Chapter 5

Walsh−Fourier coefficients properties of functions of generalized bounded variations

5.1 Order of magnitude of Walsh−Fourier coefficients of functions of generalized bounded variations

5.1.1 New result for functions of one variable

Riemann-Lebesgue lemma [8, Vol.I, p.67] says that for any function $f \in L^1(\overline{\mathbb{T}})$, its Fourier coefficients $\hat{f}(m) \to 0$ as $|m| \to \infty$. It is a fact that there is no definite rate at which Fourier coefficients tend to zero; and the study of definite rate at which Fourier coefficients tend to zero has been carried out for the functions of bounded variation as well as for the functions of generalized bounded variations. In 1949, N. J. Fine [20, Theorem VI, p.383], using second mean value theorem, carried out the study of definite rate at which Walsh−Fourier coefficients tend to zero for the functions of bounded variation. N. J. Fine proved that if $f \in BV(\bar{\mathbb{I}})$ then its Walsh−Fourier coefficients $\hat{f}(m) = O\left(\frac{1}{m}\right)$, where $\mathbb{I} = [0, 1)$. Generalizing this result in 2008 [22], the order of magnitude of Walsh−Fourier coefficients of functions of the classes $\Lambda BV^{(p)}(\bar{\mathbb{I}})$ and $\phi\Lambda BV(\bar{\mathbb{I}})$ were estimated (Theorem T and U, p.34). We have estimated the order of magnitude of Walsh−Fourier coefficients of functions of the class $\Lambda BV(p(n)\uparrow\infty,\varphi,\bar{\mathbb{I}})$ as follows.

Theorem 5.1.1.1. *If* $f\in\Lambda BV(p(n)\uparrow\infty,\varphi,\bar{\mathbb{I}})$ $(1\le p(n)\uparrow\infty$ *as* $n\to\infty)$ *then*

$$\hat{f}(2^u)=O\left(\frac{1}{\left(\sum_{j=1}^{2^u}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}}}\right),$$

where

$$\tau(m)=min\{k:k\in\mathbb{N},\varphi(k)\ge m\},\ \ m\ge 1.$$

We need the following lemma to prove this theorem.

Lemma 5.1.1.2. ([*58*, *Lemma* 3.1, *p.*217]) *If* $f\in\Lambda BV(p(n)\uparrow p,\varphi,\bar{\mathbb{I}})$ $(1\le p\le\infty)$ *then* f *is bounded on* $\bar{\mathbb{I}}$.

Proof of Theorem 5.1.1.1. In view of the above Lemma 5.1.1.2, $f\in\Lambda BV(p(n)\uparrow\infty,\varphi,\bar{\mathbb{I}})$ implies f is bounded on $\bar{\mathbb{I}}$, and hence $f\in L^1(\bar{\mathbb{I}})$.

For fixed $u\in\mathbb{N}_0$, let $h=\frac{1}{2^{u+1}}$. Put

$$g(x)=f\left(x\dot{+}\frac{1}{2^u}\dot{+}\frac{1}{2^{u+1}}\right)-f(x),\quad for\ all\ x\in\bar{\mathbb{I}}.$$

Then $g\in L^1(\bar{\mathbb{I}})$. For $m=2^u$,

$$\psi_m(h)=\psi_{2^u}\left(\frac{1}{2^{u+1}}\right)=r_u\left(\frac{1}{2^{u+1}}\right)=r_0\left(\frac{2^u}{2^{u+1}}\right)=r_0\left(\frac{1}{2}\right)=-1,$$

where r_u and r_0 are as defined earlier on page 9.

Also,

$$\psi_m\left(\frac{1}{2^u}\right)=\psi_{2^u}\left(\frac{1}{2^u}\right)=r_u\left(\frac{1}{2^u}\right)=r_0\left(\frac{2^u}{2^u}\right)=r_0(1)=1.$$

