CHAPTER V

ORDER OF MAGNITUDE OF FOURIER COEFFICIENTS AND THE ABSOLUTE CONVERGENCE WHEN THE FUNCTION SATISFIES A CERTAIN HYPOTHESIS ONLY AT A POINT

§1. The study of the order of magnitude of Fourier coefficients of the lacunary Fourier series (L) began with the consideration of the hypotheses on the function on a subinterval of $[-\pi, \pi]$. The result is due to Noble [23] who proves that ''if (L) is the Fourier series of f with $\{n_k\}$ satisfying (1.4) then $f \in \text{Lip}\,\alpha(I)$, $0 < \alpha < 1$, (respectively $f \in BV(I)$) implies (1.39) (respectively a_{n_k}, $b_{n_k} = O(n_k^{-1})$). Kennedy [17] then employed the Paley-Wiener method and proved this result under the less restrictive gap condition (1.1). He also showed later on [18] that the result doesn't hold under a still weaker gap condition (1.7). The same author then began the study by considering the hypotheses on a function on a subset E of $[-\pi, \pi]$, not necessarily a subinterval, and proved in his subsequent paper [19] that ''if $f \in \text{Lip}\,\alpha(E)$, $0 < \alpha < 1$ and E has positive measure, then (1.39) holds provided $\{n_k\}$ satisfies the Hadamard gap condition (1.2).'' He also proves that ''if $f \in \text{Lip}\,\alpha\,(E)$, $0 < \alpha < 1$ and E has positive spread, and if $\{n_k\}$ satisfies the condition (1.22) then (1.54) holds.'' He then conjectured that this result remains true if the term $\log n_k$ is suppressed in the gap condition (1.22). The answer to this, in fact considerably more, is provided by Theorem 30 due to Izumi. In the same paper [19 ; Theorem 3] Kennedy finally constructs a class of examples which shows that the estimate (1.54) is fairly sharp even when E has positive measure. The further study of the behaviour of Fourier coefficients involves the hypotheses on the function only at a point and starting with a paper of Tomić [36], the development is given in Chapter I. The entire study is really interesting and it reveals how nicely the more or less standard techniques are used to replace the hypothesis on the function from a subinterval to a subset of positive measure and then to a single point. The same techniques are further used in a nice way to better the estimates of Fourier coefficients under more general gap hypotheses. We generalize all these theorems by proving Theorem 31 in this chapter, and then we prove Theorem 32 concerning the absolute convergence. Finally, we show how our theorems generalize results of Izumi [15 ; Theorem 2] and Chao [6 ; Theorem 2].

The lacunarity condition considered by us in this chapter is the condition (1.47). Observe that the Hadamard gap condition (1.2) can be obtained from this condition by taking $F(n_k) = n_k$ for all $k \in N$. Also, with $F(n_k) = n_k^{\gamma} k^{\theta}$ $(0 < \gamma < 1 \; ; \; \theta \geq 0)$, it gives rise to the gap condition.

$$\min \{ n_{k+1} - n_k \, , \, n_k - n_{k-1} \} \geq C \, n_k^{\gamma} k^{\theta} \quad (0 < \gamma < 1 \, , \; \theta \geq 0). \quad (*)$$

We now proceed to prove our theorems.

§2. Proof of Theorem 31. Without loss of generality we may assume that $\omega^*(t)$ satisfies the required conditions (i), (ii) and (iii) in $(0, \pi)$ [40 ; P.91] and that $x_0 = 0$. Then, since $\omega^*(t)$ satisfies (i), (ii) and (iii), we have for any $\lambda \in R$, $\lambda > 1$,

$$A^{-1} \cdot \omega^*(t) \le \omega^*(\lambda t) \le B^{-1} \cdot \lambda^{\alpha} \cdot \omega^*(t) . \tag{5.1}$$

Let C_{n_k} be the n_k^{th} Complex Fourier coefficient of f, then

$$C_{n_k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; T_{M_k}(x) \; \exp(-in_k x) \; dx ,$$

where $T_{M_k}(x)$ is a trigonometric polynomial of degree $M_k = [C\, F(n_k)]$ ([ ] denotes the integral part) and with constant term 1. Then

$$C_{n_k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; T_{M_k}(x) \; \exp(-in_k x) \; dx$$

$$= - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n_k) \; T_{M_k}(x + \pi/n_k) \; \exp(-in_k x) \; dx$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) \; T_{M_k}(x) - f(x + \pi/n_k) \; T_{M_k}(x + \pi/n_k) \Big) \exp(-in_k x) dx$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\Big(f(x) - f(x + \pi/n_k)\Big)\, T_{M_k}(x)\ \exp(-in_k x)dx$$

