$\sum_{k=0}^{m}\sum_{j=0}^{m-k}A(k,j) \ = \ \sum_{j=0}^{m}\sum_{k=0}^{j}A(k,j-k), \tag{2.2.6}$$

we further have

$$\begin{aligned}\Phi(j) &= \sum_{i=0}^{j}\sum_{k=0}^{j-i}(-1)^k\binom{j}{k+i}\ \binom{k+i}{i}\\ &\quad\times\frac{(\alpha - nr + mri)\ \Gamma_p(\alpha - nr + mr(k+i) - (k+i)p + jp)}{\Gamma_p(p+\alpha - nr + (k+i)(mr-p)+ip)}\ f(i)\\ &= \sum_{i=0}^{j}\binom{j}{i}f(i)(\alpha - nr + mri)\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\\ &\quad\times\frac{\Gamma_p(\alpha - nr + mr(k+i) - (k+i)p + jp)}{\Gamma_p(p+\alpha - nr + (k+i)(mr-p)+ip)}\\ &= f(j) + \sum_{i=0}^{j-1}\binom{j}{i}f(i)(\alpha - nr + mri)\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}\\ &\quad\times\frac{\Gamma_p(\alpha - nr + mr(k+i) - (k+i)p + jp)}{\Gamma_p(p+\alpha - nr + (k+i)(mr-p)+ip)}.\end{aligned} \tag{2.2.7}$$

Here, the ratio of two p-gamma functions represents a polynomial of degree $j-i-1$ in k, that is

$$\frac{\Gamma_p(\alpha - nr + mr(k+i) - (k+i)p + jp)}{\Gamma_p(p+\alpha - nr + (k+i)(mr-p)+ip)} \ = \ \sum_{l=0}^{j-i-1} C_l\ k^l,$$

say, hence from (2.2.7), we have

$$\Phi(j) = f(j) + \sum_{i=0}^{j-1} \binom{j}{i} f(i)(\alpha - nr + mri) \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k} \sum_{l=0}^{j-i-1} C_l \; k^l.$$

Since the inner summations on the right hand side are nothing but the $(j-i)^{th}$ difference of polynomial of degree $j-i-1$, it follows from (1.3.14) that $\Phi(j) = f(j)$. This completes the proof of (2.2.4)$\Leftrightarrow$(2.2.5). $\square$

Next, we proceed to prove theorems as follow

Theorem 2.2.1. *If* $a = n \in \mathbb{N} \cup \{0\}$ *and* $b = -m,\ m \in \mathbb{N}$*, then* $n^* = \lfloor n/m \rfloor$*, and there hold the series relations*

$$\begin{aligned} u(n) &= \sum_{k=0}^{n^*} \frac{\gamma^k}{\Gamma_p(p+\alpha-nr+mrk-kp)k!} \; v(n-mk) \qquad (2.2.8)\\ &\Leftrightarrow \\ v(n) &= \sum_{k=0}^{n^*} \frac{(-\gamma)^k(\alpha-nr+mrk)\Gamma_p(\alpha-nr+kp)}{k!} \; u(n-mk). \qquad (2.2.9) \end{aligned}$$

Proof. We first show that (2.2.8)$\Rightarrow$(2.2.9). For that we denote the right hand side of (2.2.9) by $\Psi(n)$, that is

$$\Psi(n) = \sum_{k=0}^{n^*} \frac{(-\gamma)^k(\alpha-nr+mrk)\Gamma_p(\alpha-nr+kp)}{k!} \; u(n-mk),$$

and then substitute for $u(n-mk)$ from (2.2.8) and use the double series relation (2.2.6) to get

$$\begin{aligned} \Psi(n) &= \sum_{k=0}^{n^*} (-\gamma)^k \frac{(\alpha-nr+mrk)\Gamma_p(\alpha-nr+kp)}{k!} \\ &\quad \times \sum_{j=0}^{n^*-k} \frac{\gamma^j \; v(n-mk+mj)}{j! \; \Gamma_p(p+\alpha-nr+mrk+mrj-jp)}. \\ &= \sum_{j=0}^{n^*} \frac{\gamma^j}{j!} v(n-mj) \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(\alpha-nr+mrk)\Gamma_p(\alpha-nr+kp)}{\Gamma_p(p+\alpha-nr+mrj-jp+kp)}. \end{aligned} \qquad (2.2.10)$$

Now in Lemma - 2.2.1, put $g(k) = \binom{0}{k}$, then we find that $f(j) = \Gamma_p(\alpha - nr + jp)$ from the first series (2.2.4); whereas the second series (2.2.5) yields the series orthogonality relation:

$$\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(\alpha-nr+mrk)\Gamma_p(\alpha-nr+kp)}{\Gamma_p(p+\alpha-nr+mrj-jp+kp)} = \binom{0}{j}.$$

Thus we obtain from (2.2.10)

$$\Psi(n) = \sum_{j=0}^{n^*}\frac{\gamma^j}{j!}\,v(n-mj)\,\binom{0}{j} = v(n),$$

as desired.

Now for the converse part, denoting the right hand side of (2.2.8) by $\Omega(n)$ and substituting the series for $v(n-mk)$ from (2.2.9), we get

$$\begin{aligned}\Omega(n) &= \sum_{k=0}^{n^*}\frac{\gamma^k}{\Gamma_p(p+\alpha-nr+mrk-kp)\;k!}\sum_{j=0}^{n^*-k}\frac{(-\gamma)^j(\alpha-nr+mrk+mrj)}{j!}\\ &\quad\times\Gamma_p(\alpha-nr+mrk+jp)\;u(n-mk-mj).\end{aligned}$$

Again using the double series relation (2.2.6), this gives

$$\begin{aligned}\Omega(n) &= \sum_{j=0}^{n^*}(-\gamma)^j\,\frac{(\alpha-nr+mj)}{j!}\,u(n-mj)\\ &\quad\times\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{\Gamma_p(\alpha-nr+mrk+jp-kp)}{\Gamma_p(p+\alpha-nr+mrk-kp)}\\ &= u(n)+\sum_{j=1}^{n^*}\frac{(-\gamma)^j(\alpha-nr+mj)}{j!}\,u(n-mj)\sum_{k=0}^{j}(-1)^k\binom{j}{k}\\ &\quad\times\frac{\Gamma_p(\alpha-nr+mrk+jp-kp)}{\Gamma_p(p+\alpha-nr+mrk-kp)}.\end{aligned}$$

But

$$\frac{\Gamma_p(\alpha-nr+mrk+jp-kp)}{\Gamma_p(p+\alpha-nr+mrk-kp)} = \sum_{i=0}^{j-1}A_i\;k^i,$$

for appropriate coefficients A_i, hence

$$\Omega(n) = u(n)+\sum_{j=1}^{n^*}\frac{(-\gamma)^j(\alpha-nr+mj)}{j!}\,u(n-mj)\sum_{k=0}^{j}(-1)^k\binom{j}{k}\;\sum_{i=0}^{j-1}A_i\;k^i.$$

Here the inner most two summations represent the j^{th} difference of the polynomial

of degree $j-1$ which in view of (1.3.14), leads us to $\Omega(n) = u(n)$.
Thus, (2.2.8)$\Leftrightarrow$(2.2.9). $\square$

Theorem 2.2.2. *If $\{u(*)\}$ and $\{v(*)\}$ are bounded sequences, $a = n \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$, then $N = \infty$ and there hold the series relations*

$$\begin{aligned} u(n) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma_p(p+\alpha-nr-brk-kp)k!}\ v(n+bk) \qquad (2.2.11)\\ &\Leftrightarrow \\ v(n) &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{k!}\ u(n+bk). \qquad (2.2.12)\end{aligned}$$

Proof. We first show that (2.2.11)$\Rightarrow$(2.2.12). Following the method of proof of Theorem - 2.2.1, we begin with

$$\begin{aligned} \Xi(a) &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{k!}\ u(n+bk) \\ &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{k!} \\ &\quad \times \sum_{j=0}^{\infty} \frac{\gamma^j\ v(n+bk+bj)}{j!\ \Gamma_p(p+\alpha-nr-brk-brj-jp)}. \end{aligned}$$

Here, applying the double series relation

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} A(k,j) = \sum_{j=0}^{\infty}\sum_{k=0}^{j} A(k,j-k),$$

this becomes

$$\Xi(n) = \sum_{j=0}^{\infty} \frac{\gamma^j v(n+bj)}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{\Gamma_p(p+\alpha-nr-brj-jp+kp)}. \qquad (2.2.13)$$

We now put $g(k) = \binom{0}{k}$ in (2.2.4), to get $f(j) = \Gamma_p(\alpha-nr+jp)$ back and the same substitutions in (2.2.5) yields

$$\sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(\alpha-nr-brk)\Gamma_p(\alpha-nr+kp)}{\Gamma_p(p+\alpha-nr-brj-jp+kp)} = \binom{0}{j}. \qquad (2.2.14)$$

On applying (2.2.14) in (2.2.13), we find

$$\Xi(n) = \sum_{j=0}^{\infty} \frac{\gamma^j v(n+bj)}{j!} \binom{0}{j} = v(n).$$

For the converse part, we begin with

$$\begin{aligned}
\Theta(n) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma_p(p+\alpha-nr-brk-kp)k!} v(n+bk) \\
&= \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma_p(p+\alpha-nr-brk-kp)k!} \sum_{j=0}^{\infty} \frac{(-\gamma)^j(\alpha-nr-brk-bjr)}{j!} \\
&\quad \times\Gamma_p(\alpha-nr-brk+jp)\ u(n+bk+bj) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j\gamma^j(\alpha-nr-bjr)}{j!}\ u(n+bj) \\
&\quad \times \sum_{k=0}^{j}(-1)^k \binom{j}{k} \frac{\Gamma_p(\alpha-nr-brk+jp-kp)}{\Gamma_p(p+\alpha-nr-brk-kp)} \\
&= u(n) + \sum_{j=1}^{\infty} \frac{(-1)^j\gamma^j(\alpha-nr-bjr)}{j!}\ u(n+bj) \\
&\quad \times \sum_{k=0}^{j}(-1)^k \binom{j}{k} \frac{\Gamma_p(\alpha-nr-brk+jp-kp)}{\Gamma_p(p+\alpha-nr-brk-kp)}.
\end{aligned}$$

As in the earlier proof, here also the ratio of two p-gamma functions in the last expression represents a polynomial of degree $j-1$ in k, that is

$$\frac{\Gamma_p(\alpha-nr-brk+jp-kp)}{\Gamma_p(p+\alpha-nr-brk-kp)} = \sum_{i=0}^{j-1} B_i\ k^i,$$

say, then

$$\begin{aligned}
\Theta(n) &= u(n) + \sum_{j=0}^{\infty} \frac{(-1)^j\gamma^j(\alpha-nr-bjr)}{j!}\ u(n+bj) \sum_{k=0}^{j}(-1)^k \binom{j}{k} \sum_{i=0}^{j-1} B_i\ k^i \\
&= u(n) + 0 \\
&= u(n)
\end{aligned}$$

in view of (1.3.14). Thus (2.2.11)$\Leftrightarrow$(2.2.12). □

2.3 Particular cases

Here, we derived inverse series relation of a general class of polynomial (2.1.14) with its particular cases, the p-deformed version of the Wilson polynomials and the p-deformed version of the Racah polynomials along their inverse series with the help of Theorem - 2.2.1 and the p-deformed version of the Bessel function along with its inverse series with the help of Theorem - 2.2.2. Further, we obtained the p-deformed Continuous dual Hahn polynomial, the p-deformed continuous Hahn polynomial, the p-deformed Jacobi polynomial, the p-deformed Laguerre polynomial and the p-deformed Hahn polynomial as a particular cases of the p-deformed Wilson polynomials and the p-deformed Racah polynomial with their inverse series relation. Now, the replacement of $v(n)$ by $v(n)\Gamma_p(\alpha - nr + p)$ in Theorem - 2.2.1 produces the inverse pair

$$\begin{aligned} u(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} \gamma^k \frac{\Gamma_p(\alpha - nr + mrk + p)}{\Gamma_p(p + \alpha - nr + mrk - kp)k!} v(n - mk) \\ &\Leftrightarrow \\ \Gamma_p(\alpha - nr + p)v(n) &= \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k \frac{(\alpha - nr + mrk)\Gamma_p(\alpha - nr + kp)}{k!} u(n - mk). \end{aligned}$$

