CHAPTER-III

ALMOST INCREASING SEQUENCES AND THEIR APPLICATIONS

3.1 INTRODUCTION :

S.M.Mazhar [40] and Hüseyin Bor [13] studied $|C,1|_k$ and $|\overline{N},p_n|_k$ summabilities of infinite series by taking a non-decreasing sequence. A well known theorem on absolute Cesàro summability with order k of an infinite series is given by S.M.Mazhar by taking a non-decreasing sequence as under.

THEOREM 10 [40] :

Suppose (X_n) is a positive nondecreasing sequence and (λ_n) is a sequence such that

$$\lambda_m X_m = O(1) \text{ as } m \to \infty, \tag{3.1.1}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) , \tag{3.1.2}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \text{ as } m \to \infty. \tag{3.1.3}$$

Then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

Later on, Hüseyin Bor showed that the above result can be extended for more general summability $|\overline{N},p_n|_k$, $k \geq 1$. In fact, his result is as follows.

THEOREM 11 [13] :

Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad \text{as } n \to \infty. \tag{3.1.4}$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\lambda_m X_m = O(1) \text{ as } m \to \infty, \tag{3.1.5}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \ , \tag{3.1.6}$$

and

$$\sum_{n=1}^{m} \left(\frac{p_n}{P_n}\right) |t_n|^k = O(X_m) \text{ as } m \to \infty \ , \tag{3.1.7}$$

then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

It may be observed that Theorem 10 can be obtained from Theorem 11 by putting $p_n = 1$ for all values of n.

Hüseyin Bor also proved the following result on $|\overline{N}, p_n|_k$ summability.

THEOREM 12 [18] :

Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \text{ as } n \to \infty \ . \tag{3.1.8}$$

Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (λ_n) and (β_n) such that

$$|\Delta\lambda_n| \le \beta_n , \tag{3.1.9}$$

$$\beta_n \to 0 \text{ as } n \to \infty, \tag{3.1.10}$$

$$\sum_{n=1}^{\infty} nX_n |\Delta\beta_n| < \infty \quad , \tag{3.1.11}$$

and

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty . \tag{3.1.12}$$

If

$$\sum_{n=1}^{m} \left(\frac{p_n}{P_n} \right) |t_n|^k = O(X_m) \text{ as } m \to \infty \; , \tag{3.1.13}$$

where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \; , \tag{3.1.14}$$

then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$.

3.2 MAIN RESULTS:

In this chapter we intend to study the general summability $|\overline{N}, p_n, \phi_n|_k$ given by W.T.Sulaiman (see chapter-I, definition 8) of an infinite series by considering an almost increasing sequence. If we look upon the hypothesis of the above stated results, we find that the sequence (X_n), taken in theorem 10 to Theorem 12, is positive and non-decreasing and the summabilities considered were $|C,1|_k$ and $|\overline{N}, p_n|_k$, $k \ge 1$.

We will prove here two results similar to Huseyin Bor (Theorem 11 and Theorem 12), by weakening the hypothesis from nondecreasing to an almost increasing sequence on (X_n), and replacing the summability $\left|\overline{N},p_n\right|_k$ by more general summability $\left|\overline{N},p_n,\phi_n\right|_k$. In fact we shall prove the following Theorems.

THEOREM I [57] :

Let (p_n) be a sequence of positive numbers such that as $n\to\infty$

$$P_n = O(np_n). \tag{3.2.1}$$

If (X_n) be an almost increasing sequence such that the conditions (3.1.5) and (3.1.6) of Theorem 11 are satisfied and

$$\sum_{n=1}^{m}\phi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m) \text{ as } m\to\infty , \tag{3.2.2}$$

where (ϕ_n) be a sequence of positive real constants such that $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing , then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is summable $\left|\overline{N},p_n,\phi_n\right|_k$, $k\ge 1$.

THEOREM J [58] :

Let (p_n) be a sequence of positive numbers such that as $n\to\infty$

$$\sum_{\nu=1}^{n}\left(\frac{P_\nu}{\nu}\right) = O(P_{n-1}) . \tag{3.2.3}$$

Let (X_n) be an almost increasing sequence. Suppose that there exist sequences (λ_n) and (β_n) such that the conditions (3.1.9) to (3.1.12) of Theorem 12 are satisfied and

$$\sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |t_n|^k = O(X_m) \text{ as } m \to \infty , \tag{3.2.4}$$

where (ϕ_n) be a sequence of positive real constants such that $\left(\frac{\phi_n p_n}{P_n} \right)$ is nonincreasing with

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v .$$

Then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|\overline{N}, p_n, \phi_n|_k$, $k \geq 1$.