Thus,

$$\hat{g}(m)=\int_{\bar{\mathbb{I}}}g(x)\,\psi_m(x)\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{I}} \left(f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right) - f(x) \right) \psi_m(x)\, dx \\
&= \int_{\mathbb{I}} f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right) \psi_m(x)\, dx - \int_{\mathbb{I}} f(x)\, \psi_m(x)\, dx \\
&= \int_{\mathbb{I}} f(x)\, \psi_m \left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right) dx - \hat{f}(m) \\
&= \int_{\mathbb{I}} f(x)\, \psi_m(x)\, \psi_m\left(\frac{1}{2^u}\right) \psi_m\left(\frac{1}{2^{u+1}}\right) dx - \hat{f}(m) \\
&= \psi_m\left(\frac{1}{2^u}\right) \psi_m\left(\frac{1}{2^{u+1}}\right) \hat{f}(m) - \hat{f}(m) \\
&= -2\hat{f}(m)
\end{aligned}
$$

and

$$
\begin{aligned}
2|\hat{f}(m)| &\le \int_{\mathbb{I}} \left| f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right) - f(x) \right| dx \\
&= \int_{\mathbb{I}} \left| f\left(\left(x \dot{+} \frac{1}{2^{u+1}}\right) \dot{+} \left(\frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right)\right) - f\left(x \dot{+} \frac{1}{2^{u+1}}\right) \right| dx \\
&= \int_{\mathbb{I}} \left| f\left(x \dot{+} \frac{1}{2^u}\right) - f\left(x \dot{+} \frac{1}{2^{u+1}}\right) \right| dx.
\end{aligned}
$$

Similarly, we get

$$
2|\hat{f}(m)| \le \int_{\mathbb{I}} \left| f\left(x \dot{+} \frac{4}{2^{u+1}}\right) - f\left(x \dot{+} \frac{3}{2^{u+1}}\right) \right| dx
$$

and in general we have

$$
2|\hat{f}(m)| \le \int_{\mathbb{I}} \left| f\left(x \dot{+} \frac{2j}{2^{u+1}}\right) - f\left(x \dot{+} \frac{(2j-1)}{2^{u+1}}\right) \right| dx,
$$

for all $j = 1, \cdots, 2^u - 1$.

Dividing both sides of the above inequality by λ_j and then summing over $j = 1$ to $2^u - 1$, we have

$$
2|\hat{f}(2^u)| \left(\sum_{j=1}^{2^u-1} \frac{1}{\lambda_j} \right) \le \int_{\mathbb{I}} \left(\sum_{j=1}^{2^u-1} \frac{|\Delta f_j(x)|}{(\lambda_j)^{\frac{1}{p(\tau(2^u))} + \frac{1}{q(\tau(2^u))}}} \right) dx,
$$

where

$$\Delta f_j(x) = f\left(x \dotplus \frac{2j}{2^{u+1}}\right) - f\left(x \dotplus \frac{(2j-1)}{2^{u+1}}\right)$$

and $q(\tau(2^u))$ is the index conjugate to $p(\tau(2^u))$.

Applying Hölder's inequality on the right side of the above inequality, we get

$$2|\hat{f}(2^u)|\left(\sum_{j=1}^{2^u-1}\frac{1}{\lambda_j}\right) \leq \int_{\overline{\mathbb{I}}}\left(\sum_{j=1}^{2^u-1}\frac{|\Delta f_j(x)|^{p(\tau(2^u))}}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}}\left(\sum_{j=1}^{2^u-1}\frac{1}{\lambda_j}\right)^{\frac{1}{q(\tau(2^u))}} dx.$$

Hence,

$$2|\hat{f}(2^u)|\left(\sum_{j=1}^{2^u-1}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}} \leq \int_{\overline{\mathbb{I}}}\left(\sum_{j=1}^{2^u-1}\frac{|\Delta f_j(x)|^{p(\tau(2^u))}}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}} dx. \tag{5.1}$$

For any $x \in \mathbb{R}$, all these points $x \dotplus 2jh,\ x \dotplus (2j-1)h,\ for\ j = 1, \cdots, 2^u - 1$, lie in the interval of length 1. Thus, $f \in \Lambda BV(p(n) \uparrow \infty, \varphi, \overline{\mathbb{I}})$ implies

$$\left(\sum_{j=1}^{2^u-1}\frac{|\Delta f_j(x)|^{p(\tau(2^u))}}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}} = O(1).$$

This together with $\sum_{j=1}^{2^u}\frac{1}{\lambda_j} \approx \sum_{j=1}^{2^u-1}\frac{1}{\lambda_j}$ and the above inequality (5.1) imply that

$$|\hat{f}(2^u)| = O\left(\frac{1}{\left(\sum_{j=1}^{2^u}\frac{1}{\lambda_j}\right)^{\frac{1}{p(\tau(2^u))}}}\right).$$

This completes the proof of the theorem.

