CHAPTER-V

LOCAL PROPERTY OF $\gamma-\left|\bar{N},p_n\right|_k$ SUMMABILITY OF FACTORED FOURIER SERIES

5.1 INTRODUCTION:

The convergence of a Fourier series at a point depends only upon the behaviour of a generating function in the immediate neighborhood of the point considered. In other words, however small δ may be, the convergence of $s_n(t)$ (the partial sum of a Fourier series), at $t=x$, depends only upon the nature of generating function $f(t)$ in the interval $(x-\delta, x+\delta)$, and is not affected by the values which it takes outside the interval. This property of Fourier series is known as local property.

Let $f(t)$ be a periodic function with period 2π and is integrable in the Lebesgue sense over the interval $(-\pi,\pi)$, and let its Fourier series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(t), \qquad (5.1.1)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad (n = 0,1,2,-----)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \qquad (n = 1,2,-----).$$

In 1950, Mohanty [42] has demonstrated that the summability $|R, \log n, 1|$ of

$$\sum_{n=1}^{\infty} \frac{A_n(t)}{[\log(n+1)]} \tag{5.1.2}$$

at a point $t = x$ is a local property of the generating function of $\sum A_n(t)$. This result was improved by Matsumoto [43] by replacing the series (5.1.2) by

$$\sum_{n=1}^{\infty} \frac{A_n(t)}{\{\log\log(n+1)\}^{1+\varepsilon}}, \quad \varepsilon > 0.$$

Generalizing the above result, S.N.Bhatt [7] proved the following theorem.

Theorem 16 [7]:

If (λ_n) is a convex sequence such that $\sum \frac{\lambda_n}{n}$ is convergent, then the summability $|R, \log n, 1|$ of the series

$$\sum_{n=1}^{\infty} A_n(t) \lambda_n \log n$$

at a point can be ensured by a local property.

On the other hand, Mishra [44] proved Theorem 16 in the following form

Theorem 17 :

Let the sequence (p_n) be such that

$P_n = O(np_n)$ and $P_n \Delta p_n = O(p_n p_{n+1})$.

Then the summability $\left|\overline{N}, p_n\right|$ of the series

$$\sum_{n=1}^{\infty} \lambda_n \frac{P_n}{np_n} A_n(t),$$

where (λ_n) is as in theorem 16, at a point can be ensured by a local property.

Further H.Bor [19] showed that $\left|\overline{N}, p_n\right|$ summability in Mishra's result can be replaced by more general summability $\left|\overline{N}, p_n\right|_k$, $k \geq 1$. Extending all this results, H.Bor [16] generalized the above theorems under more appropriate conditions as follows:

Theorem 18 [16] :

Let $k \geq 1$, and let the sequences $(p_n)$and (λ_n) be such that

$$\Delta X_n = O\left(\frac{1}{n}\right) \tag{5.1.3}$$

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty \tag{5.1.4}$$

$$\sum_{n=1}^{\infty}(X_n^k+1)|\Delta\lambda_n|<\infty, \tag{5.1.5}$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $\left|\overline{N}, p_n\right|_k$ of the series

$$\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n \tag{5.1.6}$$

at a point can be ensured by a local property.

5.2 MAIN RESULT:

In this chapter, we establish a theorem similar to above by considering more general summability $Y-\left|\overline{N}, p_n\right|_k$ introduced by S.M.Mazhar (see chapter-I, definition 9). In fact, we shall prove the following theorem.

Theorem L:

Let $k \geq 1$, and let (p_n), $(\lambda_n)$and (Y_n) be sequences of positive real constants such that $\left(\frac{Y_n p_n}{P_n}\right)$ is non-increasing.

Suppose

$$\Delta X_n = O\left(\frac{1}{n}\right) \tag{5.2.1}$$

$$\sum_{n=1}^{\infty}\left(\frac{Y_n}{n}\right)^{k-1}\frac{|\lambda_n|^k+|\lambda_{n+1}|^k}{n}<\infty \tag{5.2.2}$$

$$\sum_{n=1}^{\infty}\left(\frac{Y_n^{k-1}P_n}{n^k p_n}+1\right)|\Delta\lambda_n|<\infty, \tag{5.2.3}$$

where $X_n = \frac{P_n}{np_n}$. Then the summability $Y-|\overline{N},p_n|_k$ of the series (5.1.6) at a point can be ensured by a local property.

