CHAPTER-IV

ABSOLUTE SUMMABILITY FACTORS OF AN INFINITE SERIES

4.1 INTRODUCTION :

It is well known that Hüseyin Bor did the pioneering work in the study of $\left|\overline{N},p_n\right|_k$ and $\left|\overline{N},p_n;\gamma\right|_k$ summability methods of an infinite series and proved many results in these directions. If we look upon the definitions of $\left|\overline{N},p_n\right|_k$ and $\left|\overline{N},p_n,\gamma\right|_k$ summability due to Hüseyin Bor (see chapter-I, definition 6 & 7), we find that, by introducing the parameter $\gamma \geq 0$, he extended $\left|\overline{N},p_n\right|_k$ summability to $\left|\overline{N},p_n,\gamma\right|_k$ summability. Likewise we will extend the definition of $\left|\overline{N},p_n,\gamma\right|_k$ summability by introducing parameter $\alpha \geq 0$ and we denote this summability by $\left|\overline{N}_p,\gamma,\alpha\right|_k$. Now, before defining the $\left|\overline{N}_p,\gamma,\alpha\right|_k$ summability, we first recall the definitions of $\left|\overline{N},p_n\right|_k$ and $\left|\overline{N},p_n,\gamma\right|_k$ summabilites given by Hüseyin Bor.

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $\left|\overline{N},p_n\right|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n - t_{n-1}\right|^k < \infty,$$

and it is said to be summable $|\overline{N}, p_n; \gamma|_k$, $k \geq 1$, $\gamma \geq 0$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\gamma k + k - 1} |t_n - t_{n-1}|^k < \infty .$$

We now define $|\overline{N}_p, \gamma, \alpha|_k$ summability as under:

Definition 12:

The series $\sum_{n=0}^{\infty} a_n$ is said to summable $|\overline{N}_p, \gamma, \alpha|_k, k \geq 1,\ \gamma \geq 0$ and $\alpha(k\gamma + k - 1) \geq k - 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\alpha(\gamma k + k - 1)} |t_n - t_{n-1}|^k < \infty . \qquad (4.1.1)$$

It is clear that, if we put

(i) $\gamma = 0$ and $\alpha = 1$ in (4.1.1) , then $|\overline{N}_p, \gamma, \alpha|_k$ summability reduces to $|\overline{N}, p_n|_k$ summability , and

(ii) if we put $\alpha = 1$ in (4.1.1) , then $|\overline{N}_p, \gamma, \alpha|_k$ summability reduces to the $|\overline{N}, p_n; \gamma|_k$ summability .

In the year 1976, F.M.Khan proved the following theorem:

THEOREM 13 [32]:

If $\sum_{n=0}^{\infty} a_n$ is $|\overline{N}, p_n|$ summable, then $\sum_{n=0}^{\infty} a_n \lambda_n$ is $|\overline{N}, q_n|$ summable provided (p_n) and (q_n) are positive sequences such that as $n \to \infty$

$$\frac{p_n}{P_n} = O\left(\frac{q_n}{Q_n}\right) \tag{4.1.2}$$

$$\frac{q_n P_n \lambda_n}{p_n Q_n} = O(1) \tag{4.1.3}$$

and

$$P_n \Delta\lambda_n = O(p_n). \tag{4.1.4}$$

This result of F.M.Khan was generalized by Huseyin Bor as follows:

THEOREM 14 [20]:

Let $k \geq 1$. If $\sum_{n=0}^{\infty} a_n$ is $|\overline{N}, p_n|_k$ summable, then $\sum_{n=0}^{\infty} a_n \lambda_n$ is $|\overline{N}, q_n|_k$ summable provided (p_n) and (q_n) are positive sequences which satisfy the conditions (4.1.2), (4.1.3) and (4.1.4).

Here it is easy to observe that, Theorem 13 can be obtained from Theorem 14 by putting k = 1.

Later on, in 1986, Hüseyin Bor extended Theorem 14 for $|\overline{N}, p_n; \gamma|_k$ summability as under:

THEOREM 15 [15]:

Let $k \geq 1$ and $\gamma \geq 0$. If $\sum_{n=0}^{\infty} a_n$ is $|\overline{N}, p_n, \gamma|_k$ summable, then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is $|\overline{N}, q_n|_k$ summable provided that (p_n) and (q_n) are positive sequences which satisfy the conditions (4.1.2), (4.1.3) and (4.1.4).

4.2 MAIN RESULT:

In this chapter we establish the following result on $|\overline{N}_p,\gamma,\alpha|_k$ summability defined by us (see definition 12).