5.1.2 New results for functions of two variables

In the Subsection 4.1.1, we have estimated the order of magnitude of double Fourier series coefficients of two variables measurable functions of generalized bounded variations in the sense of Vitali and Hardy. Here we estimate the order of magnitude of double Walsh−Fourier series coefficients of two variables measurable functions of generalized bounded variations in the sense of Vitali and Hardy.

Theorem 5.1.2.1. *If* $f \in \bigwedge BV^{(p)}(\bar{\mathbb{I}}^2) \cap L^p(\bar{\mathbb{I}}^2)$ $(p \geq 1)$ *then*

$$\hat{f}(2^u, 2^v) = O\left(\frac{1}{\left(\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p}}}\right). \tag{5.2}$$

Proof of Theorem 5.1.2.1. For fixed $u, v \in \mathbb{N}_0$, let $h_1 = \frac{1}{2^{u+1}}$ and $h_2 = \frac{1}{2^{v+1}}$. Put

$$g(x,y) = f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right) - f\left(x, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)$$
$$- f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y\right) + f(x,y),$$

for all $(x,y) \in \bar{\mathbb{I}}^2$.

For $m = 2^u$ and $n = 2^v$, $\psi_m(h_1) = \psi_n(h_2) = -1$ and $\psi_m\left(\frac{1}{2^u}\right) = \psi_n\left(\frac{1}{2^v}\right) = 1$ imply that

$$\hat{g}(m,n) = \int\int_{\bar{\mathbb{I}}^2} g(x,y)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$= \int\int_{\bar{\mathbb{I}}^2} \Bigg(f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right) - f\left(x, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)$$
$$- f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y\right) + f(x,y)\Bigg)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$= \int\int_{\bar{\mathbb{I}}^2} f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$- \int\int_{\bar{\mathbb{I}}^2} f\left(x, y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$- \int\int_{\bar{\mathbb{I}}^2} f\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}, y\right)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$+ \int\int_{\bar{\mathbb{I}}^2} f(x,y)\ \psi_m(x)\ \psi_n(y)\ dx\ dy$$
$$= \int\int_{\bar{\mathbb{I}}^2} f(x,y)\ \psi_m\left(x \dot{+} \frac{1}{2^u} \dot{+} \frac{1}{2^{u+1}}\right)\ \psi_n\left(y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)\ dx\ dy$$
$$- \int\int_{\bar{\mathbb{I}}^2} f(x,y)\ \psi_m(x)\ \psi_n\left(y \dot{+} \frac{1}{2^v} \dot{+} \frac{1}{2^{v+1}}\right)\ dx\ dy$$

$$-\int\int_{\mathbb{I}^2} f(x,y)\ \psi_m\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}\right)\ \psi_n(y)\ dx\ dy$$

$$+\hat{f}(m,n)$$

$$= \int\int_{\mathbb{I}^2} f(x,y)\ \psi_m(x)\ \psi_m\left(\frac{1}{2^u}\right)\ \psi_m\left(\frac{1}{2^{u+1}}\right)\ \psi_n(y)\ \psi_n\left(\frac{1}{2^v}\right)\ \psi_n\left(\frac{1}{2^{v+1}}\right)\ dx\ dy$$

$$-\int\int_{\mathbb{I}^2} f(x,y)\ \psi_m(x)\ \psi_n(y)\ \psi_n\left(\frac{1}{2^v}\right)\ \psi_n\left(\frac{1}{2^{v+1}}\right)\ dx\ dy$$

$$-\int\int_{\mathbb{I}^2} f(x,y)\ \psi_m(x)\ \psi_m\left(\frac{1}{2^u}\right)\ \psi_m\left(\frac{1}{2^{u+1}}\right)\ \psi_n(y)\ dx\ dy$$

$$+\hat{f}(m,n)$$

$$= \psi_m\left(\frac{1}{2^u}\right)\ \psi_m\left(\frac{1}{2^{u+1}}\right)\ \psi_n\left(\frac{1}{2^v}\right)\ \psi_n\left(\frac{1}{2^{v+1}}\right)\ \hat{f}(m,n)$$