Remark :

It can be observed that, if we take $\phi_n = \frac{P_n}{p_n}$ and the sequence (X_n) to be positive and nondecreasing in our Theorem I and Theorem J, then we get Theorem 11 and Theorem 12 due to Hüseyin Bor . In this case conditions (3.2.2) and (3.2.4) will be reduced to conditions (3.1.7) and (3.1.13), while the condition that $\left(\frac{\phi_n p_n}{P_n} \right)$ is non-increasing sequence becomes redundant.

It has been already remarked earlier in chapter-I that every increasing sequence is an almost increasing sequence but converse need not be true. Thus almost increasing sequence is a weaker condition than the increasing sequence. Moreover, we are also replacing the condition (3.1.8) by a weaker condition (3.2.3) in Theorem J, at the same time we are also considering the general summability method given by W.T.Sulaiman [50]. In view of these observations, it could be seen that our Theorem I and Theorem J are the generalizations of Theorem 11 and Theorem 12.

3.3 PROOF OF THE THEOREMS:

In order to establish the proof of our Theorems, we need the following lemmas proved by S.M.Mazhar [39].

Lemma 1

If the sequences (X_n) and (λ_n) satisfy the conditions taken in Theorem I , then

$$nX_n|\Delta\lambda_n| = O(1) \text{ as } n \to \infty \tag{3.3.1}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty \tag{3.3.2}$$

$$X_n|\lambda_n| = O(1) \text{ as } n \to \infty \tag{3.3.3}$$

Lemma 2 :

If the sequences (X_n) , (β_n) and (λ_n) satisfy the conditions taken in the Theorem J , then

$$n\beta_n X_n = O(1) \text{ as } n \to \infty \tag{3.3.4}$$

$$\sum_{n=1}^{\infty} X_n\beta_n < \infty . \tag{3.3.5}$$

PROOF OF THEOREM I:

Let (T_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum_{n=0}^{\infty} a_n \lambda_n$.

Then by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{z=0}^{v} a_z \lambda_z$$

$$= \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1}) a_v \lambda_v \text{, by (2.3.1).} \qquad (3.3.6)$$

Now, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v \lambda_v$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} \frac{P_{v-1} v a_v \lambda_v}{v} \text{, by (2.3.2)} \qquad (3.3.7)$$

Applying Able's transformation to the right hand side of (3.3.7) we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\left[\sum_{v=1}^{n-1} \Delta\left(\frac{P_v \lambda_v}{v}\right)\sum_{z=1}^{v} z a_z - \frac{P_{n-1}\lambda_n}{n}\sum_{z=1}^{n} z a_z \right].$$

But,

$$\Delta\left(\frac{\lambda_v}{v} P_{v-1}\right) = \frac{\lambda_v}{v}\Delta(P_{v-1}) - P_v \Delta\left(\frac{\lambda_v}{v}\right)$$

$$= \frac{\lambda_v}{v}\Delta(P_v - P_{v-1}) - P_v\left(\frac{1}{v}\Delta\lambda_v - \lambda_{v+1}\Delta\left(\frac{1}{v}\right)\right)$$

$$= \frac{\lambda_v}{v}(p_v) - \frac{P_v}{v}(\Delta\lambda_v) + P_v\lambda_{v+1}\Delta\left(\frac{1}{v}\right)$$

$$= \frac{\lambda_v p_v}{v} - \frac{P_v \Delta\lambda_v}{v} + \frac{P_v\lambda_{v+1}}{v(v+1)} .$$

Therefore,

$$T_n - T_{n-1} = \frac{(n+1)p_n t_n \lambda_n}{nP_n} - \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{v+1}{v}\right)p_v t_v \lambda_v +$$

$$\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{v+1}{v}\right)P_v t_v \Delta\lambda_v + \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{1}{v}\right)P_v t_v \lambda_{v+1}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ , say .} \tag{3.3.8}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \le 4^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k\right),$$

we see that, to complete the proof of the Theorem I , it is enough to show that

$$\sum_{n=1}^{\infty}\phi_n^{k-1}|T_{n,z}|^k < \infty \text{ , for } z = 1,2,3,4. \tag{3.3.9}$$

First we have,

$$\sum_{n=1}^{m}\phi_n^{k-1}|T_{n,1}|^k$$

$$= \sum_{n=1}^{m}\phi_n^{k-1}\left|\frac{(n+1)p_n t_n \lambda_n}{nP_n}\right|^k$$

$$= O(1)\sum_{n=1}^{m}\phi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k |t_n|^k |\lambda_n|^{k-1}|\lambda_n|$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\lambda_n| |t_n|^k$$

$$= O(1)\ \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v}\right)^k \ +\ O(1)\ |\lambda_m| \sum_{v=1}^{m} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v}\right)^k$$

$$= O(1)\ \sum_{n=1}^{m-1} \Delta|\lambda_n| X_n \ +\ O(1)\ |\lambda_m| X_m$$, by (3.2.2)

$$= O(1)\ \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n \ +\ O(1)\ |\lambda_m| X_m$$

$= O(1)$ as $m \to \infty$, by ((3.1.5) and(3.3.2)).

