Chapter 2

Basic properties of classes of functions of generalized bounded variations

2.1 New results for functions of one variable

Functions of bounded variation play an important role in various aspects of mathematical analysis. Their main influence is in connection with the study of Fourier series. They also appear in the theory of Riemann-Stieltjes integration, and, in particular, in characterizing the dual space of the Banach space of continuous functions on a compact interval [34, Theorem 14.5, p.245].

It is well known that the class $BV([a, b])$ is a Banach algebra with respect to the pointwise operations and the variation norm $\|f\| = \|f\|_\infty + V(f, [a, b])$, where $V(f, [a, b])$ is the total variation of the function $f \in BV([a, b])$. Looking to the feature of the class $BV([a, b])$, it has been generalized in many ways and many generalized bounded variations are introduced like $\Lambda BV^{(p)}([a, b])$, $\phi\Lambda BV([a, b])$, $\Lambda BV(p(n) \uparrow p, \varphi, [a, b])$, $r - BV([a, b])$ and $V[\nu]$. Many of these classes are linear spaces and become Banach spaces when they are equipped with suitable norms involving generalized variations. In many cases, the norms are also submultiplicative, and so the function spaces carry an additional structure of Banach algebras with respect to the pointwise operations. In 1976, D. Waterman [71, p.41] proved that the class $\Lambda BV([a,b])$ is a Banach space with respect to the pointwise operations and the $\Lambda-$variation norm, as defined earlier in (1.1) (p.26). This result was extended in 2006 by proving that the class $\Lambda BV^{(p)}([a,b])$ is a Banach space with respect to the pointwise operations and the Λ_p-variation norm, as defined earlier in (1.2) (p.26) [61, Theorem 1, p.92]. In 2010, R. Kantrowitz [31, Theorem 1, p.171] observed that the Λ_p-variation norm is submultiplicative. Thus, the class $\Lambda BV^{(p)}([a,b])$ carry an additional structure of Banach algebra with respect to the pointwise operations. While studying the spectral theory of linear operators on Banach spaces, extending the usual interval definition of a function of bounded variation to new definition of a function of bounded variation on a non-empty compact subset σ of $\mathbb{R}$, B. Ashton and I. Doust [5] in 2005 generalized the class $BV([a,b])$ to the class $BV(\sigma)$. They observed that the extended class $BV(\sigma)$ forms a Banach algebra, with respect to the pointwise operations and the variation norm as defined earlier in (1.4) (p.27), and it has some interesting applications to the Operator theory.

By straightforward extension of the Definition 1.3.1 (p.27), of a function of bounded variation on σ, and the Definition 1.2.1.1 (p.4), of a function of $p-\Lambda-$bounded variation on an interval $[a,b]$, one can define a function of $p-\Lambda-$bounded variation on σ in the following way.

Throughout the present section, σ represents non-empty compact subset of $\mathbb{R}$. $I=[a,b]$ is the smallest closed interval containing σ. $\Pi(\sigma)$ is a class of partitions of σ (that is, $\Pi(\sigma)=\{P\ :\ P=\{x_i\}_{i=1}^m$ is an increasing finite sequence in $\sigma\}$). $\mathbb{B}$ is a commutative unital Banach algebra.

Definition 2.1.1. *Given $\Lambda=\{\lambda_n\}_{n=1}^{\infty}\in\mathbb{L}$ and $p\geq 1$, a function $f:\sigma\to\mathbb{B}$ is said to be of $p-\Lambda-$bounded variation (that is, $f\in\Lambda BV^{(p)}(\sigma,\mathbb{B})$) if*

$$V_{\Lambda_p}(f,\sigma,\mathbb{B})=\sup_{P\in\Pi(\sigma)}\left\{V_{\Lambda_p}(f,\sigma,\mathbb{B},P)\right\}<\infty,$$

where

$$V_{\Lambda_p}(f,\sigma,\mathbb{B},P)=\left(\sum_i\frac{(\|\Delta f(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}},$$

in which $\Delta f(x_i)=f(x_{i+1})-f(x_i)$ and $\|\cdot\|_{\mathbb{B}}$ denotes the Banach norm in $\mathbb{B}$.

In the Definition 2.1.1, for $\sigma = I$ one gets the class $\Lambda BV^{(p)}(I, \mathbb{B})$; for $\Lambda = \{1\}$ and $p = 1$ one gets the class $BV(\sigma, \mathbb{B})$; for $p = 1$ one gets the class $\Lambda BV(\sigma, \mathbb{B})$; and for $\Lambda = \{1\}$ one gets the class $BV^{(p)}(\sigma, \mathbb{B})$. For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$ reduces to the class $\Lambda BV^{(p)}(\sigma)$.

The summary of the important similarities and differences between the class $BV(\sigma)$ and the class $BV(I)$ are listed in the paper [5]. Some of the important properties of the class $\Lambda BV^{(p)}(I)$ are listed in the paper of Vyas [61].

Some basic properties of the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$ are as followed.