Simplified form of it is given by,

$$u(n) = \sum_{k=0}^{\lfloor n/m \rfloor} \gamma^k \frac{\Gamma_p(\alpha - nr + mrk + p)}{\Gamma_p(p + \alpha - nr + mrk - kp)k!} v(n - mk) \tag{2.3.1}$$

$$\Leftrightarrow$$

$$v(n) = \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k \frac{(\alpha - nr + mrk)\Gamma_p(\alpha - nr + kp + p)}{(\alpha - nr + kp)\Gamma_p(\alpha - nr + p)k!} u(n - mk). \tag{2.3.2}$$

Next, the substitution $v(n) = (-m)^n \Gamma_p(s + p)/((c)^{n-s}\Gamma_p(s - np + p)n!)x^n$, $\alpha = s$, $r = p$ and replacing γ by γ/c in (2.3.1) and (2.3.2) provide us the p-deformed generalized Humbert polynomials (2.1.14) and its inverse series:

$$\begin{aligned} \frac{(-mx)^n}{n!} x^n &= \sum_{k=0}^{\lfloor n/m \rfloor} (-\gamma)^k c^{n-k-s} \frac{(s - np + mkp)\Gamma_p(s - np + kp + p)}{(s - np + kp)\Gamma_p(s + p)k!} \\ &\quad \times P_{n-mk,p}(m, x, \gamma, s, c). \end{aligned} \tag{2.3.3}$$

The substitutions $\gamma = 1$, $c = 1$ and $s = -\nu$ in (2.1.14) and (2.3.3) yield the inverse pair of the p-deformed Humbert polynomials:

$$\Pi^{\nu}_{n,m,p}(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-mx)^{n-mk}}{\Gamma_p(p - \nu - np + (m-1)kp)(n - mk)!k!} \tag{2.3.4}$$

$$\Leftrightarrow$$

$$\frac{(-mx)^n}{n!} = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k \frac{(-\nu - np + mkp)}{(-\nu - np + kp)\ k!} \Gamma_p(p - \nu - np + kp)\ \Pi^{\nu}_{n-mk,m,p}(x). \tag{2.3.5}$$

For $p = 1$, this coincides with (2.1.4) and its inverse series (2.1.5). In fact, this polynomial constitutes the class $\{\Pi^{\nu}_{n,m,p}(x); n = 0, 1, 2, \ldots\}$ of polynomials which include several well known polynomials as well as *not so well known* polynomials. Some worth mentioning particular polynomials are deduced below. If we substitute $\nu = 1/m$ in (2.3.4) and (2.3.5), then we get the inverse pair of the p-deformed Kinney polynomial as follows.

$$\begin{aligned} P_{n,p}(m,x) &= \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-mx)^{n-mk}}{\Gamma_p(p-1/m-np+mkp-kp)(n-mk)!k!} \\ &\Leftrightarrow \\ \frac{(-mx)^n}{n!} &= \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k \frac{(-1/m-np+mkp)}{(-1/m-np+kp)k!} \Gamma_p(p-1/m-np+kp) \\ &\quad \times P_{n-mk,p}(m,x). \end{aligned}$$

For $m = 3$ and $\nu = 1/2$, the series (2.3.4) and (2.3.5) would reduce to the p-deformed Pincherle polynomial along with its inverse series relation in the form:

$$\mathcal{P}_{n,p}(x) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{(-3x)^{n-3k}}{\Gamma_p(p-1/2-np+2kp)(n-3k)!k!} \tag{2.3.6}$$

$$\Leftrightarrow$$

$$\frac{(-3x)^n}{n!} = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \frac{(-1/2-np+3kp)}{(-1/2-np+kp)k!} \Gamma_p(p-1/2-np+kp)\mathcal{P}_{n-3k,p}(x). \tag{2.3.7}$$

The p-deformed Gegenbauer polynomial and its inverse are the special cases $m = 2$ of (2.3.4) and (2.3.5) which occur in the form:

$$C^{\nu}_{n,p}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma_p(p-\nu-np+kp)(n-2k)!k!} \tag{2.3.8}$$

$$\Leftrightarrow$$

$$\frac{(-2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(-\nu-np+2kp)}{(-\nu-np+kp)k!} \Gamma_p(p-\nu-np+kp) C^{\nu}_{n-2k,p}(x). \tag{2.3.9}$$

Further, if $\nu = 1/2$ then (2.3.8) and (2.3.9) get reduced to the p-deformed Legendre polynomial and its inverse series relation occur in the form:

$$P_{n,p}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-2x)^{n-2k}}{\Gamma_p(p-1/2-np+kp)(n-2k)!k!} \tag{2.3.10}$$

$$\Leftrightarrow$$

$$\frac{(-2x)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(-1/2-np+2kp)}{(-1/2-np+kp)k!} \Gamma_p(p-1/2-np+kp) P_{n-2k,p}(x). \tag{2.3.11}$$

All these polynomials reduce to their classical forms together with their inverse series for $p=1$ [26, p.697]. Recently, the p-Bessel function was introduced [57, Eq. (3.1)]:

$$J_{n,p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma_p(p+np+kp)k!} \left(\frac{x}{2}\right)^{n+2k}. \tag{2.3.12}$$

It is interesting to note that the set of substitutions $v(n) = (x/2)^n$, $a = n$, $b = 2$, $\gamma = -1$, $r = -p$ and $\alpha = 0$ in series (2.2.11) of Theorem - 2.2.2 leads us to the deformed Bessel function (2.3.12) with $u(n) = J_{n,p}(x)$. The series (2.2.12) then yields the p-deformed Neumann's expansion:

$$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} \frac{(np+2pk)\Gamma_p(np+kp)}{k!} J_{n+2k,p}(x) \tag{2.3.13}$$

as its inverse series. The usual Neumann's expansion occurs if $p=1$ [53, Ex.22, p. 122].

The classical orthogonal polynomials such as the Laguerre polynomial, Hermite polynomial, Legendre polynomial, Jacobi polynomial etc. are possessing the hypergeometric function forms ${}_pF_q[*]$ in which $p = 1, 2$ and $q = 0, 1$. The Hahn polynomial possesses ${}_3F_2[1]$ form. We consider the ${}_4F_3[*]$ function forms polynomials; they are the Wilson polynomials [34, Eq.(1.1.1), p.23] (also [5], [69])

$$\begin{aligned} W_n(x^2; a, b, c, d) &= (a+b)_n (a+c)_n (a+d)_n \\ \times \sum_{k=0}^{n} &\frac{(-n)_k (a+b+c+d+n-1)_k (a+ix)_k (a-ix)_k}{(a+b)_k (a+c)_k (a+d)_k k!} \end{aligned} \tag{2.3.14}$$

and the Racah polynomials (also referred to as Racah coefficients or 6-j symbols) [34, Eq.(1.2.1), p.25] (also [5], [19, Eq.(7.2.16), p.165])

$$\begin{aligned} &R_n\left(x(x+c+d+1); a, b, c, d\right) \\ &= \sum_{k=0}^{n} \frac{(-n)_k (a+b+n+1)_k (x+c+d+1)_k (-x)_k}{(1+a)_k (b+d+1)_k (c+1)_k k!}. \end{aligned} \tag{2.3.15}$$

These polynomials encompass besides the Hermite, Laguerre and Jacobi polynomials, several other polynomials such as the polynomials of Hahn, dual Hahn, continuous Hahn, continuous dual Hahn, Meixner-Pollaczek, Meixner, Krawtchouk and Charlier. The inverse series of these polynomials are given by [9]

$$\begin{aligned}\frac{(a+ix)_n(a-ix)_n}{(a+b)_n(a+c)_n(a+d)_n} &= \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2k-1)}{(a+b+c+d+k-1)_{n+1}} \\ &\quad \times \frac{W_k(x^2;a,b,c,d)}{(a+b)_k(a+c)_k(a+d)_k\ k!}\end{aligned} \tag{2.3.16}$$

and

$$\begin{aligned}\frac{(x+c+d+1)_n(-x)_n}{(a+1)_n(b+d+1)_n(c+1)_n} &= \sum_{k=0}^{n} \frac{(-n)_k\ (a+b+2k+1)}{(a+b+k+1)_{n+1}k!} \\ &\quad \times R_k\left(x(x+c+d+1);a,b,c,d\right).\end{aligned} \tag{2.3.17}$$

Here, we extend (2.3.14) and (2.3.15) in the forms:

$$\begin{aligned}&W_{n,p}(x^2;a,b,c,d) = (a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p} \\ &\times \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+np-p)_{k,p}(a+ix)_{k,p}(a-ix)_{k,p}}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}k!}\end{aligned} \tag{2.3.18}$$

and

$$\begin{aligned}&R_{n,p}\left(x(x+c+d+p);a,b,c,d\right) \\ &= \sum_{k=0}^{n} \frac{(-n)_k(a+b+np+p)_{k,p}(x+c+d+p)_{k,p}(-x)_{k,p}}{(p+a)_{k,p}(b+d+p)_{k,p}(c+p)_{k,p}k!}\end{aligned} \tag{2.3.19}$$

where $p>0$, $x\in\mathbb{R}$ and a, b, c, $d\in\mathbb{C}$ in general. We call (2.3.18) as the p-deformed Wilson polynomials and (2.3.19) as the p-deformed Racah polynomials. It is readily seen that (2.3.18) and (2.3.19) reduce to (2.3.14) and (2.3.15) respectively, if $p=1$. For obtaining inverse series relation of these polynomials, we proceed as follows. First replacing αtaking $r=2p$, $\gamma=1$, $a=n\in\mathbb{N}$ and $b=-1$, then $N=n$ in Theorem - 2.2.1 and then reversing the series, we find the pair:

$$\begin{aligned}u(n) &= \sum_{k=0}^{n} \frac{1}{\Gamma_p(p+\alpha-np-kp)(n-k)!}\ v(k) \\ &\Leftrightarrow \\ v(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(\alpha-2kp)\Gamma_p(p+\alpha-np-kp)}{(\alpha-np-kp)(n-k)!}\ u(k).\end{aligned}$$

Replacing $u(n)$ by $u(n)/\Gamma_p(p+\alpha)$ and $v(n)$ by $(-1)^n v(n)$ and using Pochhammer p-symbol, we find

$$\begin{aligned} u(n) &= \sum_{k=0}^{n} \frac{(-1)^k}{(p+\alpha)_{-(n+k),p}(n-k)!}\, v(k) \\ \Leftrightarrow & \\ (-1)^n v(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(\alpha-2kp)(p+\alpha)_{-(n+k),p}}{(\alpha-np-kp)(n-k)!}\, u(k). \end{aligned}$$