$$+ \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x+\pi/n_k)\ \Big(T_{M_k}(x) - T_{M_k}(x+\pi/n_k)\Big)\ \exp(-in_k x)dx$$

$$= I + I' \text{, say.}$$

Since the Fourier exponents of $f(x + \pi/n_k)$ with non-vanishing Fourier coefficients are the same as those of $f(x)$ and the trigonometric polynomial $T_{M_k}(x) - T_{M_k}(x + \pi/n_k)$ is of degree not exceeding M_k and with the constant term zero, we have $I' = 0$. Hence

$$c_{n_k} = I = \frac{1}{4\pi}\left(\int_{-1/M_k}^{1/M_k} + \int_{1/M_k}^{\pi} + \int_{-\pi}^{-1/M_k}\right)\cdot$$

$$\cdot\Big(f(x)-f(x+\pi/n_k)\Big)\ T_{M_k}(x)\ \exp(-in_k x)dx$$

$$= I_1 + I_2 + I_3 \text{ , say.} \tag{5.2}$$

Now, using $\pi/n_k < \pi/M_k$ and (5.1), we obtain :

if $|x| \leq 1/M_k$ then

$$|f(x) - f(x+\pi/n_k)| = |f(x)-f(0)+f(0)-f(x+\pi/n_k)|$$
$$= O(1)\left(\omega^*(|x|) + \omega^*(|x| + \pi/n_k)\right)$$
$$= O(1)\left(\omega^*(1/M_k) + \omega^*((1 + \pi)/M_k)\right)$$
$$= O(1)\ \omega^*(1/M_k) \tag{5.3}$$

and if $1/M_k < |x|$ then

$$|f(x) - f(x+\pi/n_k)| = O(1)\left(\omega^*(|x|) + \omega^*(|x| + \pi/n_k)\right)$$
$$= O(1)\left(\omega^*(|x|) + \omega^*((1+\pi)|x|)\right)$$
$$= O(1)\ \omega^*(|x|)$$
$$= O(1)\cdot|x|^{\alpha}\cdot M_k^{\alpha}\cdot\omega^*(1/M_k). \tag{5.4}$$

Case 1. In case $0 < \alpha < 1$, in the property (iii) of $\omega^*(t)$, we take $T_{M_k}(x) = 2K_{M_k}(x)$, where $K_{M_k}(x)$ is the Fejèr kernel of order M_k. Hence

$$|T_{M_k}(x)| = \frac{\sin^2((M_k+1)x/2)}{(M_k+1)\sin^2(x/2)} \le D\, M_k \quad \text{and}$$

$$|T_{M_k}(x)| \le \frac{D'}{M_k\, x^2} \qquad (0 < |x| \le \pi), \tag{5.5}$$

where D and D' are constants. Therefore from (5.2), using (5.3) we get

$$|I_1| = O(1)\cdot \omega^*(1/M_k)\cdot D\cdot M_k\cdot (2/M_k)$$

$$= O(1)\cdot \omega^*(1/M_k)$$

and using (5.4) we get

$$|I_2| = O(1)\cdot M_k^{\alpha}\cdot \omega^*(1/M_k)\cdot (D'/M_k)\cdot \int_{1/M_k}^{\pi} x^{\alpha-2}\cdot dx$$

$$= O(1)\cdot \omega^*(1/M_k) .$$

Similarly we get $|I_3| = O(1)\cdot \omega^*(1/M_k)$. Further, from the inequality

$$1/M_k \geq 1/F(n_k) \geq 1/(M_k + 1)$$

and using (5.1) repeatedly, we get

$$\omega^*(1/M_k) = O(1)\ \omega^*(1/F(n_k)) .$$

This together with (5.2) and the estimates of I_1 , I_2 and I_3 , finally gives us

$$c_{n_k} = O(1)\ \omega^*(1/F(n_k)) .$$

<u>Case 2</u>. In case $\alpha \geq 1$, in the property (iii) of $\omega^*(t)$, we take

$$T_{M_k}(x) = \left(2\ K_N(x)\right)^P \Big/ \left(\int_{-\pi}^{\pi} \left(2\ K_N(x)\right)^P\cdot dx\right) , \qquad (5.6)$$

where $K_N(x)$ is the Fejèr kernel of order $N = \left[M_k / P \right]$ and P is such that $\alpha + 1 - 2P < 0$. Then

