CHAPTER III

ABSOLUTE CONVERGENCE WHEN THE FUNCTION SATISFIES SOME HYPOTHESIS ON A SUBSET OF POSITIVE MEASURE

§1. Concerning the study of the properties of lacunary Fourier series (L), when the underlying function satisfies some hypothesis on a subset E of $[-\pi, \pi]$ of positive measure, Noble has mentioned in his paper [23] that the subinterval in some of his results could be replaced by a subset E provided his methods are modified. As we have noted in Chapter I, Kennedy [19] investigates this possibility and proves that Noble's Theorem 1(a) holds when the hypothesis is satisfied in a certain subset E provided a little more stringent gap condition is considered. We continue the study in this direction and propose to prove Theorem 16 to 22 in this chapter. The hypothesis on the function in these theorems is again in terms of either the quadratic modulus of continuity or the quadratic modulus of smoothness or the L^2- trigonometric best approximation to f - but now they are considered only on an arbitrary subset E of positive measure, not necessarily a subinterval. Consequently, for securing the absolute convergence of its Fourier series (L), we assume that the sequence $\{n_k\}$ in (L) satisfies the condition B_2 - a gap condition stronger than

(1.1) but still weaker than the Hadamard gap condition (1.2) (In fact, it is known [2 ; P.234] that any sequence satisfying the Hadamard gap condition (1.2) satisfies the condition B_2 but the converse is not true). It can also be noted that, in view of the corollary to the Zygmunds theorem [2 ; P.241]: ''If the Fourier series of a function f is lacunary with $\{n_k\}$ satisfying the condition B_2 then $\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2) < \infty$, that is, $f \in L^2[-\pi, \pi]$'' , it is not necessary to assume $f \in L^2(E)$ in the hypotheses of these theorems.

We remark that for proving these theorems, we establish a Bessel type inequality (3.2) together with some more inequalities (refer : Lemmas 2 and 3) involving either the quadratic modulus of continuity or quadratic modulus of smoothness or L^2 - trigonometric best approximation, over E - all of which have intrinsic interest in their own right.

§2. We need the following lemmas. Lemma 1 is proved by Zygmund, though not explicitely stated, assuming that the sequence $\{n_k\}$ of natural numbers satisfies the Hadamard gap condition (1.2) ; but it is easy to see from the proof there that we can as well take $\{n_k\}$ satisfying the condition B_2 .

<u>LEMMA 1</u>. [39 ; P.121]. Let $E \subset [-\pi, \pi]$ be a set of positive measure, $\{n_k\}$ satisfy the condition B_2 and $n_0 = 0$, $n_k = - n_{-k}$ (k < 0). Then there exists $\nu \in N$ with the

property : if $\{C_k\}(k \in Z)$ is any sequence of complex numbers, then for $T > \nu$ we have

$$|S_T| \leq \frac{|E|}{2} \sum_{-T}^{T} |C_k|^2 , \tag{3.1}$$

where

$$S_T = \sum_{p,q} C_p \, \overline{C}_q \int_E \exp\left(i(n_p - n_q) x\right) \cdot dx$$

in which the summation is over values of p and q such that

$$\nu < |p| , |q| \leq T \quad \text{and} \quad p \neq q.$$

<u>LEMMA 2</u>. Let E , $\{n_k\}$ and ν be as in Lemma 1 and $|E|$ denote the Lebesgue measure of the set E. Put $C_o = 0$, $C_k = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > o)$, $C_k = \overline{C}_{-k}$ $(k < o)$ and suppose that $C_k = 0$ for all k such that $|k| \leq \nu$. Then

$$\sum_{-\infty}^{\infty} |C_k|^2 \leq \frac{2}{|E|} \int_E |f(x)|^2 \, dx ; \tag{3.2}$$

and

$$\sum_{|k|>p} |C_k|^2 \leq C \, \frac{2}{|E|} \left(\omega^{(2)}(1/p , f , E)\right)^2 , \tag{3.3}$$

or more generally

$$\sum_{|k|>p} |C_k|^2 \leq C \; \frac{2}{|E|} \left(\omega_\ell^{(2)}(1/p\ ,\ f\ ,\ E) \right)^2 , \qquad (3.4)$$

where C is constant, and $\omega^{(2)}(1/p\ ,\ f\ ,\ E)$ and $\omega_\ell^{(2)}(1/p\ ,\ f\ ,\ E)$ are as in the hypothesis of Theorems 16 and 17 with n replaced by p.