Here we employ the formula (1.3.6) with $z=p+\alpha,\ n=0$ and replace k by $n+k$ to get

$$\begin{aligned} u(n) &= \sum_{k=0}^{n} \frac{(-1)^n(-\alpha)_{n+k,p}}{(n-k)!}\, v(k) \\ \Leftrightarrow & \\ v(n) &= \sum_{k=0}^{n} \frac{(-1)^n(\alpha-2kp)}{(\alpha-np-kp)(-\alpha)_{n+k,p}(n-k)!}\, u(k). \end{aligned}$$

Now, replacing $u(n)$ by $(-1)^n u(n)$ and rewriting the second series in slightly different form, it becomes

$$\begin{aligned} u(n) &= \sum_{k=0}^{n} \frac{(-\alpha)_{n+k,p}}{(n-k)!}\, v(k) \\ \Leftrightarrow & \\ v(n) &= \sum_{k=0}^{n} \frac{(-1)^{n+k}(2kp-\alpha)}{(-\alpha)_{n+k+1,p}(n-k)!}\, u(k). \end{aligned}$$

Finally, using the formula (1.3.8) with $z=-\alpha,\ m=n$ and then replacing n by k, this pair get transformed to the form:

$$u(n) = \frac{(-\alpha)_{n,p}}{n!} \sum_{k=0}^{n} (-n)_k(-\alpha+np)_{k,p}\, v(k) \tag{2.3.20}$$

$$\Leftrightarrow$$

$$v(n) = \frac{1}{(-\alpha)_{n,p} n!} \sum_{k=0}^{n} \frac{(-n)_k(2kp-\alpha)}{(-\alpha+np)_{k+1,p}}\, u(k). \tag{2.3.21}$$

The inverse series of (2.3.18) and (2.3.19) are now obtainable by comparing them with the series (2.3.20). In fact, the substitutions $\alpha=-a-b-c-d+p$ and

$$v(n) = (-1)^n(a+ix)_{n,p}(a-ix)_{n,p}/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}\ n!),$$

implies

$$u(n) = W_{n,p}(x^2;a,b,c,d)(a+b+c+d-p)_{n,p}/((a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}\ n!),$$

consequently, from the inverse series (2.3.21), we find

$$\frac{(a+ix)_{n,p}(a-ix)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}(a+d)_{n,p}} = \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}} \times \frac{W_{k,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}\ k!}. \quad (2.3.22)$$

Likewise, the inverse series of (2.3.19) occurs by putting $\alpha = -a-b-p$,

$$v(n) = (-1)^n(x+c+d+p)_{n,p}(-x)_{n,p}/((p+a)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}\ n!),$$

and

$$u(n) = R_{n,p}\left(x(x+c+d+p);a,b,c,d\right)(a+b+p)_{n,p}/\ n!$$

in (2.3.21). It is given by

$$\frac{(x+c+d+p)_{n,p}(-x)_{n,p}}{(a+p)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}} = \sum_{k=0}^{n} \frac{(-n)_k\ (a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}k!} \times R_{k,p}\left(x(x+c+d+p);a,b,c,d\right). \quad (2.3.23)$$

Next, p-Deformed Continuous dual Hahn polynomial defined by $S_{n,p}(x^2;a,b,c)$ is obtained from (2.3.18) by dividing by $(a+d)_{n,p}$ and then taking $d\to\infty$. That is,

$$\begin{aligned} S_{n,p}(x^2;a,b,c) &= \lim_{d\to\infty} \frac{W_{n,p}(x^2;a,b,c,d)}{(a+d)_{n,p}} \\ &= (a+b)_{n,p}(a+c)_{n,p} \sum_{k=0}^{n} \frac{(-n)_k(a+ix)_{k,p}(a-ix)_{k,p}}{(a+b)_{k,p}(a+c)_{k,p}k!} \\ &\quad \times \lim_{d\to\infty}\left\{\frac{(a+b+c+d+np-p)_{k,p}}{(a+d)_{k,p}}\right\} \\ &= (a+b)_{n,p}(a+c)_{n,p} \sum_{k=0}^{n} \frac{(-n)_k(a+ix)_{k,p}(a-ix)_{k,p}}{(a+b)_{k,p}(a+c)_{k,p}k!}. \end{aligned} \quad (2.3.24)$$

Now, returning to the pair (2.2.2) and (2.2.3) and substituting $br=-p$ and $a=n$, then we get

$$\begin{aligned} u(n) &= \sum_{k=0}^{N} \frac{\gamma^k}{k!\ \Gamma_p(\alpha+p)}\ v(n+bk) \\ &\Leftrightarrow \\ v(n) &= \sum_{k=0}^{N} \frac{(-\gamma)^k(\alpha+kp)\Gamma_p(\alpha+kp)}{k!}\ u(n+bk). \end{aligned}$$

Here replacing $u(n)$ by $u(n)/\Gamma_p(\alpha+p)$ and putting $\gamma=1$, $b=-1$ then it becomes

$$u(n)=\sum_{k=0}^{n}\frac{1}{k!}\ v(n-k)\Leftrightarrow v(n)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\ u(n-k).$$

If we reverse these series, then we get

$$u(n)=\sum_{k=0}^{n}\frac{1}{(n-k)!}\ v(k)\Leftrightarrow v(n)=\sum_{k=0}^{n}\frac{(-1)^{k+n}}{(n-k)!}\ u(k). \tag{2.3.25}$$

From this inverse pair we get the inverse series of the polynomial (2.3.24) by choosing

$$v(n)=(-1)^n(a+ix)_{n,p}(a-ix)_{n,p}/((a+b)_{n,p}(a+c)_{n,p}n!)$$

then

$$u(n)=S_{n,p}(x^2;a,b,c)/((a+b)_{n,p}(a+c)_{n,p}n!),$$

and consequently, we find the inverse series:

$$\frac{(a+ix)_{n,p}(a-ix)_{n,p}}{(a+b)_{n,p}(a+c)_{n,p}}=\sum_{k=0}^{n}\frac{(-n)_k}{(a+b)_{k,p}(a+c)_{k,p}k!}S_{k,p}(x^2;a,b,c)\ u(k).$$

The p-deformed continuous Hahn polynomials is obtained from (2.3.18) by dividing $(-2t)^n n!$ and replacing a by $a-it$, b by $b-it$, c by $c+it$, d by $d+it$, x by $x+t$ and then taking the limit $t\to\infty$, we have

$$\begin{aligned}
&p_{n,p}(x;a,b,c,d)\\
=&\lim_{t\to\infty}\frac{W_{n,p}((x+t)^2;a-it,b-it,c+it,d+it)}{(-2t)^n n!}\\
=&\lim_{t\to\infty}\left\{\frac{(a+b-2it)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(-2t)^n n!}\right.\\
&\left.\times\sum_{k=0}^{n}\frac{(-n)_k(a+b+c+d+np-p)_{k,p}(a+ix)_{k,p}(a-ix-2it)_{k,p}}{(a+b-2it)_{k,p}(a+c)_{k,p}(a+d)_{k,p}k!}\right\}\\
=&(a+c)_{n,p}(a+d)_{n,p}\\
&\times\sum_{k=0}^{n}\frac{(-1)^k(a+b+c+d+np-p)_{k,p}(a+ix)_{k,p}}{(a+c)_{k,p}(a+d)_{k,p}(n-k)!k!}.
\end{aligned} \tag{2.3.26}$$

Next, the choice $u(n)=p_{n,p}(x;a,b,c,d)(a+b+c+d-p)_{n,p}/((a+c)_{n,p}(a+d)_{n,p})$, corresponding to the substitutions $v(n)=(a+ix)_{n,p}/((a+c)_{n,p}(a+d)_{n,p}n!)$ and $-\alpha=a+b+c+d-p$ in (2.3.20) and (2.3.21) respectively, leads us to the polynomial (2.3.26) and its inverse series:

$$\frac{(a+ix)_{n,p}}{(a+c)_{n,p}(a+d)_{n,p}\;n!} = \times\sum_{k=0}^{n}\frac{(-1)^k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{k+1,p}(a+c)_{k,p}(a+d)_{k,p}(n-k)!}\;p_{k,p}(x;a,b,c,d).$$

The p-deformed Jacobi polynomials can be found from (2.3.18) by multiplying $1/t^{2n}n!$, the substituting $a=b=\frac{1}{2}(\alpha+1)$, $c=\frac{1}{2}(\beta+1)+it$, $d=\frac{1}{2}(\beta+1)-it$, $x\to t\sqrt{\frac{1}{2}(1-x)}$ and taking $t\to\infty$. That is,

$$\begin{aligned}
P_{n,p}^{\alpha,\beta}(x) &= \lim_{t\to\infty}\frac{W_{n,p}(\frac{1}{2}(1-x)t^2;\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+1),\frac{1}{2}(\beta+1)+it,\frac{1}{2}(\beta+1)-it)}{(t)^{2n}n!}\\
&= \lim_{t\to\infty}\Bigg\{\frac{(\alpha+1)_{n,p}\left(\frac{1}{2}(\alpha+\beta+2)+it\right)_{n,p}\left(\frac{1}{2}(\alpha+\beta+2)-it\right)_{n,p}}{t^{2n}n!}\\
&\quad\times\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{1}{2}(\alpha+1)+it\sqrt{\frac{1}{2}(1-x)}\right)_{k,p}\left(\frac{1}{2}(\alpha+1)-it\sqrt{\frac{1}{2}(1-x)}\right)_{k,p}}{(\alpha+1)_{k,p}\left(\frac{1}{2}(\alpha+\beta+2)+it\right)_{k,p}\left(\frac{1}{2}(\alpha+\beta+2)-it\right)_{k,p}k!}\\
&\quad\times(\alpha+\beta+2+np-p)_{k,p}\Bigg\}\\
&= \frac{(\alpha+1)_{n,p}}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\alpha+\beta+2+np-p)_{k,p}}{(\alpha+1)_{k,p}k!}\left(\frac{1-x}{2}\right)^k.
\end{aligned}\tag{2.3.27}$$

This reduces to the classical Jacobi polynomial [53, eq(1), page 254], when $p=1$. The inverse series is subject to the choice $v(n)=(1-x)^n/(2^n(1+\alpha)_{n,p}n!)$ and α is replaced by $-\alpha-\beta-2+p$ in (2.3.20) and (2.3.21). With these changes, we find the polynomial (2.3.27) and its inverse series

$$\left(\frac{1-x}{2}\right)^n = (1+\alpha)_{n,p}\sum_{k=0}^{n}\frac{(-n)_k(\alpha+\beta+2+2kp-p)}{(\alpha+\beta+2+kp-p)_{n+1,p}(\alpha+1)_{k,p}}\;P_{k,p}^{\alpha,\beta}(x).$$

The p-deformed Laguerre polynomial, defined in [57] can be obtained from the p-deformed Jacobi polynomial (2.3.27) by letting $x\to 1-2x/\beta$ and then making $\beta\to\infty$. That is,

$$\begin{aligned}
L_{n,p}^{(\alpha)}(x) &= \lim_{\beta\to\infty}P_{n,p}^{\alpha,\beta}\left(1-\frac{2x}{\beta}\right)\\
&= \lim_{\beta\to\infty}\frac{(\alpha+1)_{n,p}}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\alpha+\beta+2+np-p)_{k,p}}{(\alpha+1)_{k,p}k!}\left(\frac{x}{\beta}\right)^k
\end{aligned}$$

$$= \quad (\alpha+1)_{n,p} \sum_{k=0}^{n} \frac{(-1)^k}{(\alpha+1)_{k,p}(n-k)!k!}\, x^k.$$

Its inverse series

$$\frac{x^n}{(\alpha+1)_{n,p}} \quad = \quad \sum_{0}^{n} \frac{(-n)_k}{(\alpha+1)_{k,p}} L_{k,p}^{(\alpha)}(x)$$

follows from the second series in (2.3.25) with $v(n) = (-1)^n x^n/((\alpha+1)_{n,p} n!)$.