$$|T_{M_k}(x)| \leq E\, M_k \quad \text{and} \quad |T_{M_k}(x)| \leq E' \cdot M_k^{1-2P} \cdot x^{-2P} \quad (0<|x|\leq \pi), \quad (5.7)$$

where E and E' are constants (refer : [6 ; P.310]). Therefore, proceeding analogously to the case 1 and observing that

$$\alpha + 1 - 2P < 0 \text{ implies } \int_{1/M_k}^{\pi} x^{\alpha-2P} \cdot dx = O(1) \cdot M_k^{2P-\alpha-1}$$

we again get $|I_1|$, $|I_2|$, $|I_3| = O(1)\ \omega^*(1/M_k)$ and hence finally

$$c_{n_k} = O(1)\ \omega^*(1/F(n_k)) \quad .$$

Thus in both the cases we get (1.56) and hence the theorem.

Remark. Observe that this theorem generalizes Theorem 29 not only by replacing the Hadamard gap condition (1.2) by the more general gap condition (1.47), but also by suppressing the factor $\log n_k$ in the estimation (1.51) of Fourier coefficients. Further, taking $\omega^*(t)=t^{\alpha}$ $(0 < \alpha < 1)$, we see that this theorem gives theorem due to Chao [6 ; Theorem 1].

Proof of Theorem 32. For $k \in N$ put

$$f_k(x) = f(x + \pi/4n_k) - f(x - \pi/4n_k)$$

then

$$f_k(x) \sim \sum_j C_{n_j}\left(\exp\left(in_j(x+\pi/4n_k)\right) - \exp\left(in_j(x-\pi/4n_k)\right)\right)$$

$$= 2i\sum_j C_{n_j} \sin(n_j\pi/4n_k) \exp(in_jx) . \qquad (5.8)$$

If $T_{M_k}(x)$ is a trigonometric polynomial of order $[C\ F(n_k)]$ and with constant term 1, then the Fourier exponents with non-vanishing coefficients of $f_k(x)\ T_{M_k}(x)$ in the interval $(n_k , 2n_k)$ are the same as those of $f_k(x)$ in the same interval. Therefore we get from (5.8)

$$4\sum{}^{*}\left|C_{n_j}\right|^2 \sin^2\left(\frac{|n_j|\pi}{4\ n_k}\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f_k^2(x)\ T_{M_k}^2(x)\ dx ,$$

where $\sum^{*}$ signifies that summation is taken over all j satisfying

$$n_k \le |n_j| \le 2n_k .$$

Hence

$$\sum{}^{*}\left|C_{n_j}\right|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f_k^2(x)\cdot T_{M_k}^2(x)\cdot dx$$

$$= \frac{1}{2\pi}\left(\int_{|x| \le 1/M_k} + \int_{|x| > 1/M_k}\right) f_k^2(x)\ T_{M_k}^2(x)\ dx$$

$$= J_1 + J_2 \text{ , say.} \tag{5.9}$$

Now, taking $T_{M_k}(x) = 2\, K_{M_k}(x)$ and using (5.3), (5.4) and (5.5) we get

$$|J_1| = O(1)\left(\omega^*(1/M_k)\right)^2 \cdot M_k^2(2/M_k)$$

$$= O(1) \cdot M_k \cdot \left(\omega^*(1/M_k)\right)^2$$

and

$$|J_2| = O(1) \cdot \left(\omega^*(1/M_k)\right)^2 \cdot M_k^{2\alpha} \cdot \int_{|x|>1/M_k} \left(x^{2\alpha} / \left(M_k^2\, x^4\right)\right) dx$$

$$= O(1) \cdot M_k \cdot \left(\omega^*(1/M_k)\right)^2 .$$

Therefore from (5.9) we obtain

$$\sum^* |c_{n_j}|^2 = O(1)\, M_k\, \left(\omega^*(1/M_k)\right)^2$$

$$= O(1)\, F(n_k)\, \left(\omega^*(1/F(n_k))\right)^2 .$$

Hence, taking $n_k = 2^k$, an application of Hölder's inequality yields

$$\sum^* |c_{n_j}|^\beta = O(1)\, \left(\sum^* |c_{n_j}|^2\right)^{\beta/2} \left(\sum^* 1\right)^{1-\beta/2}$$

$$= O(1)\, \left(F(2^k)\right)^{\beta/2} \left(\omega^*(1/F(2^k))\right)^\beta \left(2^k / F(2^k)\right)^{1-\beta/2}$$

$$= O(1)\, (2^k)^{1-\beta/2} \left(F(2^k)\right)^{\beta-1} \left(\omega^*(1/F(2^k))\right)^\beta . \tag{5.10}$$

Finally, from (5.10), we have

$$\sum_{-\infty}^{\infty} |c_{n_j}|^{\beta} = \sum_{k=1}^{\infty} \left(\sum{}^{*} |c_{n_j}|^{\beta} \right)$$

$$= O(1) \tag{5.11}$$

on account of (1.57) and Cauchy's condensation test.