$$-\psi_n\left(\frac{1}{2^v}\right)\ \psi_n\left(\frac{1}{2^{v+1}}\right)\ \hat{f}(m,n) - \psi_m\left(\frac{1}{2^u}\right)\ \psi_m\left(\frac{1}{2^{u+1}}\right)\ \hat{f}(m,n) + \hat{f}(m,n)$$

$$= 4\hat{f}(m,n)$$

and

$$4|\hat{f}(m,n)|$$

$$\leq \int\int_{\mathbb{I}^2}\left| f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right) - f\left(x, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right.$$

$$\left. - f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y\right) + f(x,y)\right|\ dx\ dy$$

$$= \int\int_{\mathbb{I}^2}\left| f\left(\left(x \dotplus \frac{1}{2^{u+1}}\right) \dotplus \left(\frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}\right), \left(y \dotplus \frac{1}{2^{v+1}}\right) \dotplus \left(\frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right)\right.$$

$$- f\left(x \dotplus \frac{1}{2^{u+1}}, \left(y \dotplus \frac{1}{2^{v+1}}\right) \dotplus \left(\frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right)$$

$$- f\left(\left(x \dotplus \frac{1}{2^{u+1}}\right) \dotplus \left(\frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}\right), y \dotplus \frac{1}{2^{v+1}}\right)$$

$$\left. + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right)\right|\ dx\ dy$$

$$= \int\int_{\mathbb{I}^2}\left| f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^v}\right) - f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v}\right)\right.$$

$$\left. - f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^{v+1}}\right) + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right)\right|\ dx\ dy.$$

Similarly, we get

$$4|\hat{f}(m,n)| \le \int\int_{\mathbb{I}^2} \left| f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) - f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) \right.$$
$$\left. - f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right) + f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right) \right| \, dx\ dy$$

and in general we have

$$4|\hat{f}(m,n)| \le \int\int_{\mathbb{I}^2} |\Delta f_{jk}(x,y)|\ dx\ dy, \tag{5.3}$$

where

$$\Delta f_{jk}(x,y) = f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right) - f\left(x \dotplus \frac{(2j-1)}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right)$$
$$- f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{(2k-1)}{2^{v+1}}\right) + f\left(x \dotplus \frac{(2j-1)}{2^{u+1}}, y \dotplus \frac{(2k-1)}{2^{v+1}}\right),$$

for all $j = 1, \cdots, 2^u - 1$ and for all $k = 1, \cdots, 2^v - 1$.

Dividing both sides of the above inequality by $\lambda_j^1\ \lambda_k^2$ and then summing over $j = 1$ to $2^u - 1$ and $k = 1$ to $2^v - 1$, we have

$$4|\hat{f}(2^u, 2^v)| \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right) \le \int\int_{\mathbb{I}^2} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{|\Delta f_{jk}(x,y)|}{(\lambda_j^1\ \lambda_k^2)^{\frac{1}{p}+\frac{1}{q}}} \right) dx\ dy,$$

where q is the index conjugate to p.

Applying Hölder's inequality on the right side of the above inequality, we get

$$4|\hat{f}(2^u, 2^v)| \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)$$
$$\le \int\int_{\mathbb{I}^2} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{q}} dx\ dy.$$

Hence,

$$4|\hat{f}(2^u, 2^v)| \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} \le \int\int_{\mathbb{I}^2} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} dx\ dy. \tag{5.4}$$

For any $x, y \in \mathbb{R}$, all these points $x \dot{+} 2jh_1,\ x \dot{+} (2j-1)h_1,\ for\ j = 1, \cdots, 2^u - 1$, and $y \dot{+} 2kh_2,\ y \dot{+} (2k-1)h_2,\ for\ k = 1, \cdots, 2^v - 1$, lie in the interval of length 1. Thus, $f \in \bigwedge BV^{(p)}(\overline{\mathbb{I}}^2)$ implies

$$\left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{|\Delta f_{jk}(x,y)|^p}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}} = O(1).$$

This together with $\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2} \approx \sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2}$ and the above inequality (5.4) imply that

$$|\hat{f}(2^u, 2^v)| = O\left(\frac{1}{\left(\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)^{\frac{1}{p}}} \right).$$

This completes the proof of the theorem.