Proof of Lemma 2. We have

$$\sum_{-\infty}^{\infty} |C_k|\ r^{|n_k|} < \infty \qquad (0 < r < 1) \qquad (3.5)$$

and if we put

$$\phi(r,x) = \sum_{-\infty}^{\infty} C_k\ r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1) \qquad (3.6)$$

for all real x, then its existence is assured by (2.5) and we obviously get

$$\phi(r,x) = \sum_{k=1}^{\infty} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right) r^{n_k} .$$

Now, by a corollary to Zygmund's theorem [2 ; P.241] $f \in L^2 [-\pi\ ,\ \pi]$ and hence by a known theorem [39 ; P.87] it follows that

$$f(x) = L^2 - \lim_{r \to 1} \phi(r,x) \qquad \left(|x| \leq \pi \right). \qquad (3.7)$$

Again, for $T > \nu$, put

$$\phi(r,x,T) = \sum_{-T}^{T} C_p\ r^{|n_p|} \exp(in_p x) \qquad (0 < r < 1). \qquad (3.8)$$

Then

$$\int_E |\phi(r,x,T)|^2 dx$$

$$= \int_E \left(\sum_{-T}^{T} C_p \, r^{|n_p|} \exp(in_p x) \right) \left(\sum_{-T}^{T} \bar{C}_q \, r^{|n_q|} \exp(-in_q x) \right) dx$$

$$= |E| \sum_{-T}^{T} |C_p|^2 \, r^{2|n_p|}$$

$$+ \sum_{p,q} C_p \, r^{|n_p|} \, \bar{C}_q \, r^{|n_q|} \int_E \exp(i(n_p - n_q)x) \, dx$$

$$\geq |E| \sum_{-T}^{T} |C_p|^2 \, r^{2|n_p|} - \frac{|E|}{2} \sum_{-T}^{T} |C_p|^2 \, r^{2|n_p|}$$

$$= \frac{|E|}{2} \sum_{-T}^{T} |C_p|^2 \, r^{2|n_p|} \tag{3.9}$$

applying the Lemma 1, where $\sum_{p,q}$ has the same meaning as in the Lemma 1. But from (3.5), (3.6) and (3.8) it follows that for any fixed $r (0 < r < 1)$

$$\phi(r,x,T) \longrightarrow \phi(r,x) \quad \text{as} \quad T \longrightarrow \infty$$

uniformly in E. Therefore, from (3.9), we get

$$\int_E |\phi(r,x)|^2 dx \geq \frac{|E|}{2} \sum_{-\infty}^{\infty} |C_p|^2 \, r^{2|n_p|} . \tag{3.10}$$

But (3.7) implies that

$$\int_E |\phi(r,x) - f(x)|^2 \, dx \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 1$$

and by Minkowski's inequality

$$\left(\int_E |\phi(r,x)|^2 dx\right)^{1/2} \leq \left(\int_E |\phi(r,x)-f(x)|^2 dx\right)^{1/2} + \left(\int_E |f(x)|^2 dx\right)^{1/2}$$

as well as

$$\left(\int_E |f(x)|^2 dx\right)^{1/2} \leq \left(\int_E |\phi(r,x)-f(x)|^2 dx\right)^{1/2} + \left(\int_E |\phi(r,x)|^2 dx\right)^{1/2}.$$

Therefore

$$0 \leq \left| \left(\int_E |f(x)|^2 dx\right)^{1/2} - \left(\int_E |\phi(r,x)|^2 dx\right)^{1/2} \right|$$

$$\leq \left(\int_E |\phi(r,x) - f(x)|^2 dx\right)^{1/2} \longrightarrow 0 \text{ as } r \longrightarrow 1.$$

This implies that

$$\int_E |\phi(r,x)|^2 dx \longrightarrow \int_E |f(x)|^2 dx \quad \text{as} \quad r \longrightarrow 1.$$

Hence from (3.10) we get

$$\int_E |f(x)|^2 dx \geq \frac{|E|}{2} \sum_{-\infty}^{\infty} |c_p|^2$$

and hence (3.2) is proved.