Theorem 2.1.2. *If $f, g \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ then the following hold:*
(i) f and g are bounded.
(ii) $V_{\Lambda_p}(f + g, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B})$.
(iii) $V_{\Lambda_p}(\alpha f, \sigma, \mathbb{B}) = |\alpha| V_{\Lambda_p}(f, \sigma, \mathbb{B})$, for any $\alpha \in \mathbb{C}$.
(iv) $V_{\Lambda_p}(f, \sigma', \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B})$, if a non-empty compact set $\sigma' \subset \sigma$.
(v) $V_{\Lambda_p}(fg, \sigma, \mathbb{B}) \leq \|f\|_\infty V_{\Lambda_p}(g, \sigma, \mathbb{B}) + \|g\|_\infty V_{\Lambda_p}(f, \sigma, \mathbb{B})$, where

$$\|f\|_\infty = \sup_{x \in \sigma} \|f(x)\|_{\mathbb{B}} < \infty.$$

Proof of Theorem 2.1.2. Consider any $P = \{x_i\}_{i=1}^m \in \Pi(\sigma)$.

Proof of Theorem 2.1.2 (i). For any $x \in \sigma$,

$$\begin{aligned}\|f(x)\|_{\mathbb{B}} &\leq \|f(x) - f(a)\|_{\mathbb{B}} + \|f(a)\|_{\mathbb{B}} \\ &= (\lambda_1)^{\frac{1}{p}} \left(\frac{(\|f(x) - f(a)\|_{\mathbb{B}})^p}{\lambda_1} \right)^{\frac{1}{p}} + \|f(a)\|_{\mathbb{B}} \\ &\leq (\lambda_1)^{\frac{1}{p}} V_{\Lambda_p}(f, \sigma, \mathbb{B}) + \|f(a)\|_{\mathbb{B}} < \infty, \ \text{as } f \in \Lambda BV^{(p)}(\sigma, \mathbb{B}).\end{aligned}$$

Thus, the Theorem 2.1.2 (i) follows.

Proof of Theorem 2.1.2 (ii). For any $f, g \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$,

$$\left(\sum_i \frac{(\|\Delta(f+g)(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|(f+g)(x_{i+1}) - (f+g)(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|f(x_{i+1}) + g(x_{i+1}) - f(x_i) - g(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|\Delta f(x_i) + \Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i \frac{(\|\Delta f(x_i)\|_{\mathbb{B}} + \|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$= \left(\sum_i \left(\frac{\|\Delta f(x_i)\|_{\mathbb{B}}}{\lambda_i^{\frac{1}{p}}} + \frac{\|\Delta g(x_i)\|_{\mathbb{B}}}{\lambda_i^{\frac{1}{p}}} \right)^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i \frac{(\|\Delta f(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}} + \left(\sum_i \frac{(\|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

(by Minkowski's inequality)

$$\leq V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B}).$$

Thus,

$$V_{\Lambda_p}(f + g, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p}(g, \sigma, \mathbb{B}).$$

Proof of Theorem 2.1.2 (*iii*). Note that, for any $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ and $\alpha \in \mathbb{C}$,

$$\left(\sum_i \frac{(\|\Delta(\alpha f)(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}} = |\alpha| \left(\sum_i \frac{(\|\Delta f(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}.$$

Thus, the Theorem 2.1.2 (*iii*) follows.

Proof of Theorem 2.1.2 (*iv*). Let $S = \{y_i\}_{i=1}^m$ be any partition of σ'. Then $\sigma' \subset \sigma$ implies that S is also a partition of σ.

Thus,

$$V_{\Lambda_p}(f, \sigma', \mathbb{B}, S) = \left(\sum_i \frac{(\|\Delta f(y_i)\|_{\mathbb{B}})^p}{\lambda_i} \right)^{\frac{1}{p}}$$

$$\leq V_{\Lambda_p}(f, \sigma, \mathbb{B}).$$

Hence, the Theorem 2.1.2 (*iv*) follows.

Proof of Theorem 2.1.2 (v). For any $f, g \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$, in view of the Theorem 2.1.2 (i), $\|f\|_\infty < \infty$ and $\|g\|_\infty < \infty$.

Therefore,

$$\left(\sum_i \frac{(\|\Delta fg(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|fg(x_{i+1}) - fg(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|f(x_{i+1})g(x_{i+1}) - f(x_i)g(x_{i+1}) + f(x_i)g(x_{i+1}) - f(x_i)g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i \frac{(\|g(x_{i+1})\Delta f(x_i) + f(x_i)\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i \frac{(\|g(x_{i+1})\|_{\mathbb{B}}\|\Delta f(x_i)\|_{\mathbb{B}} + \|f(x_i)\|_{\mathbb{B}}\|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i \frac{(\|g\|_\infty\|\Delta f(x_i)\|_{\mathbb{B}} + \|f\|_\infty\|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i \left(\frac{\|g\|_\infty\|\Delta f(x_i)\|_{\mathbb{B}}}{\lambda_i^{\frac{1}{p}}} + \frac{\|f\|_\infty\|\Delta g(x_i)\|_{\mathbb{B}}}{\lambda_i^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i \frac{(\|g\|_\infty)^p(\|\Delta f(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}} + \left(\sum_i \frac{(\|f\|_\infty)^p(\|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