The p-deformed Hahn polynomial is obtained from the p-deformed Racah polynomials (2.3.19) by substituting $c+p = -N$ and taking $d \to \infty$. Thus,

$$\begin{aligned} Q_{n,p}(x;a,b,N) &= \lim_{d\to\infty} R_{n,p}\left(x(x+d-N);a,b,-N-p,d\right) \\ &= \lim_{d\to\infty} \sum_{k=0}^{n} \frac{(-n)_k(a+b+np+p)_{k,p}(x+d-N)_{k,p}(-x)_{k,p}}{(p+a)_{k,p}(b+d+p)_{k,p}(-N)_{k,p}k!} \\ &= \sum_{k=0}^{n} \frac{(-n)_k(a+b+np+p)_{k,p}(-x)_{k,p}}{(p+a)_{k,p}(-N)_{k,p}k!}, \end{aligned}$$

where $n = 0,1,2,\ldots,N$. When $-\alpha = a+b+p$, $v(n) = (-x)_{k,p}/((p+a)_{k,p}(-N)_{k,p}k!)$ then $u(n) = Q_{n,p}(x;a,b,N)(a+b+p)_{n,p}/n!$ in (2.3.20) and hence from (2.3.21), we obtain

$$\frac{(-x)_{n,p}}{(p+a)_{n,p}(-N)_{n,p}} \quad = \quad \sum_{k=0}^{n} \frac{(-n)_k(a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}k!}\, Q_{k,p}(x;a,b,N).$$

Similarly one can obtained p-deformed Hermite polynomial, p-deformed Meixner - Pollaczek polynomial, p-deformed Meixner polynomial, p-deformed Krawtchouk polynomial and p-deformed Charlier polynomial with their inverse series relation from (2.3.14) and (2.3.18) by assigning suitable value of parameters.

2.4 Differential equation of certain p-polynomials

In this section, we derive the differential equation of the p-deformed generalized Humbert polynomial. It was shown by Costa and Levine [7] that the homogeneous differential equation:

$$(1-x^N)y^{(N)} + A_{N-1}x^{N-1}y^{(N-1)} + A_{N-2}x^{N-2}y^{(N-2)} + \cdots + A_1 xy' + A_0 y = 0, \tag{2.4.1}$$

has a finite polynomial solutions if and only if $0 \le r < N$, $\forall r$, and $\exists\ n \ge 0$ such that $n \bmod N = r$, where n is a root of the recurrence relation and $y^{(j)}$ is a j^{th} derivative of y with respect to x for $1 \le j \le N$.

Let the sequence $(f_r)_{r=0}^{n}$ be given by $f_r = f(r)$, where

$$f(r) \quad = \quad (n-r)\left(-s+rp+\left(\frac{n-r}{m}\right)p\right)_{m-1,p}.$$

We shall use the forward difference operator: Δ and the shift operator: E which are defined by

$$\Delta f_t = f_{t+1} - f_t,\;\; E^k f_t = f_{t+k}.$$

The relation between Δ and E is given by $\Delta = E - 1$, where 1 is the identity operator defined by $1f = f$. Now, the explicit representation of (2.1.14) is given by

$$P_{n,p}(m,x,\gamma,s,c) \quad = \quad \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\gamma^k c^{s-n+mk-k}(-mx)^{n-mk}}{(p+s)_{-n+mk-k,p}(n-mk)!\; k!}. \tag{2.4.2}$$

In view of the formula:

$$(p+s)_{-n+mk-k,p} = (p+s)_{-(n-mk+k),p} = \frac{(-1)^{n-mk+k}}{(p-p-s)_{n-mk+k,p}},$$

the polynomial (2.4.2) changes to

$$P_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k \gamma^k c^{s-n+mk-k} \frac{(-s)_{n-mk+k,p}}{(n-mk)!\; k!}\,(mx)^{n-mk}. \tag{2.4.3}$$

We obtain the differential equation for this polynomial in

Theorem 2.4.1. *Let $s \in \mathbb{C}$, $p > 0$ and $m \in \mathbb{N}$. Then the polynomials $y = P_{n,p}(m,x,\gamma,s,c)$ are a particular solution of the m^{th} order differential equation in the form*

$$\gamma c^{m-1} y^{(m)} + \sum_{r=0}^{m} a_r x^r y^{(r)} \quad = \quad 0, \tag{2.4.4}$$

where $a_r = \dfrac{m^{m-1}\Delta^r f_0}{r!}$.

Proof. Let $n = ml + q$, where $\lfloor n/m \rfloor = l$ and $0 \le q \le m-1$. Now, r^{th} derivative of (2.4.3) is given by

$$D^r P_{n,p}(m,x,\gamma,s,c) \quad = \quad \sum_{k=0}^{\lfloor (n-r)/m \rfloor} \frac{(-1)^k \gamma^k c^{s-n+mk-k}(-s)_{n-mk+k,p}\, m^{n-mk} x^{n-mk-r}}{(n-mk-r)!\; k!}.$$

Hence,

$$x^r D^r P_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{\lfloor (n-r)/m \rfloor} \frac{(-1)^k \gamma^k c^{s-n+mk-k}(-s)_{n-mk+k,p}(mx)^{n-mk}}{(n-mk-r)!\ k!}. \tag{2.4.5}$$

If r is replaced by m then the from the first expression of r^{th} derivative, we immediately get

$$\begin{aligned} D^m P_{n,p}(m,x,\gamma,s,c) &= \sum_{k=0}^{\lfloor (n-m)/m \rfloor} \frac{(-1)^k \gamma^k c^{s-n+mk-k}(-s)_{n-mk+k,p} m^{n-mk} x^{n-mk-m}}{(n-mk-m)!\ k!} \\ &= \sum_{k=0}^{l-1} \frac{(-1)^k \gamma^k c^{s-n+mk-k}(-s)_{n-mk+k,p} m^m (mx)^{n-mk-m}}{(n-mk-m)!\ k!}, \end{aligned} \tag{2.4.6}$$

where

$$\left\lfloor \frac{n-r}{m} \right\rfloor = \begin{cases} l, & \text{if } r \le q \\ l-1, & \text{if } r > q \end{cases}.$$

Now substituting the expression (2.4.5) and (2.4.6) on the left hand side of the differential equation (2.4.4) and comparing the corresponding coefficients, we find that

$$\begin{aligned} \sum_{r=0}^{m} \binom{n-mk}{r} r! a_r &= \frac{m^m k(-s)_{n-mk+k+m-1,p}}{(-s)_{n-mk+k,p}} \\ &= m^m k(-s+np-mkp+kp)_{m-1,p}, \end{aligned} \tag{2.4.7}$$

where $k = 0,1,2,\ldots,l-1$, and

$$\sum_{r=0}^{q} \binom{n-ml}{r} r! a_r = m^m l(-s+np-mlp+lp)_{m-1,p}.$$

Since $n = ml + q \Rightarrow n - ml = q$, we have

$$\sum_{r=0}^{q} \binom{q}{r} r! a_r = m^{m-1}(n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p}. \tag{2.4.8}$$

Now substituting $a_r = \dfrac{m^{m-1}\Delta^r f_0}{r!}$ in (2.4.8), we get

$$\sum_{r=0}^{q} \binom{q}{r} \Delta^r f_0 = (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p},$$

that is,

$$(1+\Delta)^q f_0 = (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p}. \tag{2.4.9}$$

But $1+\Delta = E$, the shift operator, hence (2.4.9) becomes

$$E^q f_0 = f(q) = (n-q)\left(-s+qp+\left(\frac{n-q}{m}\right)p\right)_{m-1,p}.$$

For $k = 0, 1, 2, \ldots, l-1$, (2.4.7) can be written in the form

$$\sum_{r=0}^{m}\binom{n-mk}{r}\Delta^r f_0 = f_{n-mk} = mk(-s+np-mkp+kp)_{m-1,p}. \tag{2.4.10}$$

Since $t \mapsto f(t)$ is a polynomial of degree m, the equality (2.4.10) is a forward difference formula for f at the point $t = n-mk$. Thus, the proof is completed for the choice

$$a_r = \frac{m^{m-1}\Delta^r f_0}{r!} = \frac{m^{m-1}}{r!}\Delta^r\left(n\left(\frac{np-ms}{m}\right)_{m-1,p}\right).$$

□

We now illustrate the special instances of the differential equation (2.4.4). In particular, the equation of the Pincherle polynomial, Gegenbauer polynomial and thereby the Legendre polynomial. We choose $r = 0, 1, 2$ and 3, to get

$$a_0 = m^{m-1}\Delta^0 f_0 = m^{m-1} n\left(\frac{np-ms}{m}\right)_{m-1,p}, \tag{2.4.11}$$

$$\begin{aligned} a_1 &= m^{m-1}\Delta f_0 = m^{m-1}(E-1)f(0) = m^{m-1}(f(1)-f(0)) \\ &= m^{m-1}\left[(n-1)\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p}\right. \\ &\qquad\qquad \left. -n\left(\frac{np-ms}{m}\right)_{m-1,p}\right] \end{aligned} \tag{2.4.12}$$

$$\begin{aligned} a_2 &= \frac{m^{m-1}\Delta^2 f_0}{2!} = \frac{m^{m-1}}{2!}(E-1)^2 f_0 = \frac{m^{m-1}}{2!}(E^2-2E+1)f(0) \\ &= \frac{m^{m-1}}{2!}(f(2)-2f(1)+f(0)) \\ &= \frac{m^{m-1}}{2!}\left[(n-2)\left(\frac{(n-2)p+m(-s+2p)}{m}\right)_{m-1,p}\right. \end{aligned}$$

$$-2(n-1)\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p}+n\left(\frac{np-ms}{m}\right)_{m-1,p}\Bigg], \tag{2.4.13}$$

and

$$\begin{aligned} a_3 &= \frac{m^{m-1}\Delta^3 f_0}{3!}=\frac{m^{m-1}}{3!}\left[f(3)-3(f(2)+3f(1)-f(0)\right] \\ &= \frac{m^{m-1}}{3!}\Bigg[(n-3)\left(\frac{(n-3)p+m(-s+3p)}{m}\right)_{m-1,p} \\ &\quad -3(n-2)\left(\frac{(n-2)p+m(-s+2p)}{m}\right)_{m-1,p}+3(n-1) \\ &\quad \times\left(\frac{(n-1)p+m(-s+p)}{m}\right)_{m-1,p}-n\left(\frac{np-ms}{m}\right)_{m-1,p}\Bigg]. \end{aligned} \tag{2.4.14}$$

Now choosing $m=3$ in (2.4.11), (2.4.12), (2.4.13) and (2.4.14), we obtain

$$a_0 = 3^2 n\left(\frac{np-3s}{3}\right)_{2,p}=n(np-3s)(np-3(s-p)), \tag{2.4.15}$$

$$\begin{aligned} a_1 &= 3^2\left[(n-1)\left(\frac{(n-1)p+3(-s+p)}{3}\right)_{2,p}-(n)\left(\frac{np-3s}{3}\right)_{2,p}\right] \\ &= 3np(np-2s+p)-(3s-2p)(3s-5p), \end{aligned} \tag{2.4.16}$$

$$a_2 = 12ps-18p^2, \tag{2.4.17}$$

$$a_3 = -4p^2. \tag{2.4.18}$$

Further putting $m=3$, $\gamma=1$, $c=1$ and $s=-\lambda$ in the differential equation (2.4.4), then from the particular values (2.4.15), (2.4.16), (2.4.17) and (2.4.18), we arrive at the differential equation of p-deformed Pincherle polynomial. In fact, from the general form:

$$y^{(3)}+\sum_{r=0}^{3}a_r x^r y^{(r)}=0,$$

that is,

$$(1+a_3x^3)y^{(3)}+a_0y+a_1xy^{(1)}+a_2x^2y^{(2)}=0,$$

we obtain the equation:

$$\begin{aligned} &\left(1-4p^2x^3\right)y^{(3)}-6\left(2p\lambda+3p^2\right)x^2y^{(2)}+\left[3np(np+2\lambda+p)-(3\lambda+2p)(3\lambda+5p)\right]xy^{(1)} \\ &\quad +\left[n(np+3\lambda)(np+3(\lambda+p))\right]y=0. \end{aligned}$$

Here the choice $p=1$ yields the differential equation of the Pincherle polynomial due to Pierre Humbert [30, p.23].