Now put

$$g(x) = f(x+h) - f(x-h) \tag{3.11}$$

and

$$C_k^* = 2\,i\,C_k \sin n_k h\ . \tag{3.12}$$

Then $|C_k^*| \leq 2|C_k|$ and hence by (3.5) we have

$$\sum_{-\infty}^{\infty} |C_k^*|^2 \; r^{|n_k|} < \infty \qquad (0 < r < 1) \tag{3.13}$$

Put

$$g(r,x) = \sum_{-\infty}^{\infty} C_k^* \; r^{|n_k|} \; \exp(in_k x) \qquad (0 < r < 1) \tag{3.14}$$

for all real x, then its existence is assured by (3.13) and we get the identity

$$g(r,x) = \phi(r\,,\,x+h) - \phi(r\,,\,x-h)$$

and from it together with (3.7) and (3.11) we get

$$g(x) = L^2 - \lim_{r \to 1} g(r,x) \qquad (|x| \leq \pi). \tag{3.15}$$

We now apply (3.2), with C_k and $f(x)$ replaced by C_k^* and $g(x)$ respectively, to get

$$\sum_{-\infty}^{\infty} |C_k^*|^2 \leq \frac{2}{|E|} \int_E |g(x)|^2 dx\ .$$

Hence, by (3.11) and (3.12) we obtain

$$4 \sum_{-\infty}^{\infty} |C_k|^2 \cdot \sin^2 |n_k| h \le \frac{2}{|E|} \int_E |f(x+h) - f(x-h)|^2 dx . \qquad (3.16)$$

Integrating both the sides of (3.16) with respect to h over $(0, \pi/p)$ $(p \in N)$, we shall have

$$4 \sum_{-\infty}^{\infty} |C_k|^2 \int_0^{\pi/p} \sin^2 |n_k| h \, dh$$

$$\le \frac{2}{|E|} \cdot \int_0^{\pi/p} dh \left(\int_E |f(x+h) - f(x-h)|^2 dx \right) . \qquad (3.17)$$

Now, by Lemma 3 of Chapter II, taking $s = p$ in it, we get

$$\int_0^{\pi/p} \sin^2 |n_k| h \, dh > \frac{\pi}{4p} \quad \text{when} \quad p \le |n_k| .$$

Therefore, from (3.17), using

$$\omega^{(2)}(\lambda/n , f , E) \le C(\lambda) \; \omega^{(2)}(1/n , f , E)$$

$(\lambda > 0$, $C(\lambda)$ is a constant depending on λ), we get

$$\frac{\pi}{p} \sum_{|k| \ge p} |C_k|^2 \le \frac{2}{|E|} \frac{\pi}{p} \left(\omega^{(2)}(\pi/p , f , E) \right)^2$$

$$\le \frac{2}{|E|} \cdot \frac{\pi}{p} \cdot C \cdot \left(\omega^{(2)}(1/p , f , E) \right)^2$$

from which (3.3) follows.

Further, if we put

$$g(x) = \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+(2j-\ell)h) \qquad (3.11)'$$

and

$$C_k^* = C_k \exp(-in_k \ell h) \left(\exp(2in_k h)-1\right)^{\ell}$$

$$= 2^{\ell} C_k \exp(-i\ell n_k h)(-1)^{\ell} \exp\left(i\ell(n_k h - \pi/2)\right) \sin^{\ell} n_k h \;, \qquad (3.12)'$$

then $|C_k^*| \leq 2^{\ell} |C_k|$ and hence by (3.5) we have

$$\sum_{-\infty}^{\infty} |C_k^*| \; r^{|n_k|} < \infty \qquad (0 < r < 1). \qquad (3.13)'$$

Put

$$g(r,x) = \sum_{-\infty}^{\infty} C_k^* \, r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1) \qquad (3.14)'$$

for all real x, then its existence is assured by $(3.13)'$ and we get the identity

$$g(r,x) = \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \phi(r \;,\; x + (2j - \ell)h).$$

This together with (3.7) and $(3.11)'$ gives

$$g(x) = L^2\text{-}\lim_{r \to 1} g(r,x) \qquad \left(|x| \leq \pi \right). \qquad (3.15)'$$