Again applying Hölder's inequality with indices k and k', where 1/k +1/k'=1, and using the fact that

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{1}{P_v}\right).$$

we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,2}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left| -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v}\right) p_v t_v \lambda_v \right|^k$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} p_v |t_v| |\lambda_v|\right)^k$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\}$$

$$= O(1)\ \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \left(\frac{\phi_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\ \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v}\right)^{k-1} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\ \sum_{v=1}^{m} \phi_v^{k-1} |\lambda_v| \left(\frac{p_v}{P_v}\right)^k |t_v|^k$$

$$= O(1)\ \sum_{v=1}^{m-1} \Delta|\lambda_n| \sum_{i=1}^{v} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k + O(1)\ |\lambda_m| \sum_{i=1}^{m} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= O(1)\ \sum_{v=1}^{m-1} \Delta|\lambda_v| X_v + O(1)\ |\lambda_m| X_m \quad \text{, by (3.2.2)}$$

$$= O(1)\ \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)\ |\lambda_m| X_m$$

$$= O(1) \quad \text{as } m \to \infty \quad \text{, by ((3.1.5) and(3.3.2)).}$$

Similary , we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,3}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v}\right) P_v t_v \Delta\lambda_v \right|^k$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} P_v |t_v| |\Delta\lambda_v| \right)^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} v|\Delta\lambda_v|^k p_v |t_v|^k\right\} \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}, \text{ by (3.2.1)}$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{\sum_{v=1}^{n-1} v|\Delta\lambda_v|^k p_v |t_v|^k\right\}$$

$$= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v|^k p_v |t_v|^k \sum_{n=1}^{v+1} \left(\frac{\phi_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v}\right)^{k-1} v|\Delta\lambda_v|^k p_v |t_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \phi_v^{k-1} v|\Delta\lambda_v| \left(\frac{p_v}{P_v}\right)^k |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \left|(v\Delta|\lambda_n|)\right| \sum_{i=1}^{v} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k + O(1) \; m|\Delta\lambda_m| \sum_{i=1}^{m} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= O(1) \sum_{v=1}^{m-1} v\Delta|\lambda_v| X_v + O(1) \; |\lambda_m| X_m \text{ , by (3.2.2)}$$

$$= O(1) \sum_{v=1}^{m-2} vX_v |\Delta^2\lambda_v| + \sum_{v=1}^{m-1} \Delta|\lambda_{v+1}| X_v + O(1) \; m|\Delta\lambda_m| X_m$$

$= O(1)$ as $m \to \infty$, by ((3.1.6) , (3.3.3) and (3.3.1)).

Finally, by using the fact that $P_n = O(np_n)$ and as in $T_{n,3}$, we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,4}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left|\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{1}{v}\right) P_v t_v \lambda_{v+1}\right|^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{v=1}^{n-1} p_v |t_v| |\lambda_{v+1}| \right)^k$$

$$= O(1) \sum_{v=1}^{m} \phi_v^{k-1} |\lambda_{v+1}| \left(\frac{p_v}{P_v} \right)^k |t_v|^k$$

$= O(1)$ as $m \to \infty$, by ((3.1.6) , (3.3.3) and(3.3.1)).

Therefore, we get

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |T_{n,z}|^k < \infty \quad \text{, for } z = 1,2,3,4.$$

This completes the proof of theorem I.

PROOF OF THEOREM J :

Let (T_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum_{n=0}^{\infty} a_n \lambda_n$. Then, by (3.3.6) , (3.3.7) and (3.3.8) we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{z=0}^{v} a_z \lambda_z ,$$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} v a_v \lambda_v}{v},$$

and

$$T_n - T_{n-1} = \frac{(n+1) p_n t_n \lambda_n}{n P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v} \right) p_v t_v \lambda_v +$$

$$\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v} \right) P_v t_v \Delta\lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{1}{v} \right) P_v t_v \lambda_{v+1}$$

$$= \quad T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ , say .}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k\right),$$

to complete the proof of the theorem J , it is enough to show that

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |T_{n,z}|^k < \infty \quad \text{, for } z = 1,2,3,4. \tag{3.3.10}$$

First we have ,

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,1}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left| \frac{(n+1) p_n t_n \lambda_n}{n P_n} \right|^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |t_n|^k |\lambda_n|^{k-1} |\lambda_n|$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |\lambda_n| |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v} \right)^k + O(1) \ |\lambda_m| \sum_{v=1}^{m} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v} \right)^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| X_n + O(1) \ |\lambda_m| X_m \quad \text{, by (3.2.4)}$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) \ |\lambda_m| X_m$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \ |\lambda_m| X_m \quad \text{by (3.1.9)}$$

$= O(1)$ as $m \to \infty$, by ((3.1.12) and (3.3.5)).