THEOREM K [56]:

Suppose

$$k \geq 1,\ \gamma \geq 0 \text{ and } \alpha(k\gamma+k-1) \geq k-1 . \tag{4.2.1}$$

If the series $\sum_{n=0}^{\infty} a_n$ is $|\overline{N_p},\gamma,\alpha|_k$ summable, then the series $\sum_{n=0}^{\infty} a_n\lambda_n$ is $|\overline{N},q_n|_k$ summable provided (p_n) and (q_n) are positive sequences which satisfy the conditions (4.1.2), (4.1.3) and (4.1.4).

REMARK :

It is interesting to observe that, if we put $\alpha=1$, $\gamma=0$, and $k=1$ in our theorem K, then we get theorem 13 due to F.M.Khan. Further if we put $\alpha=1$, $\gamma=0$ and $\alpha=1$ in our theorem K then we get theorem 14 and theorem 15 respectively. Thus we observe that, our Theorem generalizes Theorem 13 to Theorem 15.

4.3 PROOF OF THEOREM K :

Since the series $\sum_{n=0}^{\infty} a_n$ is summable $|\overline{N}_p,\gamma,\alpha|_k$ it follows that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\alpha(\gamma k+k-1)}|t_n - t_{n-1}|^k < \infty . \qquad (4.3.1)$$

Let (t_n) be a sequence of $(\overline{N},p_n)$ means of the series $\sum_{n=0}^{\infty} a_n$. Then, by (2.2.1), we have

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v . \qquad (4.3.2)$$

Then for $n \geq 1$, we have

$$\begin{aligned}
\Delta t_{n-1} &= t_n - t_{n-1} = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})a_v - \frac{1}{P_{n-1}}\sum_{v=1}^{n-1}(P_{n-1} - P_{v-1})a_v \\
&= \frac{1}{P_n}\sum_{v=0}^{n}P_n a_v - \frac{1}{P_n}\sum_{v=0}^{n}P_{v-1}a_v - \frac{1}{P_{n-1}}\sum_{v=0}^{n-1}P_{n-1}a_v + \frac{1}{P_{n-1}}\sum_{v=0}^{n-1}P_{v-1}a_v \\
&= \sum_{v=0}^{n}a_v - \frac{1}{P_n}\sum_{v=0}^{n}P_{v-1}a_v - \sum_{v=0}^{n-1}a_v + \frac{1}{P_{n-1}}\sum_{v=0}^{n-1}P_{v-1}a_v \\
&= \frac{1}{P_{n-1}}\sum_{v=0}^{n-1}P_{v-1}a_v - \frac{1}{P_n}\sum_{v=0}^{n}P_{v-1}a_v + a_n \\
&= \frac{1}{P_{n-1}}\sum_{v=0}^{n}P_{v-1}a_v - \frac{1}{P_n}\sum_{v=0}^{n}P_{v-1}a_v \\
&= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{v=1}^{n}P_{v-1}a_v \\
&= -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v . \qquad (4.3.3)
\end{aligned}$$

Therefore

$$a_n = -\frac{P_n}{p_n}\Delta t_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-2}\ , \text{ by (2.2.9)} \qquad (4.3.4)$$

Similarly, if (T_n) denotes the $(\overline{N}, q_n)$ means of the series $\sum_{n=0}^{\infty} a_n$. Then, by (4.3.2) and (4.3.3), we have

$$T_n = \frac{1}{Q_n}\sum_{v=0}^{n}(Q_n - Q_{v-1})a_v \ , \tag{4.3.5}$$

and

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1} a_v \lambda_v \ . \tag{4.3.6}$$

Therefore

$$T_n - T_{n-1} = -\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1}\lambda_v\left[-\frac{P_v}{p_v}\Delta t_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta t_{v-2}\right], \quad \text{by (4.3.4)}$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n}\frac{P_v}{p_v} Q_{v-1}\lambda_v \Delta t_{v-1} - \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n}\frac{P_{v-2}}{p_{v-1}} Q_{v-1}\lambda_v \Delta t_{v-2}$$

$$= \frac{q_n P_n \lambda_n}{p_n Q_n}\Delta t_{v-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{\Delta t_{v-1}}{p_v}\{P_v Q_{v-1}\lambda_v - P_{v-1}Q_v\lambda_{v+1}\}$$