Next, for obtaining the equation for the p-deformed Gegenbauer polynomial, we put $m = 2, \gamma = 1, c = 1$ and $s = -\nu$ in (2.4.4) to get

$$y^{(2)} + \sum_{r=0}^{2} a_r x^r y^{(r)} = 0,$$

or equivalently,

$$(1 + a_2x^2)y'' + a_1xy' + a_0y = 0.$$

Now from (2.4.11), (2.4.12) and (2.4.13), we have

$$\begin{aligned}
a_0 &= n\,(np - 2s)\,, \\
a_1 &= 2\left[(n-1)\left(\frac{(n-1)p + 2(-s+p)}{2}\right)_{1,p} - (n)\left(\frac{np-2s}{2}\right)_{1,p}\right] \\
&= 2\left[(n-1)\left(\frac{(n-1)p + 2(-s+p)}{2}\right) - (n)\left(\frac{np-2s}{2}\right)\right] = 2s - p, \\
a_2 &= -p.
\end{aligned}$$

With these a_0, a_1 and a_2, the above equation takes the precise form given by

$$(1 - px^2)y'' + n(np + 2\nu)y - (2\nu + p)xy' = 0,$$

where $y = C^{\nu}_{n,p}(x)$ given by (2.3.8). When $p = 1$, this reduces to the differential equation of the Gegenbauer polynomial [53, Eq.(1.4), p.279]. The well known special case $\nu = 1/2$ of this equation corresponds to the differential equation:

$$(1 - px^2)P''_{n,p}(x) - (1 + p)xP'_{n,p}(x) + n(np + 1)P_{n,p}(x) = 0$$

of p-deformed Legendre polynomial (2.3.10). Here also, for $p = 1$, this reduces to the differential equation of Legendre polynomial $P_n(x)$ (cf. [53, Eq.(5), p.161]).

2.5 Generating function relations

Objective of this section is to derive generating function relation or GFRs of the p-deformed generalized Humbert polynomials(2.1.14). This will be accomplished with the help of the p-deformed version of the identity:

$$\frac{(1+z)^{a+1}}{1-zb} = \sum_{n=0}^{\infty} \binom{a+bn+n}{n} w^n,$$

where $a,\ b \in \mathbb{C}$ and $w = z(1 + z)^{-b-1}$ due to G. Pólya at el [52, Ex. 212 and Ex. 216, p. 146],

Theorem 2.5.1. *For $p > 0$, $a, b \in \mathbb{C}$ and $w = \dfrac{z}{(1+z)^{b+1}}$.*

$$\frac{(1+z)^{a/p+1}}{1-zb} = \sum_{n=0}^{\infty} \frac{\Gamma_p(a+bnp+np+p)}{\Gamma_p(a+bnp+p)n!} p^{-n} w^n. \tag{2.5.1}$$

Proof. Here we use a technique of Lagrange's series[62, Eq.(3), p.354] due to Lagrange.

$$\frac{f(z)}{1-wg'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[f(z)(g(z))^n\right]_{z=z_0}, \quad D = \frac{d}{dz},$$

where $w = \dfrac{z-z_0}{g(z)}$.
In order to derive (2.5.1), take $z_0 = 0$, $f(z) = (1+z)^{a/p}$ and $g(z) = (1+z)^{b+1} \Rightarrow w = \dfrac{z}{(1+z)^{b+1}}$

$$\begin{aligned}
\frac{f(z)}{1-wg'(z)} &= \frac{(1+z)^{a/p}}{1-w(b+1)(1+z)^b} \\
&= \frac{(1+z)^{a/p}}{1-\dfrac{z}{(1+z)^{b+1}}(b+1)(1+z)^b} \\
&= \frac{(1+z)^{a/p}}{1-\dfrac{z}{(1+z)}(b+1)} \\
&= \frac{(1+z)^{a/p+1}}{1-zb} \\
\sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[f(z)(g(z))^n\right]_{z=z_0} &= \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \left[(1+z)^{\frac{a}{p}+bn+n}\right]_{z=0} \\
&= \sum_{n=0}^{\infty} \frac{w^n}{n!} \left(\frac{a}{p}+bn+n\right)\left(\frac{a}{p}+bn+n-1\right) \\
&\quad \times \cdots \left(\frac{a}{p}+bn+n-n+1\right) \\
&= \sum_{n=0}^{\infty} \frac{w^n}{p^n\ n!} (a+bnp+np)(a+bnp+np-p) \\
&\quad \times \cdots (a+bnp+np-np+p) \\
&= \sum_{n=0}^{\infty} \frac{w^n(-1)^n}{p^n\ n!} (-a-bnp-np)_{n,p} \\
&= \sum_{n=0}^{\infty} \frac{w^n(-1)^n}{p^n\ n!} (p-a-bnp-p-np)_{n,p}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{w^n}{p^n\ n!}(a+bnp+p)_{n,p}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma_p(a+bnp+np+p)}{\Gamma_p(a+bnp+p)n!}p^{-n}w^n$$

This completes the proof. (2.5.1) provides p-version of a result given by G. Pólya at el [52, p. 146, Ex. 216 and Ex. 212](cf. with $p=1$). □

We define a function

$$R(A_n,\alpha,\gamma,r,m,p) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(-\alpha+mrk+p)}{\Gamma_p(-\alpha+mrk-kp+p)k!}\gamma^k p^{-k} A_{n-mk} \tag{2.5.2}$$

which will be required to derive the following GFR.

Theorem 2.5.2. *For $m \in \mathbb{N}$, $G(z) = \sum\limits_{n=0}^{\infty} A_n z^n, A_0 \neq 0,\ p>0,\ w = t(1+\gamma w^m)^{-\beta/p}$,*

$$\sum_{n=0}^{\infty} R(A_n,\alpha+\beta n,\gamma,r,m,p)t^n = \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} G\left[\frac{w}{(1+\gamma w^m)^{r/p}}\right], \tag{2.5.3}$$

where $\{A_n\}$ is an arbitrary sequence such that $\sum\limits_{i=0}^{\infty}|A_i| < \infty$ and other parameters unrestricted in general.

Proof. In (2.5.2) replacing α by $\alpha+\beta n$ and then making both sides as coefficients of infinite series in t^n, we get

$$\sum_{n=0}^{\infty} R(A_n,\alpha+\beta n,\gamma,r,m,p)t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor} \frac{\Gamma_p(p-\alpha-\beta n-nr+mrk)}{\Gamma_p(p-\alpha-\beta n-nr+mrk-kp)k!}\gamma^k p^{-k} A_{n-mk} t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma_p(p-\alpha-\beta n-\beta mk-nr)}{\Gamma_p(p-\alpha-\beta n-\beta mk-nr-kp)k!}\gamma^k p^{-k} A_n t^{n+mk}$$
$$= \sum_{n=0}^{\infty} A_n t^n \sum_{k=0}^{\infty} \frac{\Gamma_p(p-\alpha-\beta n-\beta mk-nr)}{\Gamma_p(p-\alpha-\beta n-\beta mk-nr-kp)k!}\gamma^k p^{-k} t^{mk}.$$

The inner series on the right hand side here, in view of the p-deformed Polya's sum (2.5.1) yields

$$\sum_{n=0}^{\infty} R(A_n,\alpha+\beta n,\gamma,r,m,p)t^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} A_n t^n \sum_{k=0}^{\infty} \frac{\Gamma_p(p-\alpha-\beta n-\beta mk-nr)}{\Gamma_p(p-\alpha-\beta n-\beta mk-nr-kp)k!}\gamma^k p^{-k} t^{mk} \\
&= \sum_{n=0}^{\infty} A_n t^n \frac{(1+v)^{(-\alpha-\beta n-nr)/p+1}}{1+\left(\frac{\beta m}{p}+1\right)v} \\
&= \frac{(1+v)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)v} \sum_{n=0}^{\infty} A_n \frac{\left(t(1+v)^{-\beta/p}\right)^n}{(1+v)^{nr/p}},
\end{aligned}
$$

wherein $v = \gamma t^m (1+v)^{-\beta m/p}$. If we replace v by γw^m then $w = t(1+\gamma w^m)^{-\beta/p}$ and we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} R(A_n, \alpha+\beta n, \gamma, r, m, p) t^n &= \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} \sum_{n=0}^{\infty} A_n \frac{\left(t(1+\gamma w^m)^{-\beta/p}\right)^n}{(1+\gamma w^m)^{nr/p}} \\
&= \frac{(1+\gamma w^m)^{(p-\alpha)/p}}{1+\left(\frac{\beta m}{p}+1\right)\gamma w^m} \sum_{n=0}^{\infty} A_n \left[\frac{w}{(1+\gamma w^m)^{r/p}}\right]^n.
\end{aligned}
$$

This completes the proof of GFR (2.5.3). □

The substitutions $\alpha = -s$, $r = p$, $A_n = (-m)^n (c)^{s-n} \frac{\Gamma_p(p+s)}{\Gamma_p(p+s-np)n!} x^n$ and replacement of γ by $\gamma p/c$ in (2.5.3) yields the GFR of the p-deformed generalized Humbert polynomials (or briefly $pGHP$) as

$$
\sum_{n=0}^{\infty} P_{n,p}(m, x, \gamma, s+\beta n, c) t^n = \frac{(1+\gamma p w^m/c)^{(p+s)/p}}{1+\left(\frac{\beta m}{p}+1\right)\frac{\gamma p w^m}{c}} G\left[\frac{w}{\left(1+\frac{\gamma p w^m}{c}\right)}\right], \tag{2.5.4}
$$

where $G(u) = \sum_{n=0}^{\infty} (-m)^n (c)^{s-n} \frac{\Gamma_p(p+s)}{\Gamma_p(p+s-np)n!} x^n u^n$ and $w = t\left(1+\frac{\gamma p w^m}{c}\right)^{-\beta/p}$. We note that $\beta = 0 \Leftrightarrow w = t$ and hence the p-binomial series (1.3.12) yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty} P_{n,p}(m, x, \gamma, s, c) t^n \\
&= (1+\gamma p t^m/c)^{s/p} G\left[\frac{t}{\left(1+\frac{\gamma p t^m}{c}\right)}\right] \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-m)^n \Gamma_p(p+s)}{\Gamma_p(p+s-np)n!} \left(\frac{xt}{c+\gamma p t^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-m)^n}{(p+s)_{-n,p} n!} \left(\frac{xt}{c+\gamma p t^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma p t^m)^{s/p} \sum_{n=0}^{\infty} \frac{(-m)^n (-1)^n (-s)_{n,p}}{n!} \left(\frac{xt}{c+\gamma p t^m}\right)^n
\end{aligned}
$$

$$
\begin{aligned}
&= c^{(1-1/p)s}(c+\gamma pt^m)^{s/p}\sum_{n=0}^{\infty}\frac{(m)^n(-s)_{n,p}}{n!}\left(\frac{xt}{c+\gamma pt^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma pt^m)^{s/p}\sum_{n=0}^{\infty}\frac{(-s)_{n,p}}{n!}\left(\frac{mxt}{c+\gamma pt^m}\right)^n \\
&= c^{(1-1/p)s}(c+\gamma pt^m)^{s/p}\left(1-\frac{pmxt}{c+\gamma pt^m}\right)^{s/p} \\
&= c^{(1-1/p)s}\left(c+\gamma pt^m-pmxt\right)^{s/p}. \qquad (2.5.5)
\end{aligned}
$$

This generalizes the generating function relation (2.1.2).
We now give a computation formula of Fibonacci-type polynomials of order n in the following statement with the help of (2.5.5) as follows.