Again applying Hölder's inequality with indices k and k', where 1/k +1/k'=1, we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,2}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left| -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v} \right) p_v t_v \lambda_v \right|^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \left(\frac{\phi_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v} \right)^{k-1} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \phi_v^{k-1} |\lambda_v| \left(\frac{p_v}{P_v} \right)^k |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_n| \sum_{i=1}^{v} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i} \right)^k + O(1) |\lambda_m| \sum_{i=1}^{m} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i} \right)^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| X_v + O(1) |\lambda_m| X_m \text{ , by (3.2.4)}$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m$$

$$= O(1)\ \sum_{v=1}^{m-1} \beta_n X_v + O(1)\ |\lambda_m| X_m \quad \text{by (3.1.9)}$$

$$= O(1) \quad \text{as } m \to \infty \text{ , by ((3.1.12) and(3.3.5)).}$$

Again, we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,3}|^k$$

$$= \sum_{n=1}^{m} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{v+1}{v} \right) P_v t_v \Delta\lambda_v \right|^k$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} P_v |t_v|\ |\Delta\lambda_v| \right\}^k$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} \frac{P_v}{v} v |t_v|\ \beta_v \right\}^k \quad \text{by (3.1.9)}$$

$$= O(1)\ \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} (v\beta_v)^k p_v |t_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{v} \right\}^{k-1} \text{, by (3.3.3)}$$

$$= O(1)\ \sum_{v=1}^{m} (v\beta_v)^k (v\beta_v) p_v |t_v|^k \sum_{n=v+1}^{\infty} \left(\frac{\phi_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\ \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v} \right)^{k-1} (v\beta_v) p_v |t_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\ \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v} \right)^{k-1} (v\beta_v) p_v |t_v|^k \frac{1}{P_v}$$

$$= O(1)\ \sum_{v=1}^{m} \phi_v^{k-1} (v\beta_v) \left(\frac{p_v}{P_v} \right)^k |t_v|^k$$

$$= O(1)\ \sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\phi_i^{k-1}|t_i|^k\left(\frac{p_i}{P_i}\right)^k + O(1)\ m\beta_m\sum_{i=1}^{m}\phi_i^{k-1}|t_i|^k\left(\frac{p_i}{P_i}\right)^k$$

$$= O(1)\ \sum_{v=1}^{m-1}vX_v|\Delta\beta_v| + O(1)\sum_{v=1}^{m-1}X_v\beta_{v+1} + O(1)\ |\lambda_m|X_m \quad \text{, by (3.2.4)}$$

$$= O(1) \quad \text{as } m\to\infty \text{ , by [(3.1.11) , (3.1.12) and(3.3.5)].}$$

Finally, we have

$$\sum_{n=1}^{m}\phi_n^{k-1}|T_{n,4}|^k$$

$$= \sum_{n=1}^{m}\phi_n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{1}{v}\right)P_vt_v\lambda_{v+1}\right|^k$$

$$= O(1)\ \sum_{n=1}^{m}\phi_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}\lambda_{v+1}|t_v|\right\}^k$$

$$= O(1)\ \sum_{n=1}^{m}\phi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}|^k|t_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right\}^{k-1} \text{, by (3.3.3)}$$

$$= O(1)\ \sum_{n=1}^{m}\phi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}|^k|t_v|^k\right\}$$

$$= O(1)\ \sum_{v=1}^{m}p_v|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{\infty}\left(\frac{\phi_np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\ \sum_{v=1}^{m}\phi_v^{k-1}|\lambda_v|\left(\frac{p_v}{P_v}\right)^k|t_v|^k$$

$$= O(1) \quad \text{as } m\to\infty.$$

Therefore, we get

$$\sum_{n=1}^{\infty}\phi_n^{k-1}|T_{n,z}|^k < \infty \quad \text{, for } z = 1,2,3,4.$$

This completes the proof of theorem J.

3.4 APPLICATIONS OF OUR THEOREMS:

1. If we take p_n=1 and ϕ_n=1 for all values of n in our Theorem I and Theorem J , then we get results concerning the $|C,1|_k$ summability method.
2. If we take $\phi_n = n$ for all values of n in our Theorem I and Theorem J, then we get results concerning the Absolute Ries'z summability method of order k (i.e. $|R, p_n|_k$) .