Then it follows from $(3.13)'$, $(3.14)'$ and $(3.15)'$ that we can apply the inequality (3.2), with C_k and $f(x)$ replaced by C_k^* and $g(x)$ respectively, to get

$$\sum_{-\infty}^{\infty} |C_k^*|^2 \leq \frac{2}{|E|} \int_E |g(x)|^2 dx .$$

In view of $(3.11)'$ and $(3.12)'$, we obtain from this

$$2^{2\ell} \sum_{-\infty}^{\infty} |C_k|^2 \sin^{2\ell} |n_k| h$$

$$\leq \frac{2}{|E|} \int_E \left| \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \; f(x+(2j-\ell)h) \right|^2 dx. \quad (3.16)'$$

Further, by Lemma 3 of Chapter II, taking $s = p$ in it, we have

$$\int_0^{\pi/p} \sin^{2\ell} |n_k| h \; dh \; > \; \frac{\pi}{2^{\ell+1} p} \qquad \text{when} \quad p \leq |n_k| \; .$$

Using this along with $(3.16)'$, replacing $\omega^{(2)}(1/p \; , \; f \; , \; E)$ by $\omega_\ell^{(2)}(1/p \; , \; f \; , \; E)$ and proceeding analogously as in the proof of the inequality (3.3), the inequality (3.4) is proved. This completes the proof of Lemma 2.

<u>Proof of Theorem 16</u>. Put $n_o = 0$, $n_k = -n_{-k}$ $(k < o)$; $C_o = 0$, $C_k = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > o)$, $C_k = \overline{C}_{-k}$ $(k < o)$. We assume

throughout, without loss of generality, that $C_k = 0$ for all k such that $|k| \leq \nu$, where ν is as in the Lemma 1. Then putting

$$r_p = \sum_{|k|>p} |C_k|^2$$

in the inequality (3.3) of Lemma 2, we obtain

$$r_p^{\beta/2} \leq C \left(\omega^{(2)}(1/p\ ,\ f\ ,\ E)\right)^{\beta},$$

where C is some constant depending on E. This implies

$$\sum_{p=1}^{\infty} (r_p / p)^{\beta/2} < \infty$$

on account of (1.24).

Finally,

$$\sum_{-\infty}^{\infty} |C_k|^{\beta} = \sum_{|k|=1}^{\infty} \left(|k| \frac{|C_k|^{\beta}}{|k|}\right)$$

$$= \sum_{|k|=1}^{\infty} \left(\sum_{p=1}^{|k|} \left(\frac{|C_k|^{\beta}}{|k|}\right)\right)$$

$$= \sum_{p=1}^{\infty} \left(\sum_{|k|=p}^{\infty} \frac{|C_k|^{\beta}}{|k|}\right)$$

$$\leq \sum_{p=1}^{\infty}\left(\left(\sum_{|k|=p}^{\infty}|c_k|^2\right)^{\beta/2}\cdot\left(\sum_{|k|=p}^{\infty}\left(\frac{1}{|k|}\right)^{2/(2-\beta)}\right)^{1-\beta/2}\right)$$

$$\leq \sum_{p=1}^{\infty}\left(r_p^{\beta/2}\cdot\frac{A}{p^{\beta/2}}\right) \qquad (A = \text{constant})$$

$$< \infty .$$

Hence (1.6) follows, completing the proof of Theorem 16.

Remark. With $\beta = 1$, $E = [-\pi , \pi]$ and without the lacunarity condition, this is Theorem 6 due to Szàsz ; while $\beta = 1$, $E = I$ and with the lacunarity condition (1.1), this is Theorem 9 due to the author.

Proof of Theorem 17. Applying the inequality (3.4) instead of (3.3), replacing $\omega^{(2)}(1/p , f , E)$ by $\omega_{\ell}^{(2)}(1/p , f , E)$ and proceeding analogously as in the proof of Theorem 16, this theorem is proved.

Remark. With $\ell = 1$, this is Theorem 16. With $\beta = 1$, $E = I$ and with lacunarity condition (1.1) instead of the condition B_2 this is Theorem 11 due to the author.