Note that in case $\alpha \geq 1$, in the property (iii) of $\omega^{*}(t)$, we take for $T_{M_k}(x)$ the polynomial considered in (5.6), use (5.7) and proceed as above to get (5.10) and hence (5.11). This completes the proof of Theorem 32.

We shall now show that the following theorems due to Chao [6 ; Theorem 2] and M. Izumi and S. I. Izumi [15 ; Theorem 2] can easily be obtained from our results.

<u>THEOREM A</u>. (Chao). If $f \in \mathrm{Lip}\,\alpha(P)$ $(0 < \alpha < 1)$ and if $\{n_k\}$ satisfies (*), then the Fourier series (L) of f converges absolutely provided $\alpha\gamma + \alpha\theta + \gamma > 1$.

<u>THEOREM B</u>. (Izumi). If $f \in \mathrm{Lip}\,\alpha(P)$ $(0 < \alpha < 1)$ and if $\{n_k\}$ satisfies (*) with $\theta = 0$, then the Fourier series (L) of f converges absolutely provided $\alpha > \min\{1/2\gamma , \gamma^{-1} - 1\}$.

Observe that the case when $\alpha > 1/\gamma - 1$ in Theorem B follows from Theorem A when $\theta = 0$.

We need the following Lemma. It is due to Chao [6 ; Proof of Theorem 2] but is not explicitely stated there.

LEMMA. If $\{n_k\}$ satisfies (*) then

$$n_k > A\, k^{\delta}$$

for any $\delta < (1+\theta)/(1-\gamma)$ and for all sufficiently large k, where A is some constant.

Now, suppose the hypothesis of Theorem A holds. Therefore $\alpha\gamma + \alpha\theta + \gamma > 1$ and hence $(1-\alpha\theta)/\alpha\gamma < (1+\theta)/(1-\gamma)$. Choose δ such that

$$(1-\alpha\theta)/\alpha\gamma < \delta < (1+\theta)/(1-\gamma). \tag{5.12}$$

Since $f \in \mathrm{Lip}\,\alpha(P)$, in view of (1.55) we take $\omega^*(t) = t^{\alpha}$ in the hypothesis of Theorem 31. Hence, observing that $F(n_k) = n_k^{\gamma}\, k^{\theta}$, we get using the Lemma:

$$\omega^*(1/F(n_k)) = O(1)\left(1/(n_k^{\gamma}\, k^{\theta})^{\alpha}\right)$$

$$= O(1)\left(1/k^{\alpha\delta\gamma + \alpha\theta}\right).$$

Therefore, using (5.12) and applying Theorem 31, we obtain

$$\sum_{-\infty}^{\infty} |c_{n_k}| = O(1) \sum_{k=1}^{\infty} \omega^*(1/F(n_k))$$

$$= O(1) \sum_{k=1}^{\infty} \left(1/k^{\alpha\delta\gamma + \alpha\theta}\right)$$

$$= O(1).$$

Thus, the Fourier series (L) of f converges absolutely and hence the Theorem A.

The case $\alpha > (\gamma^{-1} - 1)$ of Theorem B is obviously included in this.

Again, when $f \in \text{Lip}\,\alpha(P)$, $F(n_k) = n_k^{\gamma}$ and $\alpha > 1/2\gamma$ then taking $\omega^*(t)=t^{\alpha}$ and $\beta = 1$ in Theorem 32, we obtain

$$\sum_{n=1}^{\infty} \left(\frac{\omega^*(1/F(n))}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \left(1 / n^{\alpha\gamma + 1/2}\right)$$

$$= O(1)$$

which implies the absolute convergence of (L). Thus the case $\alpha > 1/2\gamma$ of Theorem B is established.

Finally, in view of our Theorems 18 and 21 of Chapter III which involve the hypothesis on a function in terms of either the modulus of continuity or the modulus of smoothness, considered on a subset E of $[-\pi, \pi]$ of positive measure, we pose the question whether $\omega^*(1/F(n))$ in our Theorem 32 can be replaced by the modulus of continuity considered only at a point. We investigate this problem in the next chapter.