Theorem 2.5.3. *For the pGHP $P_{n,p}(m,x,\gamma,s,c)$ defined by (2.1.14),*

$$P_{n,p}(m,x,\gamma,s_1+s_2,c)=\sum_{k=0}^{n}P_{n-k,p}(m,x,\gamma,s_1,c)\ P_{k,p}(m,x,\gamma,s_2,c). \qquad (2.5.6)$$

Proof. With s is replaced by s_1+s_2 in the generating function relation (2.5.5) gives

$$
\begin{aligned}
&\sum_{n=0}^{\infty}P_{n,p}(m,x,\gamma,s_1+s_2,c)t^n \\
&= (c)^{(1-1/p)(s_1+s_2)}\left(c+\gamma pt^m-pmxt\right)^{s_1+s_2/p} \\
&= (c)^{(1-1/p)s_1}\left(c+\gamma pt^m-pmxt\right)^{s_1/p}(c)^{(1-1/p)s_2}\left(c+\gamma pt^m-pmxt\right)^{s_2/p} \\
&= \sum_{n=0}^{\infty}P_{n,p}(m,x,\gamma,s_1,c)t^n\sum_{k=0}^{\infty}P_{k,p}(m,x,\gamma,s_2,c)t^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{n,p}(m,x,\gamma,s_1,c)P_{k,p}(m,x,\gamma,s_2,c)t^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}P_{n-k,p}(m,x,\gamma,s_1,c)P_{k,p}(m,x,\gamma,s_2,c)t^n.
\end{aligned}
$$

On comparing coefficient of t^n, this yields (2.5.6). □

The GFR of p-deformed Humbert polynomial occurs as a special case of (2.5.4) with the substitutions $\gamma=1,\ c=1,\ s=-\mu$ which is given by

$$\sum_{n=0}^{\infty}\Pi_{n,m,p}^{\mu+\beta n}(x)t^n = \frac{(1+pw^m)^{(p-\mu)/p}}{1+(\beta m+p)\,w^m}G\left[\frac{w}{(1+pw^m)}\right], \qquad (2.5.7)$$

where $G(z)=\sum_{n=0}^{\infty}(-m)^n\frac{\Gamma_p(p-\mu)}{\Gamma_p(p-\mu-np)n!}x^nz^n$ and $w=t\,(1+pw^m)^{-\beta/p}$.
The case $\beta=0$ and p-binomial series (1.3.12) yields the GFR

$$\sum_{n=0}^{\infty}\Pi^{\mu}_{n,m,p}(x)t^n \;=\; (1+pt^m-pmxt)^{-\mu/p}. \tag{2.5.8}$$

This extends the generating function relation given by Humbert [30, p.24]. The GFR (2.5.8) readily leads us to the computation formula of Fibonacci-type polynomials of order n stated as

Theorem 2.5.4. *In the usual notations and meaning,*

$$\Pi^{\mu_1+\mu_2}_{n,m,p}(x) \;=\; \sum_{k=0}^{n}\Pi^{\mu_1}_{n-k,m,p}(x)\Pi^{\mu_2}_{k,m,p}(x).$$

Now, the GFR of the p-deformed Kinney polynomial is the special case $\gamma=1,\ c=1,\ s=-1/m$ of (2.5.4) which is given by

$$\sum_{n=0}^{\infty}P_{n,p}(m,\beta n,x)t^n \;=\; \frac{(1+pw^m)^{(p-1/m)/p}}{1+(\beta m+p)\,w^m}G\left[\frac{w}{(1+pw^m)}\right],$$

where $G(z)=\sum_{n=0}^{\infty}(-m)^n\frac{\Gamma_p(p-\frac{1}{m})}{\Gamma_p(p-\frac{1}{m}-np)n!}x^nz^n$ and $w=t\,(1+pw^m)^{-\beta/p}$.
The GFR of the p-deformed Pincherle polynomial is obtained by taking $m=3,\ \gamma=1,\ c=1,$ and $s=-\lambda$ in (2.5.4) and it is given by

$$\sum_{n=0}^{\infty}\mathcal{P}^{\lambda+\beta n}_{n,p}(x)t^n \;=\; \frac{(1+pw^3)^{(p-\lambda)/p}}{1+\left(\frac{\beta 3}{p}+1\right)pw^3}G\left[\frac{w}{(1+pw^3)}\right],$$

where $G(z)=\sum_{n=0}^{\infty}(-3)^n\frac{\Gamma_p(p-\lambda)}{\Gamma_p(p-\lambda-np)n!}x^nz^n$ and $w=t\,(1+pw^3)^{-\beta/p}$.
Similarly,

$$\sum_{n=0}^{\infty}C^{\nu+\beta n}_{n,p}(x)t^n \;=\; \frac{(1+pw^2)^{(p-\nu)/p}}{1+\left(\frac{\beta 2}{p}+1\right)pw^2}G\left[\frac{w}{(1+pw^2)}\right], \tag{2.5.9}$$

where $G(z)=\sum_{n=0}^{\infty}(-2)^n\frac{\Gamma_p(p-\nu)}{\Gamma_p(p-\nu-np)n!}x^nz^n$ and $w=t\,(1+pw^2)^{-\beta/p}$ is the GFR of the p-deformed Gegenbauer polynomial obtained from (2.5.4) by putting $m=2,\ \gamma=1,\ c=1$ and $s=-\nu$. Further taking $\nu=1/2$ in (2.5.9), we get the GFR of the

p-deformed Legendre polynomial or briefly pLP given by

$$\sum_{n=0}^{\infty} P_{n,p}(x)t^n = \frac{(1+pw^2)^{(2p-1)/2p}}{1+\left(\frac{\beta 2}{p}+1\right)pw^2} G\left[\frac{w}{(1+pw^2)}\right],$$

where $G(z) = \sum_{n=0}^{\infty} (-2)^n \frac{\Gamma_p(p-\frac{1}{2})}{\Gamma_p(p-\frac{1}{2}-np)n!} x^n z^n$ and $w = t\,(1+pw^2)^{-\beta/p}$.

2.6 Recurrence relations and Differential recurrence relations

In this section the differential recurrence relations and the mixed recurrence relations of the p-deformed generalized Humbert polynomials are derived.
First we denote $(c+\gamma pt^m - pmxt)^{s/p}$ by $A(t;m,x,\gamma,s,c,p)$ and rewrite (2.5.5) in the form:

$$A(t;m,x,\gamma,s,c,p) = c^{(1/p-1)s} \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c)t^n, \tag{2.6.1}$$

then with $D_x = d/dx$, we have

$$\begin{aligned} D_x(A(t;m,x,\gamma,s,c,p)) &= D_x\left((c+\gamma pt^m - pmxt)^{s/p}\right) \\ &= -mts\,(c+\gamma pt^m - pmxt)^{s/p-1}. \end{aligned}$$

Taking $sq = -p, q \in \mathbb{N}$, this gives

$$\begin{aligned} D_x\left(A(t;m,x,\gamma,-p/q,c,p)\right) &= \frac{mtp}{q}\,(c+\gamma pt^m - pmxt)^{s/p+sq/p} \\ &= \frac{mtp}{q} A(t;m,x,\gamma,-p/q,c,p)^{1+q}. \end{aligned}$$

The successive differentiation yields

$$\begin{aligned} D_x^2(A(t;m,x,\gamma,-p/q,c,p)) &= D_x\left(\frac{mtp}{q}A(t;m,x,\gamma,-p/q,c,p)^{1+q}\right) \\ &= \frac{mtp}{q} D_x\left((A(t;m,x,\gamma,-p/q,c,p))^{1+q}\right) \\ &= \frac{mtp}{q}(1+q)\left(A(t;m,x,\gamma,-p/q,c,p)\right)^q \\ &\quad \times D_x\left(A(t;m,x,\gamma,-p/q,c,p)\right) \\ &= \left(\frac{mtp}{q}\right)^2 (1+q)\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+2q}, \\ D_x^3(A(t;m,x,\gamma,-p/q,c,p)) &= \left(\frac{mtp}{q}\right)^3 (1+q)(1+2q) \end{aligned}$$

$$\times\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+3q},$$

and in general,

$$D_x^j(A(t;m,x,\gamma,-p/q,c,p)) = \left(\frac{mtp}{q}\right)^j\left\{\prod_{i=0}^{j-1}(1+iq)\right\} \times\left(A(t;m,x,\gamma,-p/q,c,p)\right)^{1+jq}. \quad (2.6.2)$$

Next taking j^{th} derivative with respect to x in (2.6.1) yields

$$\begin{aligned}
&D_x^j A(t;m,x,\gamma,s,c,p)\\
&= c^{(1/p-1)s}\sum_{n=0}^{\infty}t^n D_x^j P_{n,p}(m,x,\gamma,s,c)\\
&= c^{(1/p-1)s}\sum_{n=0}^{\infty}t^n\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\gamma^k c^{s-n+mk-k}\frac{(-s)_{n-mk+k,p}}{k!(n-mk)!}m^{n-mk}D_x^j(x)^{n-mk}\\
&= c^{(1/p-1)s}\sum_{n=j}^{\infty}t^n\sum_{k=0}^{\lfloor\frac{n-j}{m}\rfloor}(-1)^k\gamma^k c^{s-n+mk-k}\frac{(-s)_{n-mk+k,p}}{k!(n-mk-j)!}m^{n-mk}x^{n-mk-j}\\
&= c^{(1/p-1)s}\sum_{n=0}^{\infty}t^{n+j}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\gamma^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!(n-mk)!}m^j(mx)^{n-mk}.
\end{aligned}\quad (2.6.3)$$