(by Minkowski's inequality)

$$= \|g\|_\infty \left(\sum_i \frac{(\|\Delta f(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}} + \|f\|_\infty \left(\sum_i \frac{(\|\Delta g(x_i)\|_{\mathbb{B}})^p}{\lambda_i}\right)^{\frac{1}{p}}$$

$$\leq \|g\|_\infty \, V_{\Lambda_p}(f, \sigma, \mathbb{B}) + \|f\|_\infty \, V_{\Lambda_p}(g, \sigma, \mathbb{B}).$$

Thus,

$$V_{\Lambda_p}(fg, \sigma, \mathbb{B}) \leq \|f\|_\infty \, V_{\Lambda_p}(g, \sigma, \mathbb{B}) + \|g\|_\infty \, V_{\Lambda_p}(f, \sigma, \mathbb{B}).$$

This completes the proof of the theorem.

Corollary 2.1.3. *Let σ_1 and σ_2 be non-empty compact subsets of $\mathbb{R}$ such that $\sigma_1 \subset \sigma_2$. If $f \in \Lambda BV^{(p)}(\sigma_2, \mathbb{B})$ then $f|\sigma_1 \in \Lambda BV^{(p)}(\sigma_1, \mathbb{B})$ and $\|f|\sigma_1\|_{\Lambda_p(\sigma_1,\mathbb{B})} \leq \|f\|_{\Lambda_p(\sigma_2,\mathbb{B})}$, where*

$$\|f\|_{\Lambda_p(\sigma,\mathbb{B})} = \|f\|_\infty + V_{\Lambda_p}(f, \sigma, \mathbb{B}).$$

The Corollary 2.1.3 can be easily followed from the above Theorem 2.1.2 (iv).

For many of the properties of the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$, it is easy to embed the class $\Lambda BV^{(p)}(\sigma, \mathbb{B})$ into the class $\Lambda BV^{(p)}(I, \mathbb{B})$ and then use the classical theory.

Definition 2.1.4. *Given a function $f : \sigma \to \mathbb{B}$, define the function $E_f : I \to \mathbb{B}$ by $E_f(x) = f(\alpha(x))$, where*

$$\alpha(x) = \begin{cases} x, & \text{if } x \in \sigma, \\ \sup \{t \ : \ [x,t] \subset I \setminus \sigma\}, & \text{otherwise.} \end{cases}$$

Obviously, E_f is an extension of f and is constant on the gaps in σ.

Theorem 2.1.5. *If $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ then $V_{\Lambda_p}(f, \sigma, \mathbb{B}) = V_{\Lambda_p}(E_f, I, \mathbb{B})$, where $I = [a, b]$ is the smallest closed interval containing σ.*

Proof of Theorem 2.1.5. I is the smallest closed interval containing σ and $E_f|\sigma = f$ implies $V_{\Lambda_p}(f, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(E_f, I, \mathbb{B})$.

For any $P = \{x_i\}_{i=1}^m \in \Pi(I)$, define $S = P \cap \sigma \in \Pi(\sigma)$. For simplicity of the proof, suppose that there is only one $x_k \in P \setminus S$ and $(x_{k-1}, x_k) \cap \sigma = (x_k, x_{k+1}) \cap \sigma = \emptyset$. Then

$$\sum_P \frac{(\|\Delta E_f(x_i)\|_{\mathbb{B}})^p}{\lambda_i} = \sum_{i=1}^{k-2} \frac{(\|\Delta E_f(x_i)\|_{\mathbb{B}})^p}{\lambda_i} + \frac{(\|\Delta E_f(x_{k-1})\|_{\mathbb{B}})^p}{\lambda_{k-1}} + \frac{(\|\Delta E_f(x_k)\|_{\mathbb{B}})^p}{\lambda_k}$$

$$+ \sum_{i \geq k+1} \frac{(\|\Delta E_f(x_i)\|_{\mathbb{B}})^p}{\lambda_i}.$$

Since $E_f(x_k) = f(\alpha(x_k))$, for $S^* = S \cup \{\alpha(x_k)\}$, we get

$$\sum_P \frac{(\|\Delta E_f(x_i)\|_{\mathbb{B}})^p}{\lambda_i} \leq \sum_{S^*} \frac{(\|\Delta f(y_i)\|_{\mathbb{B}})^p}{\lambda_i}.$$

Hence, the theorem follows from $V_{\Lambda_p}(E_f, I, \mathbb{B}) \leq V_{\Lambda_p}(f, \sigma, \mathbb{B})$.

Corollary 2.1.6. *Given a function $f : \sigma \to \mathbb{B}$, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ if and only if $E_f \in \Lambda BV^{(p)}(I, \mathbb{B})$.*

Proof of Corollary 2.1.6. In view of the above Theorem 2.1.5, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$ implies $V_{\Lambda_p}(f, \sigma, \mathbb{B}) = V_{\Lambda_p}(E_f, I, \mathbb{B})$. Hence, $E_f \in \Lambda BV^{(p)}(I, \mathbb{B})$. Since $\sigma \subset I$ and $E_f|\sigma = f$, in view of the Theorem 2.1.2 (*iv*) (p.37), we have $V_{\Lambda_p}(f, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(E_f, I, \mathbb{B})$. Thus, $E_f \in \Lambda BV^{(p)}(I, \mathbb{B})$ implies $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$.
This completes the proof of the corollary.