Remark 1 :

It can be observed that, if we put $Y_n = \frac{P_n}{p_n}$ in our theorem L , then we get Theorem 18 due to H.Bor. In this case the conditions (5.2.2) and (5.2.3) reduces to the conditions (5.1.8) and (5.1.9). Moreover the condition that the sequence $\left(\frac{Y_n p_n}{P_n}\right)$ is nonincreasing becomes redundant.

Remark 2 :

It is also interesting to observe that, if we put $Y_n = n$ for all n in our theorem L, then we get result for absolute Reisz summability $|R,p_n|_k$, $k \geq 1$.

5.3 LEMMA:

In order to establish the proof of our theorem L, we first prove the following lemma.

LEMMA:

Let $k \geq 1$, and let (p_n), $(\lambda_n)$and (Y_n) be sequences as in Theorem L such that conditions (5.2.1), (5.2.2) and (5.2.3) are satisfied. If the sequence (s_n) of the partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded, then the series

$$\sum_{n=1}^{\infty} a_n X_n \lambda_n \qquad (5.3.1)$$

is summable $Y-\left|\bar{N}, p_n\right|_k$.

PROOF OF LEMMA:

Let (t_n) denote the $(\bar{N}, p_n)$ means of the series (5.3.1). Then, by definition, we have

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{z=0}^{v} a_z \lambda_z X_z$$

$$= \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1}) a_v \lambda_v X_v, \qquad X_0 = 0. \qquad (5.3.2)$$

Then, for $n \geq 1$, we have

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v \lambda_v X_v. \qquad (5.3.3)$$

Applying Able's transformation on the right hand side of (5.3.3), we get

$$t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v \lambda_v X_v$$

$$= \frac{p_n}{P_n P_{n-1}}\left[\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v X_v)\sum_{r=1}^{v} a_r + P_{n-1}\lambda_n X_n \sum_{v=1}^{n} a_v\right]$$

$$= \frac{p_n}{P_n P_{n-1}}\left[\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v X_v) s_v + P_{n-1}\lambda_n X_n s_n\right]$$

$$= \frac{p_n}{P_n P_{n-1}} \left[\sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v X_v) s_v \right] + \frac{p_n \lambda_n s_n X_n}{P_n}$$

$$= \frac{p_n}{P_n P_{n-1}} \left[\sum_{v=1}^{n-1} (\lambda_v X_v \Delta P_{v-1} + P_v \Delta(\lambda_v X_v)) s_v \right] + \frac{p_n \lambda_n s_n X_n}{P_n}$$

$$= \frac{p_n}{P_n P_{n-1}} \left[\sum_{v=1}^{n-1} (\lambda_v X_v (P_{v-1} - P_v) + P_v (X_v \Delta\lambda_v + \lambda_{v+1} \Delta X_v)) s_v \right] + \frac{p_n \lambda_n s_n X_n}{P_n}$$

$$= \frac{p_n}{P_n P_{n-1}} \left[\sum_{v=1}^{n-1} (\lambda_v X_v (-p_v) + P_v (X_v \Delta\lambda_v + \lambda_{v+1} \Delta X_v)) s_v \right] + \frac{p_n \lambda_n s_n X_n}{P_n}$$

$$= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v X_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v X_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_{v+1} \Delta X_v + \frac{p_n s_n \lambda_n X_n}{P_n}.$$

$$= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}, \text{ say}.$$

Since by Minkowski's inequality

$$|t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}|^k \le 4^k \left(|t_{n,1}|^k + |t_{n,2}|^k + |t_{n,3}|^k + |t_{n,4}|^k \right),$$

it follows that, to complete the proof of the lemma , it is sufficient to show that

$$\sum_{n=2}^{\infty} Y_n^{k-1} |t_{n,r}|^k < \infty \quad , \text{ for } r = 1,2,3,4.$$

By applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} Y_n^{k-1} |t_{n,1}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v X_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{v=1}^{n-1} p_v |s_v||\lambda_v| X_v\right\}^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} p_v |s_v|^k |\lambda_v|^k X_v^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} p_v |s_v|^k |\lambda_v|^k X_v^k\right\}$$

$$= O(1)\sum_{v=1}^{m} p_v |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} p_v |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1}\left(\frac{Y_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_n p_n}{P_n}\right)^{k-1} p_v |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right)|\lambda_v|^k X_v^k$$

$$= O(1)\sum_{v=1}^{m} Y_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right)|\lambda_v|^k X_v^k$$

$$= O(1)\sum_{v=1}^{m} Y_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}|\lambda_v|^k X_v^k$$

$$= O(1)\sum_{v=1}^{m} Y_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}\left(\frac{P_v}{v p_v}\right)^{k}|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v}{v}\right)^{k-1}\frac{|\lambda_v|^k}{v}$$

$= O(1)$ as $m \to \infty$, by (5.2.2).

Again, we have

$$\sum_{n=2}^{m+1} Y_n^{k-1} |t_{n,2}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v X_v \right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v |s_v|^k |\Delta\lambda_v|^k X_v^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \right\}^{k-1}$$

Since

$$\sum_{v=2}^{n-1} P_v |\Delta\lambda_v| \leq P_{v-1} \sum_{v=2}^{n-1} |\Delta\lambda_v| \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v|$$

$$\leq \sum_{v=1}^{n-1} |\Delta\lambda_z|$$

$$= O(1) \text{ , by (5.2.3).}$$

Therefore,

$$\sum_{n=2}^{m+1} Y_n^{k-1} |t_{n,2}|^k$$

$$= O(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v |s_v|^k |\Delta\lambda_v|^k X_v^k \right\}$$

$$= O(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} Y_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Y_v p_v}{P_v}\right)^{k-1} P_v |\Delta\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v p_v}{P_v}\right)^{k-1}|\Delta\lambda_v|^k X_v^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{v p_v}\right)^{k}|\Delta\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v^{k-1} P_v}{v^k p_v}\right)|\Delta\lambda_v|^k$$

$= O(1)$ as $m \to \infty$, by (5.2.3).

Using the fact that $\Delta X_n = O\left(\frac{1}{n}\right)$, we have

$$\sum_{n=2}^{m+1} Y_n^{k-1}|t_{n,3}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1}\left|-\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v s_v \lambda_{v+1}\Delta X_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1} P_v|\lambda_{v+1}|\Delta X_v\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{v}\right)|\lambda_{v+1}|\right\}^k$$

$$= O(1)\sum_{v=1}^{m} p_v|\lambda_{v+1}|^k X_v^k \sum_{n=v+1}^{m+1} Y_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} Y_v^{k-1}\left(\frac{p_v X_v}{P_v}\right)^k|\lambda_{v+1}|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Y_v}{v}\right)^{k-1}\frac{|\lambda_{v+1}|^k}{v}$$

$= o(1)$ as $m \to \infty$, by (5.2.2).

Finally,we have

$$\sum_{n=2}^{m+1} Y_n^{k-1} |t_{n,4}|^k$$

$$= \sum_{n=2}^{m+1} Y_n^{k-1} \left| \frac{p_n s_n \lambda_{nX_n}}{P_n} \right|^k$$

$$= o(1) \sum_{n=2}^{m+1} Y_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |\lambda_n|^k X_n^k$$

$$= o(1) \sum_{n=2}^{m+1} \left(\frac{Y_n}{n} \right)^k \frac{|\lambda_n|^k}{n}$$

$= o(1)$ as $m \to \infty$, by (5.1.12).

This completes the proof of lemma.

5.4 PROOF OF THEOREM:

Since the convergence of Fourier series at a point is a local property of its generating function f, our Theorem follows from the above lemma.