Corollary 5.1.2.2. *If* $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{I}}^2)$ $(p \geq 1)$ *then* (5.2) *holds true.*

Proof of Corollary 5.1.2.2. In view of the Lemma 4.1.1.4 (p.62), $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{I}}^2)$ implies f is bounded on $\overline{\mathbb{I}}^2$, and hence $f \in L^p(\overline{\mathbb{I}}^2)$, for all $p \geq 1$. Thus, $\bigwedge^* BV^{(p)}(\overline{\mathbb{I}}^2)$ $\subset \bigwedge BV^{(p)}(\overline{\mathbb{I}}^2) \cap L^p(\overline{\mathbb{I}}^2)$. Therefore, the corollary follows from the Theorem 5.1.2.1.

Theorem 5.1.2.3. *If* ϕ *satisfies* Δ_2 *condition and* $f \in \phi \bigwedge BV(\overline{\mathbb{I}}^2) \cap L^1(\overline{\mathbb{I}}^2)$, *then*

$$\hat{f}(2^u, 2^v) = O\left(\phi^{-1} \left(\frac{1}{\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2}} \right) \right). \tag{5.5}$$

Proof of Theorem 5.1.2.3. For fixed $u, v \in \mathbb{N}_0$, put

$$\begin{aligned} \Delta f_{jk}(x,y) = f&\left(x \dot{+} \frac{2j}{2^{u+1}}, y \dot{+} \frac{2k}{2^{v+1}} \right) - f\left(x \dot{+} \frac{(2j-1)}{2^{u+1}}, y \dot{+} \frac{2k}{2^{v+1}} \right) \\ &- f\left(x \dot{+} \frac{2j}{2^{u+1}}, y \dot{+} \frac{(2k-1)}{2^{v+1}} \right) + f\left(x \dot{+} \frac{(2j-1)}{2^{u+1}}, y \dot{+} \frac{(2k-1)}{2^{v+1}} \right), \end{aligned}$$

for all $j = 1, \cdots, 2^u - 1$ and for all $k = 1, \cdots, 2^v - 1$.

Then, proceeding as in the proof of the Theorem 5.1.2.1, we get (5.3)

$$\begin{aligned} |\hat{f}(2^u, 2^v)| &\leq \frac{1}{4} \int\int_{\overline{\mathbb{I}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy \\ &\leq \int\int_{\overline{\mathbb{I}}^2} |\Delta f_{jk}(x,y)|\ dx\ dy. \end{aligned}$$

For $c > 0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(2^u, 2^v)|) \leq \int\int_{\overline{\mathbb{I}}^2} \phi(c|\Delta f_{jk}(x, y)|)\ dx\ dy.$$

Dividing both sides of the above inequality by $\lambda_j^1\ \lambda_k^2$ and then summing over $j = 1$ to $2^u - 1$ and $k = 1$ to $2^v - 1$, we get

$$\phi(c|\hat{f}(2^u, 2^v)|) \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right) \leq \int\int_{\overline{\mathbb{I}}^2} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{\phi(c|\Delta f_{jk}(x, y)|)}{\lambda_j^1\ \lambda_k^2} \right)\ dx\ dy.$$

For any $x, y \in \mathbb{R}$, all these points $x \dotplus 2jh_1,\ x \dotplus (2j-1)h_1,\ for\ j = 1, \cdots, 2^u - 1$, and $y \dotplus 2kh_2,\ y \dotplus (2k-1)h_2,\ for\ k = 1, \cdots, 2^v - 1$, lie in the interval of length 1, where h_1 and h_2 are as defined earlier in the Theorem 5.1.2.1. Thus,

$$\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{\phi(c|\Delta f_{jk}(x, y)|)}{\lambda_j^1\ \lambda_k^2} \leq V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2),$$

as ϕ satisfies Δ_2 condition implies $cf \in \phi \bigwedge BV(\overline{\mathbb{I}}^2)$.

Therefore,

$$\phi(c|\hat{f}(2^u, 2^v)|) \leq \frac{V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2)}{\left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2} \right)}. \tag{5.6}$$

Since ϕ is convex and $\phi(0) = 0$, for $c \in (0, 1]$ we have $\phi(cx) \leq c\phi(x)$ and hence we can choose sufficiently small $c \in (0, 1]$ such that $V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2) \leq 1$. This together with $\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2} \approx \sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\ \lambda_k^2}$ and the above inequality (5.6) imply that

$$|\hat{f}(2^u, 2^v)| \leq \frac{1}{c} \phi^{-1} \left(\frac{1}{\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\ \lambda_k^2}} \right).$$

This completes the proof of the theorem.