But

$$P_v Q_{v-1}\lambda_v - P_{v-1}Q_v\lambda_{v+1}$$

$$= (Q_v - q_v)P_v\lambda_v - (P_v - p_v)Q_v\lambda_{v+1}$$

$$= Q_v P_v \lambda_v - q_v P_v \lambda_v - P_v Q_v \lambda_{v+1} + p_v Q_v \lambda_{v+1}$$

$$= -q_v P_v \lambda_v + (\lambda_v - \lambda_{v+1}) + p_v Q_v \lambda_{v+1}$$

$$= -q_v P_v \lambda_v + P_v Q_v \Delta\lambda_v + Q_v p_v \lambda_{v+1} .$$

Thus

$$\Delta T_{n-1} = \frac{q_n P_n \lambda_n}{Q_n p_n}\Delta t_{n-1} - \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right) q_v \lambda_v \Delta t_{v-1} + \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right) Q_v \Delta\lambda_v \Delta t_{v-1}$$

$$+ \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} Q_v \lambda_{v+1}\Delta t_{v-1}$$

$$= \quad T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ , say.}$$

Since

$$\left|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\right|^k \le 4^k\left(\left|T_{n,1}\right|^k + \left|T_{n,2}\right|^k + \left|T_{n,3}\right|^k + \left|T_{n,4}\right|\right),$$

it follows that, to complete the proof of Theorem K , it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|T_{n,i}\right|^k < \infty, \text{ for } i = 1,2,3,4. \tag{4.3.7}$$

Firstly, we have

$$\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|T_{n,1}\right|^k$$

$$= \sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{q_n P_n}{Q_n p_n}\Delta t_{n-1}\right|^k$$

$$\le \sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{q_n P_n}{Q_n p_n}\right)^k\left|\Delta t_{n-1}\right|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\Delta t_{n-1}\right|^k \text{, by (4.1.3)}$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\Delta t_{n-1}\right|^k \text{, by (4.1.2)}$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}\left(\frac{p_n}{P_n}\right)^{\alpha(k\gamma+k-1)+1-k}|\Delta t_{n-1}|^k .$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}|\Delta t_{n-1}|^k, \quad \text{by ((1.2.12) and (1.2.13))}$$

$$= O(1) \quad \text{as } m\to\infty, \text{(4.3.1).}$$

Again by applying Hölder's inequality, we have

$$\sum_{n=1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,2}|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)q_v\lambda_v\Delta t_{v-1}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)\left(\frac{q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)q_v|\lambda_v||\Delta t_{v-1}|\right\}^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k q_v|\lambda_v||\Delta t_{v-1}|^k\right\}\left\{\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k q_v|\Delta t_{v-1}|^k\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k q_v|\Delta t_{v-1}|^k\left\{\sum_{n=v+1}^{m+1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\left(\frac{q_v}{Q_v}\right)|\Delta t_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} |\Delta t_{v-1}|^k \text{ , by (4.1.3)}$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)} \left(\frac{p_n}{P_n}\right)^{\alpha(k\gamma+k-1)+1-k} |\Delta t_{n-1}|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)} |\Delta t_{n-1}|^k \text{ , by ((1.2.12) and (1.2.13))}$$

$$= O(1) \text{ as } m \to \infty \text{ , by (4.3.1).}$$

Similarly, we have

$$\sum_{n=1}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} |T_{n,3}|^k$$

$$= \sum_{n=2}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) Q_v \Delta\lambda_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v}\right) q_v |\Delta t_{v-1}| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{q_n}{Q_n Q_{n-1}}\right) \left\{ \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta t_{v-1}|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta t_{v-1}|^k \left\{ \sum_{n=v+1}^{m+1} \left(\frac{q_n}{Q_n Q_{n-1}}\right) \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} |\Delta t_{v-1}|^k \text{ , by (4.1.2)}$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}\left(\frac{p_n}{P_n}\right)^{\alpha(k\gamma+k-1)+1-k}|\Delta t_{n-1}|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}|\Delta t_{n-1}|^k \text{ , by (1.2.12)}$$

$$= O(1) \text{ as } m\to\infty \text{ , by (4.3.1)}$$

Finally, we have

$$\sum_{n=1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,4}|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}Q_v\Delta\lambda_{v+1}\Delta t_{v-1}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{v=1}^{n-1}\left(\frac{Q_v}{q_v}\right)q_v|\Delta t_{v-1}|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)\left\{\sum_{v=1}^{n-1}\left(\frac{Q_v}{q_v}\right)^k q_v|\Delta t_{v-1}|^k\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{Q_v}{q_v}\right)^k q_v|\Delta t_{v-1}|^k\left\{\sum_{n=v+1}^{m+1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta t_{v-1}|^k \text{ , by (4.1.2)}$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}\left(\frac{p_n}{P_n}\right)^{\alpha(k\gamma+k-1)+1-k}|\Delta t_{n-1}|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\alpha(k\gamma+k-1)}|\Delta t_{n-1}|^k \text{ , by (1.2.12)}$$

$$= O(1) \text{ as } m\to\infty \text{ , by (4.3.1)}$$

Hence, we get

$$\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,i}|^k < \infty, \text{ for } i=1,2,3,4.$$

This completes the proof of theorem K.