Corollary 2.1.7. *The map $F : \Lambda BV^{(p)}(\sigma, \mathbb{B}) \to \Lambda BV^{(p)}(I, \mathbb{B})$, defined as $F(f) = E_f$ for all $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$, is a linear isometry.*

The Corollary 2.1.7 can be easily followed from the Theorem 2.1.5.

Theorem 2.1.8. *$\big(\Lambda BV^{(p)}(\sigma, \mathbb{B}), \|\cdot\|_{\Lambda_p(\sigma,\mathbb{B})}\big)$ is a commutative unital Banach algebra with respect to the pointwise operations.*

Proof of Theorem 2.1.8. Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\Lambda BV^{(p)}(\sigma, \mathbb{B})$. Therefore, it converges uniformly to some function say f on σ. For any $P \in \Pi(\sigma)$, we get

$$\begin{aligned} V_{\Lambda_p}(f_k, \sigma, \mathbb{B}, P) &\leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}, P) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B}, P) \\ &\leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B}). \end{aligned}$$

This implies,

$$V_{\Lambda_p}(f_k, \sigma, \mathbb{B}) \leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\Lambda_p}(f_l, \sigma, \mathbb{B})$$

and

$$|V_{\Lambda_p}(f_k, \sigma, \mathbb{B}) - V_{\Lambda_p}(f_l, \sigma, \mathbb{B})| \leq V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}) \to 0 \ as \ k, l \to \infty.$$

Hence, $\{V_{\Lambda_p}(f_k, \sigma, \mathbb{B})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore,

$$\begin{aligned} V_{\Lambda_p}(f, \sigma, \mathbb{B}, P) &= \lim_{k\to\infty} V_{\Lambda_p}(f_k, \sigma, \mathbb{B}, P) \\ &\leq \lim_{k\to\infty} V_{\Lambda_p}(f_k, \sigma, \mathbb{B}) \leq M < \infty. \end{aligned}$$

Thus, $f \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$.

Since $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$V_{\Lambda_p}(f_k - f_l, \sigma, \mathbb{B}, P) < \epsilon, \quad for\ all\ k, l \geq n_0.$$

Letting $l \to \infty$ and taking supremum on both sides of the above inequality, we get $V_{\Lambda_p}(f_k - f, \sigma, \mathbb{B}) < \epsilon, \quad for\ all\ k \geq n_0$.

Thus, $\|f_k - f\|_{\Lambda_p(\sigma,\mathbb{B})} \to 0 \ \ as \ \ k \to \infty$.

Hence, $\big(\Lambda BV^{(p)}(\sigma, \mathbb{B}), \|\cdot\|_{\Lambda_p(\sigma,\mathbb{B})}\big)$ is a Banach space.

For any $g_1, g_2 \in \Lambda BV^{(p)}(\sigma, \mathbb{B})$,

$\|g_1 g_2\|_{\Lambda_p(\sigma,\mathbb{B})} \ = \ \|g_1 g_2\|_\infty + V_{\Lambda_p}(g_1 g_2, \sigma, \mathbb{B})$

$\leq \|g_1\|_\infty \ \|g_2\|_\infty + \|g_1\|_\infty \ V_{\Lambda_p}(g_2, \sigma, \mathbb{B}) + \|g_2\|_\infty \ V_{\Lambda_p}(g_1, \sigma, \mathbb{B})$ $(by\ the\ Theorem\ 2.1.2\ (v))$

$\leq \|g_1\|_\infty(\|g_2\|_\infty + V_{\Lambda_p}(g_2, \sigma, \mathbb{B})) + \|g_2\|_\infty \ V_{\Lambda_p}(g_1, \sigma, \mathbb{B}) + V_{\Lambda_p}(g_1, \sigma, \mathbb{B}) \ V_{\Lambda_p}(g_2, \sigma, \mathbb{B})$

$= (\|g_1\|_\infty + V_{\Lambda_p}(g_1, \sigma, \mathbb{B})) \ (\|g_2\|_\infty + V_{\Lambda_p}(g_2, \sigma, \mathbb{B}))$

$= \|g_1\|_{\Lambda_p(\sigma,\mathbb{B})} \ \|g_2\|_{\Lambda_p(\sigma,\mathbb{B})}$.

This completes the proof of the theorem.