Proof of Theorem 18. Observe that

$$\omega^{(2)}(1/n , f , E) = \sup_{0\leq h\leq 1/n}\left\{\left(\int_E |f(x+h)-f(x-h)|^2 dx\right)^{1/2}\right\}$$

$$\leq \sup_{0\leq h\leq 1/n}\left\{\left(\sup_{x\in E}\left\{|f(x+h)-f(x-h)|\right\}\right)|E|^{1/2}\right\}$$

$$= |E|^{1/2}\,\omega(1/n\,,\, f\,,\, E).$$

Similarly, we shall obtain

$$\omega_\ell^{(2)}(1/n\,,\, f\,,\, E) \leq |E|^{1/2}\,\omega_\ell(1/n\,,\, f\,,\, E).$$

Using these and applying Theorems 16 and 17, Theorem 18 follows as a corollary.

Remark. With $E = [-\pi\,,\,\pi]$, $\beta = 1$ and without the gap condition, the first part of this theorem is the classical Theorem 3, due to Bernšteĭn, for the absolute convergence of the Fourier series $\sigma(f)$ of a function f.

We now proceed to prove Theorems 19 to 22. This requires the sharpened form of the inequalities we have proved in Lemma 2 —which is done by a slight modification in the proof. We also prove a similar inequality involving the trigonometric best approximation to f in the space $L^2(E)$. This, in turn, gives us the condition on f in terms of the best approximation to ensure the absolute convergence of (L). We need the following lemma. The inequalities (3.19) and (3.20) in this lemma are the sharpened versions of our inequalities (3.3) and (3.4) of Lemma 2 and their proofs are merely outlined.

LEMMA 3. Let E, $\{n_k\}$ and ν be as in Lemma 1 and $|E|$ denote the Lebesgue measure of the set E. Put $C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$, $C_{n_k} = \overline{C}_{n_{-k}}$ $(k < 0)$ and suppose that $C_{n_k} = 0$ for all k such that $|k| \leq \nu$.

Then

$$\sum_{-\infty}^{\infty} |C_{n_k}|^2 \leq \frac{2}{|E|} \int_E |f(x)|^2 \, dx ; \tag{3.18}$$

$$\sum_{|k| \geq p} |C_{n_k}|^2 \leq C \frac{2}{|E|} \left(\omega^{(2)} (1/n_p , f , E) \right)^2 , \tag{3.19}$$

or more generally

$$\sum_{|k| \geq p} |C_{n_k}|^2 \leq C \frac{2}{|E|} \left(\omega_\ell^{(2)} (1/n_p , f , E) \right)^2 , \tag{3.20}$$

and

$$\sum_{|k| \geq p} |C_{n_k}|^2 \leq C \frac{2}{|E|} \left(E_{n_p}^{(2)} (f , E) \right)^2 , \tag{3.21}$$

where C is some constant and $E_{n_p}^{(2)}(f , E)$ is as in the hypothesis of Theorem 22 with k replaced by p.

<u>Proof of Lemma 3</u>. We have (2.3), (2.4) and (2.4)$'$. Now, by a corollary to Zygmund's theorem [2 ; P.241] $f \in L^2[-\pi , \pi]$ and hence by a known theorem [39 ; P.87] we get (2.5). Then the inequality (3.18) is infact the inequality (3.2) of Lemma 2. Also, proceeding as in the proof of Lemma 2 we shall obtain the inequality (3.16) with C_k now denoted by C_{n_k}. Instead of integrating both the sides of (3.16) with respect to h over $(0 , \pi/p)$, $p \in N$, we now integrate them over $(0 , \pi/n_p)$. Then, observing that $|k| \geq p$ implies $|n_k| \geq n_p$,

in view of (*) of Chapter II we see that the inequality (3.19) is proved proceeding as in the proof of the inequality (3.3) of Lemma 2. Similarly, we shall also get the inequality $(3.16)'$ with C_k denoted by C_{n_k}. Integrating both the sides of $(3.16)'$ now over $(0, \pi/n_p)$ and observing (**) of Chapter II we see that the inequality (3.20) is proved proceeding as in the proof of the inequality (3.19) and replacing $\omega^{(2)}(1/n_p, f, E)$ by $\omega_\ell^{(2)}(1/n_p, f, E)$ throughout.