Corollary 5.1.2.4. *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge^* BV(\overline{\mathbb{I}}^2)$, then* (5.5) *holds true.*

Proof of Corollary 5.1.2.4. In view of the earlier Lemma 4.1.1.10 (p.66), $f \in \phi \bigwedge^* BV(\overline{\mathbb{I}}^2)$ implies f is bounded on $\overline{\mathbb{I}}^2$, and hence $f \in L^1(\overline{\mathbb{I}}^2)$. Thus, $\phi \bigwedge^* BV(\overline{\mathbb{I}}^2) \subset \phi \bigwedge BV(\overline{\mathbb{I}}^2) \cap L^1(\overline{\mathbb{I}}^2)$. Therefore, the corollary follows from the Theorem 5.1.2.3.

Definition 5.1.2.5. *Given a function $f \in L^p(\overline{\mathbb{I}}^2)$, where $p \geq 1$, the dyadic $p-$integral modulus of continuity of f is defined as*

$$\omega^{(p)}(f;\delta_1,\delta_2)$$

$$= \sup\left\{\left(\int\int_{\overline{\mathbb{I}}^2} |\Delta f(x,y;h_1,h_2)|^p \; dx \; dy\right)^{\frac{1}{p}} : \; 0 \leq h_1 < \delta_1, \; 0 \leq h_2 < \delta_2\right\},$$

where

$$\Delta f(x,y;h_1,h_2) = f(x \dotplus h_1, y \dotplus h_2) - f(x, y \dotplus h_2) - f(x \dotplus h_1, y) + f(x,y).$$

For $p \geq 1$ and $\alpha_1, \alpha_2 \in (0,1]$, we say that $f \in Lip(p;\alpha_1,\alpha_2)(\overline{\mathbb{I}}^2)$ if

$$\omega^{(p)}(f;\delta_1,\delta_2) = O(\delta_1^{\alpha_1}\delta_2^{\alpha_2}).$$

Theorem 5.1.2.6. *If $f \in Lip(p;\alpha_1,\alpha_2)(\overline{\mathbb{I}}^2)$ $(p \geq 1, \;\alpha_1,\alpha_2 \in (0,1])$ then*

$$\hat{f}(2^u, 2^v) = O\left(\frac{1}{2^{u\alpha_1 + v\alpha_2}}\right).$$

Proof of Theorem 5.1.2.6. For fixed $u, v \in \mathbb{N}_0$, put

$$\Delta f(x,y) = f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right) - f\left(x, y \dotplus \frac{1}{2^{v+1}}\right) - f\left(x \dotplus \frac{1}{2^{u+1}}, y\right) + f(x,y).$$

For $m = 2^u$ and $n = 2^v$, $\psi_m\left(\frac{1}{2^{u+1}}\right) = \psi_n\left(\frac{1}{2^{v+1}}\right) = -1$ implies

$$\widehat{\Delta f}(m,n) = \psi_m\left(\frac{1}{2^{u+1}}\right)\psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n) - \psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n)$$

$$- \psi_m\left(\frac{1}{2^{u+1}}\right)\hat{f}(m,n) + \hat{f}(m,n)$$

$$= 4\hat{f}(m,n)$$

and

$$|\hat{f}(m,n)| \leq \frac{1}{4}\int\int_{\overline{\mathbb{I}}^2} |\Delta f(x,y)| \; dx \; dy.$$

Applying Hölder's inequality on the right side of the above inequality, we have

$$|\hat{f}(2^u,2^v)| = O(1) \left(\int\int_{\overline{\mathbb{I}}^2} |\Delta f(x,y)|^p \; dx \; dy\right)^{\frac{1}{p}}$$

$$= O\left(\frac{1}{2^{u\alpha_1+v\alpha_2}}\right),$$

as $f \in Lip(p;\alpha_1,\alpha_2)(\overline{\mathbb{I}}^2)$.

Hence, the theorem follows.