Remark 2.1.9. *The Theorem 2.1.8, with $\sigma = [a, b]$ and $\mathbb{B} = \mathbb{R}$, reduces to the Theorem B (p.27) as a particular case. Also, the Theorem 2.1.8, with $p = 1$, $\Lambda = \{1\}$ and $\mathbb{B} = \mathbb{C}$, reduces to B. Ashton and I. Doust Theorem C (p.27) as a particular case.*

2.2 New results for functions of two variables

The notion of bounded variation is extended from a function of one variable to a function of two variables in different ways. Several definitions are given under which a function of two or more independent variables shall be said to be of bounded variation. Some of these definitions are associated with mathematicians namely Hardy, Vitali, Arzelà, Pierpont, Fréchet and Tonelli. For functions of two variables, G. Vitali [56] introduced the class $BV_V([a, b] \times [c, d])$ and G. H. Hardy [28] introduced the class $BV_H([a, b] \times [c, d])$. In 1984, E. Berkson and T. Gillespie [9, Theorem 3, p.310] observed that the class $BV_H([a, b] \times [c, d])$ is a commutative unital Banach algebra with respect to the pointwise operations and the variation norm, as defined earlier in (1.5) (p.27). Considering the natural analogue of that of bounded variation for a function of one variable, Arzelà [4] introduced the class $BV_A([a,b]\times[c,d])$. The inter-relations between these classes are studied by two mathematicians namely C. R. Adams and J. A. Clarkson [1]. The classes are further generalized in many ways and many generalized bounded variations are introduced. Generalizing the class $BV(\sigma)$ and the class $BV_A([a,b]\times[c,d])$, B. Ashton and I. Doust [5] introduced the class $BV(\rho)$ of two variables functions of bounded variation over a non-empty compact subset ρ of $\mathbb{C}$. They proved that the class $BV(\rho)$ forms a Banach algebra with respect to the pointwise operations and the suitable variation norm [5, Theorem 3.8].

One can extend the class $\Lambda BV^{(p)}(\sigma,\mathbb{B})$ to the classes $\bigwedge BV^{(p)}(\sigma_1\times\sigma_2,\mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma_1\times\sigma_2,\mathbb{B})$ in the sense of Vitali and Hardy respectively in the following way.

Throughout the present section, σ_1 and σ_2 represent non-empty compact subsets of $\mathbb{R}$ and $R^2 = I\times J\subset\mathbb{R}^2$ is the smallest closed rectangle containing $\sigma=\sigma_1\times\sigma_2$, where $I=[a,b]$ and $J=[c,d]$ are the smallest closed intervals containing σ_1 and σ_2 respectively.

Definition 2.2.1. *Given $\bigwedge=(\Lambda^1,\Lambda^2)$, where $\Lambda^k=\{\lambda_n^k\}_{n=1}^{\infty}\in\mathbb{L}$, for $k=1,\ 2,$ and $p\geq 1$, a function $f:\sigma\to\mathbb{B}$ is said to be of $p-\bigwedge-$bounded variation (that is, $f\in\bigwedge BV^{(p)}(\sigma,\mathbb{B})$) if*

$$V_{\bigwedge_p}(f,\sigma,\mathbb{B})=\sup_P\left\{V_{\bigwedge_p}(f,\sigma,\mathbb{B},P)\right\}<\infty,$$

where $P=P_1\times P_2$ is a rectangular grid on σ obtained from partitions $P_1=\{x_i\}_{i=1}^m\in\Pi(\sigma_1)$ and $P_2=\{y_j\}_{j=1}^n\in\Pi(\sigma_2)$, and

$$V_{\bigwedge_p}(f,\sigma,\mathbb{B},P)=\left(\sum_i\sum_j\frac{(\|\Delta f(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}},$$

in which

$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_{j+1})-f(x_{i+1},y_j)+f(x_i,y_j)$$

and $\|\cdot\|_{\mathbb{B}}$ is as defined earlier in the Definition 2.1.1 (p.36).

This class is further generalized to the class $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ in the sense of Hardy as follows.

If $f \in \bigwedge BV^{(p)}(\sigma, \mathbb{B})$ is such that the marginal functions $f(.,c) \in \Lambda^1 BV^{(p)}(\sigma_1, \mathbb{B})$ and $f(a,.) \in \Lambda^2 BV^{(p)}(\sigma_2, \mathbb{B})$ then f is said to be of $p - \bigwedge^*$ −bounded variation (that is, $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$).

If $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ then each of the marginal functions $f(.,s) \in \Lambda^1 BV^{(p)}(\sigma_1, \mathbb{B})$ and $f(t,.) \in \Lambda^2 BV^{(p)}(\sigma_2, \mathbb{B})$, where $s \in \sigma_2$ and $t \in \sigma_1$ are fixed.

Note that, for $\sigma = R^2$, the classes $\bigwedge BV^{(p)}(\sigma, \mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ reduce to the classes $\bigwedge BV^{(p)}(R^2, \mathbb{B})$ and $\bigwedge^* BV^{(p)}(R^2, \mathbb{B})$ respectively. For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the classes $\bigwedge BV^{(p)}(\sigma, \mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ reduce to the classes $\bigwedge BV^{(p)}(\sigma)$ and $\bigwedge^* BV^{(p)}(\sigma)$ respectively.

Now, we extend our earlier results of the Section 2.1 for functions of two variables in the following way.