But since,

$$\begin{aligned}
&D_x^j P_{n+j,p}(m,x,\gamma,s,c)\\
&= \sum_{k=0}^{\lfloor\frac{n+j}{m}\rfloor}(-1)^k\gamma^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!(n+j-mk)!}\,m^{n+j-mk}D_x^j(x)^{n+j-mk}\\
&= \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\gamma^k c^{s-n-j+mk-k}\frac{(-s)_{n+j-mk+k,p}}{k!(n-mk)!}m^{n+j-mk}x^{n-mk},
\end{aligned}\quad (2.6.4)$$

we have from (2.6.3),

$$D_x^j A(t;m,x,\gamma,s,c,p) = c^{(1/p-1)s}\sum_{n=0}^{\infty}t^{n+j}D_x^j P_{n+j,p}(m,x,\gamma,s,c) \quad (2.6.5)$$

Taking $s=-p/q$ in (2.6.5) and using (2.6.2), we get

$$(c)^{(1/p-1)s}\sum_{n=0}^{\infty}t^n D_x^j P_{n+j,p}(m,x,\gamma,-p/q,c)$$

$$= \left(\frac{mp}{q}\right)^j \left\{\prod_{i=0}^{j-1}(1+iq)\right\} (A(t;m,x,\gamma,-p/q,c,p))^{1+jq}.$$

Now replacing $A(t;m,x,\gamma,-p/q,c,p)$ by its series expansion from (2.6.1), this becomes

$$\begin{aligned} & c^{(1/p-1)s}\sum_{n=0}^{\infty} t^n D_x^j P_{n+j,p}(m,x,\gamma,-p/q,c) \\ = & \left(\frac{mp}{q}\right)^j \left\{\prod_{i=0}^{j-1}(1+iq)\right\} \left(c^{(1/p-1)s}\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,-p/q,c)t^n\right)^{1+jq}, \end{aligned}$$

that is,

$$\begin{aligned} & \sum_{n=0}^{\infty} t^n D_x^j P_{n+j,p}(m,x,\gamma,-p/q,c) \\ = & \left(\frac{mp}{q}\right)^j c^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\} \left(\sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,-p/q,c)t^n\right)^{1+jq} \\ = & \left(\frac{mp}{q}\right)^j c^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\} \sum_{n=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{1+jq}=n} P_{i_1,p}P_{i_2,p}\cdots P_{i_{1+jq},p}t^n, \end{aligned}$$

where $q \in \mathbb{N}$. On comparing coefficients of t^n yields

$$\begin{aligned} & D_x^j P_{n+j,p}(m,x,\gamma,-p/q,c) \\ = & \left(\frac{mp}{q}\right)^j (c)^{(1/p-1)sjq}\left\{\prod_{i=0}^{j-1}(1+iq)\right\} \sum_{i_1+i_2+\cdots+i_{1+jq}=n} P_{i_1,p}P_{i_2,p}\cdots P_{i_{jq+1},p}. \end{aligned}$$

This provides p-deformed version of the result due to Gould[26, Eq.(3.4), p.702](cf. with $p=1$). Further, multiplying (2.6.4) by t^n and taking sum from $n=0$ to ∞, produces

$$\begin{aligned} & \sum_{n=0}^{\infty} D_x^j P_{n+j,p}(m,x,\gamma,s,c)t^n \\ = & \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k c^{s-n-j+mk-k}\gamma^k \frac{(-s)_{n+j-mk+k,p}}{k!(n-mk)!}\, m^{n+j-mk}x^{n-mk}t^n \\ = & \sum_{n=0}^{\infty} c^{jp-j}(-s)_{j,p}m^j \sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\gamma^k c^{s-jp-n+mk-k}\frac{(-s+jp)_{n-mk+k,p}}{k!(n-mk)!}\,(mx)^{n-mk}t^n \\ = & \sum_{n=0}^{\infty} c^{(p-1)j}(-s)_{j,p}\, m^j P_{n,p}(m,x,\gamma,s-jp,c)\; t^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} c^{(p-1)j}(-s)_{j,p}\; m^j P_{n,p}(m,x,\gamma,s-jp,c)\; t^n.$$

On comparing the coefficient of t^n, gives

$$D_x^j P_{n+j,p}(m,x,\gamma,s,c) \;=\; c^{(p-1)j}(-s)_{j,p}\; m^j P_{n,p}(m,x,\gamma,s-jp,c). \quad (2.6.6)$$

This generalizes the formula given by Gould[26, Eq.(3.5), p.702](cf. with $p = 1$). Also, taking $j = 1, m = 2, \gamma = 1, c = 1, s = -\nu$ and replacing n by $n-1$ in (2.6.6) yields

$$D_x C^{\nu}_{n,p}(x) \;=\; 2\nu C^{\nu}_{n-1,p}(x).$$

This generalizes the familiar formula by E. T. Whittaker and G. N. Watson[66, (III), p.330](cf. with $p = 1$) involving the p-Gegenbauer polynomial which for $\nu = 1/2$ reduces to

$$D_x P_{n,p}(x) \;=\; P_{n-1,p}(x)$$

involving the p-Legendre polynomial. Next,

$$\begin{aligned}
& (c+\gamma p t^m - pmxt)\, tD_t A(t;m,x,\gamma,s,c,p) \\
= \;& (c+\gamma p t^m - pmxt)\, tD_t \left(c+\gamma p t^m - pmxt\right)^{s/p} \\
= \;& (c+\gamma p t^m - pmxt)\, t\frac{s}{p} \left(c+\gamma p t^m - pmxt\right)^{s/p-1} \left(\gamma p m t^{m-1} - pmx\right) \\
= \;& (-ms)(xt-\gamma t^m) A(t;m,x,\gamma,s,c,p).
\end{aligned}$$

Here substituting for $A(t;m,x,\gamma,s,c,p)$ from (2.6.1), we get

$$\begin{aligned}
& (c+\gamma p t^m - pmxt)\; t\; D_t \left(c^{(1/p-1)s} \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c) t^n \right) \\
&\qquad = \; (-ms)(xt-\gamma t^m)\; c^{(1/p-1)s} \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c) t^n.
\end{aligned}$$

Simplifying this and abbreviating $P_{n,p}(m,x,\gamma,s,c)$ by $P_{n,p}(x)$, we have

$$\begin{aligned}
& (c+\gamma p t^m - pmxt) \sum_{n=0}^{\infty} P_{n,p}(x) n t^n \;=\; -ms(xt-\gamma t^m) \sum_{n=0}^{\infty} P_{n,p}(x) t^n \\
\Rightarrow & \sum_{n=0}^{\infty} cn P_{n,p}(x) t^n + \gamma p t^m \sum_{n=0}^{\infty} P_{n,p}(x) n t^n - pmxt \sum_{n=0}^{\infty} P_{n,p}(x) n t^n
\end{aligned}$$

$$
\begin{aligned}
&= -msxt\sum_{n=0}^{\infty} P_{n,p}(x)t^n + ms\gamma t^m \sum_{n=0}^{\infty} P_{n,p}(x)t^n \\
\Rightarrow & \sum_{n=0}^{\infty} cnP_{n,p}(x)t^n + \gamma p\sum_{n=0}^{\infty} P_{n,p}(x)nt^{n+m} - pmx\sum_{n=0}^{\infty} P_{n,p}(x)nt^{n+1} \\
&= -msx\sum_{n=0}^{\infty} P_{n,p}(x)t^{n+1} + ms\gamma\sum_{n=0}^{\infty} P_{n,p}(x)t^{n+m} \\
\Rightarrow & \sum_{n=0}^{\infty} cnP_{n,p}(x)t^n + \gamma p\sum_{n=m}^{\infty} P_{n-m,p}(x)(n-m)t^n \\
& -pmx\sum_{n=1}^{\infty} P_{n-1,p}(m,x,\gamma,s,c)(n-1)t^n \\
&= -msx\sum_{n=0}^{\infty} P_{n-1,p}(x)t^n + ms\gamma\sum_{n=0}^{\infty} P_{n-m,p}(x)t^n
\end{aligned}
$$

$$
\Rightarrow \sum_{n=m}^{\infty} \left(cnP_{n,p}(x) + mx(s-np+p)P_{n-1,p}(x) + \gamma(np-mp-ms)P_{n-m,p}(x)\right)t^n = 0.
$$

This yields for $n \geq m \geq 1$, the recurrence relation:

$$
cnP_{n,p}(x) + mx(s-np+p)P_{n-1,p}(x) + \gamma(np-mp-ms)P_{n-m,p}(x) = 0. \tag{2.6.7}
$$

This identity provides p-deformed version of a recurrence relation derived by Gould[26, Eq.(2.3), p.700](cf. with $p=1$). On differentiating (2.5.5) with respect to x, we get

$$
\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n = -smt(c)^{(1-1/p)s}\left(c+\gamma pt^m - pmxt\right)^{s/p-1}, \tag{2.6.8}
$$

whereas differentiating (2.5.5) with respect to t and making use of (2.6.8), we find

$$
\begin{aligned}
& \sum_{n=0}^{\infty} P_{n,p}(m,x,\gamma,s,c)nt^{n-1} \\
&= \frac{s}{p}\, c^{(1-1/p)s}(\gamma pmt^{m-1} - pmx)\left(c+\gamma pt^m - pmxt\right)^{s/p-1} \\
&= -s\ c^{(1-1/p)s}\frac{(\gamma mt^{m-1} - mx)}{smt(c)^{(1-1/p)s}}\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n \\
&= \frac{(x-\gamma t^{m-1})}{t}\sum_{n=0}^{\infty} D_x P_{n,p}(m,x,\gamma,s,c)t^n
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} nP_{n,p}(m,x,\gamma,s,c)t^n \\
= & \; x\sum_{n=0}^{\infty} D_xP_{n,p}(m,x,\gamma,s,c)t^n - \gamma t^{m-1}\sum_{n=0}^{\infty} D_xP_{n,p}(m,x,\gamma,s,c)t^n \\
= & \; x\sum_{n=0}^{\infty} D_xP_{n,p}(m,x,\gamma,s,c)t^n - \gamma \sum_{n=0}^{\infty} D_xP_{n,p}(m,x,\gamma,s,c)t^{n+m-1} \\
= & \; x\sum_{n=0}^{\infty} D_xP_{n,p}(m,x,\gamma,s,c)t^n - \gamma \sum_{n=m-1}^{\infty} D_xP_{n-m+1,p}(m,x,\gamma,s,c)t^n.
\end{aligned}$$

Equating the coefficients of t^n in this identity, we obtain for $n \geq m-1$,

$$nP_{n,p}(m,x,\gamma,s,c) = xD_xP_{n,p}(m,x,\gamma,s,c) - \gamma D_xP_{n-m+1,p}(m,x,\gamma,s,c).$$

This provides a p-deformed version of the recurrence relation due to Gould[26, Eq.(2.5), p.700]. Similarly, one can obtain other recurrence relations of the p-deformed generalized Humbert polynomial.