Theorem 5.1.2.7. *If $f \in AC(\overline{\mathbb{I}}^2)$ then*

$$\hat{f}(2^u, 2^v) = o\left(\frac{1}{2^{u+v}}\right).$$

Proof of Theorem 5.1.2.7. The Theorem 5.1.2.7 can be proved in a similar way to the proof of the Theorem 5.1.2.1.

5.1.3 New results for functions of N−variables

Now, we extend the results of the Subsection 5.1.2 for functions of $N-$variables in the following way.

Theorem 5.1.3.1. *If $f \in \bigwedge BV^{(p)}(\overline{\mathbb{I}}^N) \cap L^p(\overline{\mathbb{I}}^N)$ $(p \geq 1)$ then*

$$\hat{f}(2^{u_1}, \cdots, 2^{u_N}) = O\left(\frac{1}{\left(\sum_{r_1=1}^{2^{u_1}} \cdots \sum_{r_N=1}^{2^{u_N}} \frac{1}{\lambda^1_{r_1}\cdots\lambda^N_{r_N}}\right)^{\frac{1}{p}}}\right). \tag{5.7}$$

Corollary 5.1.3.2. *If a measurable function $f \in \bigwedge^* BV^{(p)}(\overline{\mathbb{I}}^N)$ $(p \geq 1)$ then* (5.7) *holds true.*

Theorem 5.1.3.3. *If ϕ satisfies Δ_2 condition and $f \in \phi\bigwedge BV(\overline{\mathbb{I}}^N) \cap L^1(\overline{\mathbb{I}}^N)$, then*

$$\hat{f}(2^{u_1}, \cdots, 2^{u_N}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{2^{u_1}} \cdots \sum_{r_N=1}^{2^{u_N}} \frac{1}{\lambda^1_{r_1}\cdots\lambda^N_{r_N}}}\right)\right). \tag{5.8}$$

Corollary 5.1.3.4. *If ϕ satisfies Δ_2 condition and a measurable function $f \in \phi\bigwedge^* BV(\overline{\mathbb{I}}^N)$, then* (5.8) *holds true.*

Definition 5.1.3.5. *Given $\mathbf{x} = (x_1, \cdots, x_N) \in \overline{\mathbb{I}}^N$ and $f \in L^p(\overline{\mathbb{I}}^N)$, where $p \geq 1$, the dyadic $p-$integral modulus of continuity of f is defined as*

$\omega^{(p)}(f;\delta_1,\cdots,\delta_N)$

$$= \sup\left\{ \left(\int \cdots \int_{\overline{\mathbb{I}}^N} |\Delta f(x_1,\cdots,x_N;h_1,\cdots,h_N)|^p \; d\mathbf{x} \right)^{\frac{1}{p}} \right.$$

$$\left. : \; 0 \le h_i < \delta_i \; for \; all \; i = 1,2,\cdots,N \right\}, \; where$$

$\Delta f(x_1,\cdots,x_N;h_1,\cdots,h_N)$

$$= \sum_{u_1=0}^{1} \cdots \sum_{u_N=0}^{1} (-1)^{u_1+\cdots+u_N} \; f(x_1 \dot{+} (1-u_1)h_1,\cdots,x_N \dot{+} (1-u_N)h_N).$$

For $p \ge 1$ and $\alpha_i \in (0,1]$, for all $i = 1,2,\cdots,N$, we say that $f \in Lip(p;\alpha_1,\cdots,\alpha_N)(\overline{\mathbb{I}}^N)$ if

$$\omega^{(p)}(f;\delta_1,\cdots,\delta_N) = O(\delta_1^{\alpha_1}\cdots\delta_N^{\alpha_N}).$$

Theorem 5.1.3.6. *If* $f \in Lip(p;\alpha_1,\cdots,\alpha_N)(\overline{\mathbb{I}}^N)$ $(p \ge 1, \; \alpha_1,\cdots,\alpha_N \in (0,1])$ *then*

$$\hat{f}(2^{u_1},\cdots,2^{u_N}) = O\left(\frac{1}{2^{u_1\alpha_1+\cdots+u_N\alpha_N}}\right).$$

Theorem 5.1.3.7. *If* $f \in AC(\overline{\mathbb{I}}^N)$ *then*

$$\hat{f}(2^{u_1},\cdots,2^{u_N}) = o\left(\frac{1}{2^{u_1+\cdots+u_N}}\right).$$

All extended results of this subsection can be proved in the same way as the results in the Subsection 5.1.2.