Theorem 2.2.2. *If $f, g \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ then the following hold:*
(i) *f and g are bounded.*
(ii) $V_{\bigwedge_p}(f + g, \sigma, \mathbb{B}) \leq V_{\bigwedge_p}(f, \sigma, \mathbb{B}) + V_{\bigwedge_p}(g, \sigma, \mathbb{B})$.
(iii) $V_{\bigwedge_p}(\alpha f, \sigma, \mathbb{B}) = |\alpha| V_{\bigwedge_p}(f, \sigma, \mathbb{B})$, *for any* $\alpha \in \mathbb{C}$.
(iv) $V_{\bigwedge_p}(f, \sigma', \mathbb{B}) \leq V_{\bigwedge_p}(f, \sigma, \mathbb{B})$, *if* $\sigma' = \sigma_1' \times \sigma_2' \subset \sigma$, *in which* σ_1' *and* σ_2' *are non-empty compact subsets of* $\mathbb{R}$.

Proof of Theorem 2.2.2. Consider any rectangular grid $P = P_1 \times P_2$ of σ obtained from partitions $P_1 = \{x_i\}_{i=1}^m \in \Pi(\sigma_1)$ and $P_2 = \{y_j\}_{j=1}^n \in \Pi(\sigma_2)$.

Proof of Theorem 2.2.2 (i). For any $(x, y) \in \sigma$,

$$\|f(x,y)\|_{\mathbb{B}}$$

$$\leq \|f(x,y) - f(a,y) - f(x,c) + f(a,c)\|_{\mathbb{B}} + \|f(x,c) - f(a,c)\|_{\mathbb{B}} + \|f(a,y) - f(a,c)\|_{\mathbb{B}} + \|f(a,c)\|_{\mathbb{B}}$$

$$= (\lambda_1^1\, \lambda_1^2)^{\frac{1}{p}} \left(\frac{(\|f(x,y) - f(a,y) - f(x,c) + f(a,c)\|_{\mathbb{B}})^p}{\lambda_1^1\, \lambda_1^2} \right)^{\frac{1}{p}}$$

$$+ (\lambda_1^1)^{\frac{1}{p}} \left(\frac{(\|f(x,c) - f(a,c)\|_{\mathbb{B}})^p}{\lambda_1^1} \right)^{\frac{1}{p}} + (\lambda_1^2)^{\frac{1}{p}} \left(\frac{(\|f(a,y) - f(a,c)\|_{\mathbb{B}})^p}{\lambda_1^2} \right)^{\frac{1}{p}} + \|f(a,c)\|_{\mathbb{B}}$$

$$\leq (\lambda_1^1\ \lambda_1^2)^{\frac{1}{p}}\ V_{\bigwedge_p}(f,\sigma,\mathbb{B}) + (\lambda_1^1)^{\frac{1}{p}}\ V_{\Lambda_p^1}(f(.,c),\sigma_1,\mathbb{B}) + (\lambda_1^2)^{\frac{1}{p}}\ V_{\Lambda_p^2}(f(a,.),\sigma_2,\mathbb{B})$$

$$+ \|f(a,c)\|_{\mathbb{B}} < \infty, \quad as\ f \in \bigwedge{}^{*} BV^{(p)}(\sigma,\mathbb{B}).$$

Thus, the Theorem 2.2.2 (i) follows.

Proof of Theorem 2.2.2 (ii). For any $f, g \in \bigwedge^{*} BV^{(p)}(\sigma,\mathbb{B})$,

$$\left(\sum_i\sum_j \frac{(\|\Delta(f+g)(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i\sum_j \frac{(\|\Delta f(x_i,y_j) + \Delta g(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i\sum_j \frac{(\|\Delta f(x_i,y_j)\|_{\mathbb{B}} + \|\Delta g(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}}$$

$$= \left(\sum_i\sum_j \left(\frac{\|\Delta f(x_i,y_j)\|_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{\frac{1}{p}}} + \frac{\|\Delta g(x_i,y_j)\|_{\mathbb{B}}}{(\lambda_i^1\ \lambda_j^2)^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_i\sum_j \frac{(\|\Delta f(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}} + \left(\sum_i\sum_j \frac{(\|\Delta g(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}}$$

(by Minkowski's inequality)

$$\leq V_{\bigwedge_p}(f,\sigma,\mathbb{B}) + V_{\bigwedge_p}(g,\sigma,\mathbb{B}).$$

Thus,

$$V_{\bigwedge_p}(f+g,\sigma,\mathbb{B}) \leq V_{\bigwedge_p}(f,\sigma,\mathbb{B}) + V_{\bigwedge_p}(g,\sigma,\mathbb{B}).$$

Proof of Theorem 2.2.2 (iii). Note that, for any $f \in \bigwedge^{*} BV^{(p)}(\sigma,\mathbb{B})$ and $\alpha \in \mathbb{C}$,

$$\left(\sum_i\sum_j \frac{(\|\Delta(\alpha f)(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}} = |\alpha| \left(\sum_i\sum_j \frac{(\|\Delta f(x_i,y_j)\|_{\mathbb{B}})^p}{\lambda_i^1\ \lambda_j^2}\right)^{\frac{1}{p}}.$$

Thus, the Theorem 2.2.2 (iii) follows.