Finally, let $T_{n_p}(x)$ and $T_{n_p}(r,x)$ be as in (2.21) and (2.22) respectively. We shall then get (2.23). Putting $g(x)$, A_k and $g(r,x)$ as in (2.24), (2.25) and (2.27) respectively, we shall get (2.26) and (2.28) proceeding analogously. Then, instead of applying the lemma quoted by Kennedy, if we now apply the inequality (3.18) we shall get

$$\sum_{|k| \geq p} |C_{n_k}|^2 \leq \frac{2}{|E|} \int_E |f(x) - T_{n_p}(r,x)|^2 dx . \tag{3.22}$$

since (3.22) holds for arbitrary trigonometric polynomial of order not higher than n_p, we get the inequality (3.21). This completes the proof of Lemma 3.

Remark. The inequalities (3.19) and (3.20) generalize our inequalities (2.16) and (2.17) respectively, of Lemma 5, Chapter II. Observe that in the inequalities (2.16) and (2.17) the gap condition involved is (1.1) and the quadratic modulus of continuity or the quadratic modulus of smoothness is considered on a subinterval I ; while in the inequalities (3.19) and (3.20) we consider the gap condition as the condition B_2 but the quadratic modulus of continuity or the quadratic modulus of smoothness is considered on a subset E of $[-\pi, \pi]$ of positive measure. It may be noted here that any sequence satisfying the Hadamard gap condition (1.2) satisfies the condition B_2 as well as the gap condition (1.1).

<u>Proof of Theorem 19</u>. Define $\{n_k\}$ $(k \in Z)$ and $\{C_{n_k}\}$ $(k \in Z)$ as in the hypothesis of Lemma 3. We assume throughout, without loss of generality, that $C_{n_k} = 0$ for all k such that $|k| \leq \nu$, where ν is as in Lemma 1. Then putting

$$r_{n_p} = \sum_{|k| \geq p} |C_{n_k}|^2$$

in the inequality (3.19) of Lemma 3, we obtain

$$r_{n_p}^{\beta/2} \leq C \left(\omega^{(2)}(1/n_p , f , E)\right)^{\beta} , \qquad (3.23)$$

where C is some constant depending on E.

Then, using (3.23) and (1.30) instead of (2.31) and (1.19) respectively and proceeding as in the proof of Theorem 12, Chapter II, this theorem is proved.

Proof of Theorems 20 and 22. Applying the inequalities (3.20) and (3.21) instead of the inequality (3.19), throughout replacing $\omega^{(2)}(1/n_k\ ,\ f\ ,\ E)$ by $\omega_\ell^{(2)}(1/n_k\ ,\ f\ ,\ E)$ and $E_{n_k}^{(2)}(f\ ,\ E)$ respectively and proceeding analogously as in the proof of Theorem 19, this theorem is proved.

Remark. With $\ell = 1$ Theorem 20 is Theorem 19.

Proof of Theorem 21. As in the proof of Theorem 18, we have

$$\omega^{(2)}(1/n_k\ ,\ f\ ,\ E) \leq |E|^{1/2}\,\omega(1/n_k\ ,\ f\ ,\ E)$$

and

$$\omega_\ell^{(2)}(1/n_k\ ,\ f\ ,\ E) \leq |E|^{1/2}\,\omega_\ell(1/n_k\ ,\ f\ ,\ E).$$

Hence, applying now Theorems 19 and 20, Theorem 21 follows immediately.

Note: In view of the fact that $\omega^{(2)}(\delta\ ,\ f\ ,\ E)$ and $\omega_\ell^{(2)}(\delta\ ,\ f\ ,\ E)$ are non-decreasing functions of δ , it may be noted that Theorems 19, 20 and 21 are sharpened versions of Theorems 16, 17 and 18.

Remark 1. With $E = I$ and with the gap condition (1.1) instead of the condition B_2 , Theorems 19, 20 and 21 are our Theorems 12, 13 and 14 respectively.

Remark 2. With $\ell = \beta = 1$, $E = [-\pi , \pi]$, without the gap condition and taking $\{n_k\}$ as an arbitrary sequence of natural numbers, Theorem 20 is Theorem 8 due to Stečkin. This means ''if $\{n_k\}$ satisfies the condition B_2 and (L) is a Fourier series of f then (L) converges absolutely evenwhen the hypothesis in Stečkin's theorem is satisfied only in a subset E of $[-\pi , \pi]$ of positive measure.