2.7 Summation formulas

In this section, we use the inverse series (2.3.3), (2.3.22), (2.3.23) in obtaining certain summation formulas. While deducing the summation formulas involving the p-Wilson polynomials and the p-Racah polynomials, we shall need the p-deformation of Gauss's summation formula which we state and prove here as

Lemma 2.7.1. *If* $p > 0$, $c \neq -p, -2p, \ldots$ *and* $\Re(c-a-b) > 0$ *then*

$${}_2F_1((a,b),p,(c),p)(1/p) = \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-a)\Gamma_p(c-b)}. \tag{2.7.1}$$

Proof. The p-Beta function (1.3.13)[17] is given by

$$B_p(a,b) = \frac{1}{p}\int_0^1 t^{\frac{a}{p}-1}(1-t)^{\frac{b}{p}-1}dt = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)},$$

where $p > 0$, a, $b \in \mathbb{C}$, $\Re(a,b) \neq 0, -p, -2p, \ldots$.
Also for $|x| < \dfrac{1}{p}$, they showed that [17]

$$(1-px)^{-\frac{a}{p}} = \sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!}x^n.$$

If we replace x by xt/p then it becomes

$$(1-xt)^{-\frac{a}{p}} = \sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!p^n}(xt)^n. \tag{2.7.2}$$

Now, multiplying both sides of (2.7.2) by $t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-p}{p}}$ and then integrating from $t=0$ to $t=1$, we get

$$\begin{aligned}
\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-p}{p}}(1-xt)^{-\frac{a}{p}}dx &= \sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!p^n}x^n \int_0^1 t^{\frac{b}{p}+n-1}(1-t)^{\frac{c-b-p}{p}}dt \\
&= p\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!p^n}x^n B_p(b+np, c-b) \\
&= p\sum_{n=0}^{\infty} \frac{(a)_{n,p}}{n!p^n}x^n \frac{\Gamma_p(b+np)\Gamma_p(c-b)}{\Gamma_p(c+np)}.
\end{aligned}$$

Thus we obtain the integral representation for deformed hypergeometric function ${}_2F_1[*]$ as follows.

$$\begin{aligned}
&\frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-p}{p}}(1-xt)^{-\frac{a}{p}}dx \\
&= \sum_{n=0}^{\infty} \frac{(a)_{n,p}(b)_{n,p}}{(c)_{n,p}n!p^n}x^n = {}_2F_1((a,b),p,(c),p)(x/p).
\end{aligned} \tag{2.7.3}$$

This may be regarded as p-deformed Euler integral formula.
When $p=1$, this reduces to the Euler integral representation of ${}_2F_1[*]$ [53].
In order to obtain the deformation of Gauss summation formula [53, Theorem 18, p. 49], we allow $x \to 1$ in (2.7.3), to get

$$\begin{aligned}
{}_2F_1((a,b),p,(c),p)(1/p) &= \frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-p}{p}}(1-t)^{-\frac{a}{p}}dx \\
&= \frac{\Gamma_p(c)}{p\Gamma_p(b)\Gamma_p(c-b)}\int_0^1 t^{\frac{b}{p}-1}(1-t)^{\frac{c-b-a-p}{p}}dx \\
&= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} B_p(b, c-b-a) \\
&= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(c-b)} \frac{\Gamma_p(b)\Gamma_p(c-b-a)}{\Gamma_p(c-a)}.
\end{aligned}$$

thus the lemma. □

The substitution $p=1$ yields the classical Gauss sum.
An immediate consequence of this lemma is the p-Chu-Vandermonde identity

$$ {}_2F_1((-np,b),p,(c),p)(1/p) = \frac{\Gamma_p(c)\Gamma_p(c-b+np)}{\Gamma_p(c+np)\Gamma_p(c-b)} = \frac{(c-b)_{n,p}}{(c)_{n,p}} \qquad (2.7.4) $$

which occurs for $a=-np$ in (2.7.1) (cf. [53, Ex. 4, p.69] for $p=1$). We now obtain certain summation formulas. For that we multiply the inverse series (2.3.3) by $(a)_{n,l}$ and transform the factor $(-m)^n$ to the left hand side and then take the summation from $n=0$ to ∞ both sides, then for $l>0$, $|x|<1/l$ and $p\neq\nu$, we have

$$ \sum_{n=0}^{\infty}\frac{(a)_{n,l}}{(-m)^n}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k c^{n-k-s}\frac{(s-np+mkp)\Gamma_p(s-np+kp+p)}{(s-np+kp)\Gamma_p(s+p)k!} \times P_{n-mk,p}(m,x,\gamma,s,c) = (1-lx)^{-a/l}. $$

The substitutions $\gamma=1,\ c=1$ and $s=-\nu$ in this summation formula yields

$$ \sum_{n=0}^{\infty}\frac{(a)_{n,l}}{(-m)^n}\sum_{mk=0}^{n}(-1)^k\frac{\Gamma_p(p-\nu-np+kp)(-\nu-np+mkp)}{(-\nu-np+kp)k!} \times \Pi^{\nu}_{n-mk,m,p}(x) = (1-lx)^{-a/l}. $$

Again in (2.3.3), taking the summation from $n=0$ to ∞ we get

$$ \sum_{n=0}^{\infty}(-m)^{-n}\sum_{k=0}^{\lfloor n/m\rfloor}(-\gamma)^k c^{n-k-s}\frac{(s-np+mkp)\Gamma_p(s-np+kp+p)}{(s-np+kp)\Gamma_p(s+p)k!} \times P_{n-mk,p}(m,x,\gamma,s,c) = e^x. $$

The substitutions $\gamma=1,\ c=1$ and $s=-\nu$ in this summation formula yields

$$ \sum_{n=0}^{\infty}(-m)^{-n}\sum_{k=0}^{\lfloor n/m\rfloor}(-1)^k\frac{\Gamma_p(p-\nu-np+kp)(-\nu-np+mkp)}{(-\nu-np+kp)\ k!} \times \Pi^{\nu}_{n-mk,m,p}(x) = e^x. $$

We now derive certain summation formulas involving the p-deformed Wilson polynomials and the p-deformed Racah polynomials. We first multiply both sides of (2.3.22) by $p^{-n}/n!$ and then take the summation from $n=0$ to ∞, to get

$$ \sum_{n=0}^{\infty}\frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ n!}\sum_{k=0}^{n}\frac{(-n)_k\ (a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}\ (a+b)_{k,p}} $$

$$\times \frac{W_{k,p}(x^2;a,b,c,d)}{(a+c)_{k,p}(a+d)_{k,p}\ k!} = \sum_{n=0}^{\infty} \frac{(a+ix)_{n,p}(a-ix)_{n,p}}{(a+b)_{n,p}\ p^n\ n!}.$$

This with the aid of p-Gauss sum (2.7.1), gets simplified to the sum:

$$\sum_{n=0}^{\infty} \frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ n!} \sum_{k=0}^{n} \frac{(-n)_k\ (a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}\ (a+b)_{k,p}}$$
$$\times \frac{W_{k,p}(x^2;a,b,c,d)}{(a+c)_{k,p}(a+d)_{k,p}\ k!} = \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{\Gamma_p(b-ix)\Gamma_p(b+ix)}.$$

Here, the case $x=0$ is worth mentioning; since

$$\frac{W_{k,p}(0;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}}$$
$$= {}_4F_3((-k,a+b+c+d+k-1,a,a),p,(a+b,a+c,a+d),p)(1),$$

we find the summation formula:

$$\sum_{n=0}^{\infty} \frac{(a+c)_{n,p}(a+d)_{n,p}}{p^n\ n!} \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}\ k!}$$
$$\times {}_4F_3((-k,a+b+c+d+k-1,a,a),p,(a+b,a+c,a+d),p)(1)$$
$$= \frac{\Gamma_p(a+b)\Gamma_p(b-a)}{[\Gamma_p(b)]^2}.$$

Now, if both sides of (2.3.22) are multiplied by $(-jp)_{n,p}\ p^{-n}/n!$ and then the summation from $n=0$ to j is taken, then we find

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}p^n n!} \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}k!}$$
$$\times \frac{W_{k,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}} = \sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+ix)_{n,p}}{(a+b)_{n,p}p^n n!}.$$

Here the left hand series when summed up by using p-Chu-Vandermonde's sum (2.7.4), we obtain

$$\sum_{n=0}^{j} \frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}p^n n!} \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}k!}$$
$$\times \frac{W_{k,p}(x^2;a,b,c,d)}{(a+b)_{k,p}(a+c)_{k,p}(a+d)_{k,p}} = \frac{(b-ix)_{j,p}}{(a+b)_{j,p}}.$$

Once again the choice $x=0$ gives rise to the sum

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(a+c)_{n,p}(a+d)_{n,p}}{(a-ix)_{n,p}p^n n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+c+d+2kp-p)}{(a+b+c+d+kp-p)_{n+1,p}\ k!}$$
$$\times{}_4F_3((-k,a+b+c+d+k-1,a,a),p,(a+b,a+c,a+d),p)(1)=\frac{(b)_{j,p}}{(a+b)_{j,p}}.$$

Similarly, From the inverse series (2.3.23) of the p-deformed Racah polynomials, we obtain the following formula when its both sides are multiplied by $p^{-n}/n!$ and then the summation from $n=0$ to ∞ is taken.

$$\sum_{n=0}^{\infty}\frac{(b+d+p)_{n,p}(c+p)_{n,p}}{p^n n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}\ k!}$$
$$\times R_{k,p}\left(x(x+c+d+p);a,b,c,d\right)\ =\ \sum_{n=0}^{\infty}\frac{(x+c+d+p)_{n,p}(-x)_{n,p}}{(p+a)_{n,p}\ p^n\ n!}.$$

This in view of p-Gauss sum (2.7.1), simplifies to

$$\sum_{n=0}^{\infty}\frac{(b+d+p)_{n,p}(c+p)_{n,p}}{p^n n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}\ k!}$$
$$\times R_{k,p}\left(x(x+c+d+p);a,b,c,d\right)\ =\ \frac{\Gamma_p(p+a)\Gamma_p(a-c-d)}{\Gamma_p(a-x-c-d)\Gamma_p(p+a+x)}.$$

If we multiply both sides of (2.3.23) by $(-jp)_{n,p}\ p^{-n}/n!$ and then take the summation from $n=0$ to j, then we have

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}k!}$$
$$\times R_{k,p}\left(x(x+c+d+p);a,b,c,d\right)\ =\ \sum_{n=0}^{j}\frac{(-jp)_{n,p}(-x)_{n,p}}{(p+a)_{n,p}p^n n!}.$$

Applying p-Chu-Vandermonde's sum (2.7.4) on the right hand side, gives

$$\sum_{n=0}^{j}\frac{(-jp)_{n,p}(b+d+p)_{n,p}(c+p)_{n,p}}{(x+c+d+p)_{n,p}\ p^n\ n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+2kp+p)}{(a+b+kp+p)_{n+1,p}k!}$$
$$\times R_{k,p}\left(x(x+c+d+p);a,b,c,d\right)\ =\ \frac{(x+a+p)_{j,p}}{(a+p)_{j,p}}.$$

2.8 Companion matrix

We first make the leading coefficient of the polynomial (2.4.3); unity thereby obtain the monic form $\widetilde{P}_{n,p}(m,x,\gamma,s,c)$ to get

$$\widetilde{P}_{n,p}(m,x,\gamma,s,c) = \sum_{k=0}^{N} \delta_k \; x^{n-mk},$$

where

$$\delta_k \;\; = \;\; \frac{(-1)^{k+mk}(-n)_{mk}\gamma^k c^{mk-k}(-s+np)_{-mk+k,p}}{m^{mk}k!}.$$

With this $\delta_{k,p}$, $C\left(\widetilde{P}_{n,p}(m,x,\gamma,s,c)\right)$ assumes the form as stated in Definition 1.3.1. The eigen values of this matrix will be then precisely the zeros of $\widetilde{P}_{n,p}(m,x,\gamma,s,c)$ (see [48, p. 39]).

We shall revisit some alternative forms of the general inversion pair of GIP for the purpose of deducing the p-versions of Riordan's inverse pairs belonging to the table 1.1 to table 1.6 (which are listed in chapter 1) in chapter 8.

POLYNOMIALS' REDUCIBILITY