Proof of Theorem 2.2.2 (iv). Let $S = S_1 \times S_2$ be any rectangular grid of σ' obtained from partitions $S_1 = \{u_i\}_{i=1}^m \in \Pi(\sigma_1')$ and $S_2 = \{v_j\}_{j=1}^n \in \Pi(\sigma_2')$. Then

$\sigma^{'} \subset \sigma$ implies that S is also a rectangular grid of σ.

Thus,

$$V_{\bigwedge_p}(f, \sigma^{'}, \mathbb{B}, S) = \left(\sum_i \sum_j \frac{(\|\Delta f(u_i, v_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \; \lambda_j^2} \right)^{\frac{1}{p}}$$
$$\leq V_{\bigwedge_p}(f, \sigma, \mathbb{B}).$$

Hence, the Theorem 2.2.2 (iv) follows.

This completes the proof of the theorem.

Corollary 2.2.3. *Let σ_1, σ_2, τ_1 and τ_2 be non-empty compact subsets of $\mathbb{R}$ such that $\sigma = \sigma_1 \times \sigma_2 \subset \tau = \tau_1 \times \tau_2$. If $f \in \bigwedge^* BV^{(p)}(\tau, \mathbb{B})$ then $f|\sigma \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ and $\|f|\sigma\|_{\bigwedge_p(\sigma,\mathbb{B})} \leq \|f\|_{\bigwedge_p(\tau,\mathbb{B})}$, where*

$$\|f\|_{\bigwedge_p(\sigma,\mathbb{B})} = \|f\|_\infty + V_{\bigwedge_p}(f, \sigma, \mathbb{B}) + V_{\Lambda_p^1}(f(., c), \sigma_1, \mathbb{B}) + V_{\Lambda_p^2}(f(a, .), \sigma_2, \mathbb{B}),$$

in which

$$\|f\|_\infty = \sup_{(x,y)\in\sigma} \|f(x, y)\|_{\mathbb{B}} < \infty.$$

The Corollary 2.2.3 can be easily followed from the above Theorem 2.2.2 (iv) and the Corollary 2.1.3 (p.40).

For many of the properties of the class $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$, it is easy to embed the class $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ into the class $\bigwedge^* BV^{(p)}(R^2, \mathbb{B})$ and then use the classical theory.

Definition 2.2.4. *Given a function $f : \sigma \to \mathbb{B}$, define the function $E_f : R^2 \to \mathbb{B}$ by $E_f(x, y) = f(\alpha(x), \alpha(y))$, where*

$$\alpha(x) = \begin{cases} x, & \text{if } x \in \sigma_1, \\ \sup \; \{t \; : \; [x, t] \subset I \setminus \sigma_1\}, & \text{otherwise}; \end{cases}$$

and

$$\alpha(y) = \begin{cases} y, & \text{if } y \in \sigma_2, \\ \sup \; \{t \; : \; [y, t] \subset J \setminus \sigma_2\}, & \text{otherwise}. \end{cases}$$

Theorem 2.2.5. *If $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ then $V_{\bigwedge_p}(f, \sigma, \mathbb{B}) = V_{\bigwedge_p}(E_f, R^2, \mathbb{B})$, where $R^2 = I \times J = [a, b] \times [c, d]$ is the smallest closed rectangle containing $\sigma = \sigma_1 \times \sigma_2$.*

Proof of Theorem 2.2.5. R^2 is the smallest closed rectangle containing σ and $E_f|\sigma = f$ implies $V_{\bigwedge_p}(f, \sigma, \mathbb{B}) \leq V_{\bigwedge_p}(E_f, R^2, \mathbb{B})$.

Let $P = P_1 \times P_2$ be any rectangular grid of R^2, where $P_1 = \{x_i\}_{i=1}^m \in \Pi(I)$ and $P_2 = \{y_j\}_{j=1}^n \in \Pi(J)$. Consider $S_1 = P_1 \cap \sigma_1 \in \Pi(\sigma_1)$ and $S_2 = P_2 \cap \sigma_2 \in \Pi(\sigma_2)$. Then $S = S_1 \times S_2$ is a rectangular grid of σ. For simplicity of the proof, suppose there is only one $(x_k, y_l) \in P$ such that $(x_k, y_l) \in P \setminus S$, where $(x_{k-1}, x_k) \cap \sigma_1 = (x_k, x_{k+1}) \cap \sigma_1 = \emptyset$ and $(y_{l-1}, y_l) \cap \sigma_2 = (y_l, y_{l+1}) \cap \sigma_2 = \emptyset$. Then

$$\sum_{P_1 \times P_2} \frac{(\|\Delta E_f(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2} = \sum_{i=1}^{k-2} \sum_{j=1}^{l-2} \frac{(\|\Delta E_f(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2} + \frac{(\|\Delta E_f(x_{k-1}, y_{l-1})\|_{\mathbb{B}})^p}{\lambda_{k-1}^1 \ \lambda_{l-1}^2}$$
$$+ \frac{(\|\Delta E_f(x_{k-1}, y_l)\|_{\mathbb{B}})^p}{\lambda_{k-1}^1 \ \lambda_l^2} + \frac{(\|\Delta E_f(x_k, y_{l-1})\|_{\mathbb{B}})^p}{\lambda_k^1 \ \lambda_{l-1}^2}$$
$$+ \frac{(\|\Delta E_f(x_k, y_l)\|_{\mathbb{B}})^p}{\lambda_k^1 \ \lambda_l^2} + \sum_{i \geq k+1} \sum_{j \geq l+1} \frac{(\|\Delta E_f(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2}.$$

Since $E_f(x_k, y_l) = f(\alpha(x_k), \alpha(y_l))$, for $(S_1 \times S_2)^* = (S_1 \times S_2) \cup \{(\alpha(x_k), \alpha(y_l))\}$, we get

$$\sum_{P_1 \times P_2} \frac{(\|\Delta E_f(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2} \leq \sum_{(S_1 \times S_2)^*} \frac{(\|\Delta f(u_i, v_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2}.$$

Hence, the theorem follows from $V_{\bigwedge_p}(E_f, R^2, \mathbb{B}) \leq V_{\bigwedge_p}(f, \sigma, \mathbb{B})$.

Corollary 2.2.6. *Given a function $f : \sigma \to \mathbb{B}$, $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ if and only if $E_f \in \bigwedge^* BV^{(p)}(R^2, \mathbb{B})$.*

Proof of Corollary 2.2.6. In view of the above Theorem 2.2.5, $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ implies $V_{\bigwedge_p}(f, \sigma, \mathbb{B}) = V_{\bigwedge_p}(E_f, R^2, \mathbb{B})$. Hence, in view of the Corollary 2.1.6 (p.41), $E_f \in \bigwedge^* BV^{(p)}(R^2, \mathbb{B})$. Since $\sigma \subset R^2$ and $E_f|\sigma = f$, in view of the Theorem 2.2.2 (*iv*) (p.44), we have $V_{\bigwedge_p}(f, \sigma, \mathbb{B}) \leq V_{\bigwedge_p}(E_f, R^2, \mathbb{B})$. Thus, in view of the Corollary 2.1.6, $E_f \in \bigwedge^* BV^{(p)}(R^2, \mathbb{B})$ implies $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$.

This completes the proof of the corollary.

Corollary 2.2.7. *The map $F : \bigwedge^* BV^{(p)}(\sigma, \mathbb{B}) \to \bigwedge^* BV^{(p)}(R^2, \mathbb{B})$, defined as $F(f) = E_f$ for all $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$, is a linear isometry.*

The Corollary 2.2.7 can be easily followed from the Theorem 2.2.5 and the Corollary 2.1.7 (p.41).

Theorem 2.2.8. $\left(\bigwedge^* BV^{(p)}(\sigma, \mathbb{B}), \|\cdot\|_{\bigwedge_p(\sigma,\mathbb{B})}\right)$ *is a Banach space with respect to the pointwise operations.*

Proof of Theorem 2.2.8. Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$. Therefore, it converges uniformly to some function say f on σ. From the Theorem 2.1.8 (p.41), we get

$$\lim_{k\to\infty} V_{\Lambda_p^1}((f_k(.,c) - f(.,c)), \sigma_1, \mathbb{B}) = 0 \tag{2.1}$$

and

$$\lim_{k\to\infty} V_{\Lambda_p^2}((f_k(a,.) - f(a,.)), \sigma_2, \mathbb{B}) = 0. \tag{2.2}$$

Now, for any rectangular grid P of σ

$$\begin{aligned} V_{\bigwedge_p}(f_k, \sigma, \mathbb{B}, P) &\le V_{\bigwedge_p}(f_k - f_l, \sigma, \mathbb{B}, P) + V_{\bigwedge_p}(f_l, \sigma, \mathbb{B}, P) \\ &\le V_{\bigwedge_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\bigwedge_p}(f_l, \sigma, \mathbb{B}). \end{aligned}$$

This implies,

$$V_{\bigwedge_p}(f_k, \sigma, \mathbb{B}) \le V_{\bigwedge_p}(f_k - f_l, \sigma, \mathbb{B}) + V_{\bigwedge_p}(f_l, \sigma, \mathbb{B})$$

and

$$|V_{\bigwedge_p}(f_k, \sigma, \mathbb{B}) - V_{\bigwedge_p}(f_l, \sigma, \mathbb{B})| \le V_{\bigwedge_p}(f_k - f_l, \sigma, \mathbb{B}) \to 0 \ as \ k, l \to \infty.$$

Hence, $\{V_{\bigwedge_p}(f_k, \sigma, \mathbb{B})\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore,

$$\begin{aligned} V_{\bigwedge_p}(f, \sigma, \mathbb{B}, P) &= \left(\sum_i \sum_j \frac{(\|\Delta f(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2}\right)^{\frac{1}{p}} \\ &= \lim_{k\to\infty} \left(\sum_i \sum_j \frac{(\|\Delta f_k(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2}\right)^{\frac{1}{p}} \\ &\le \lim_{k\to\infty} V_{\bigwedge_p}(f_k, \sigma, \mathbb{B}) \le M < \infty. \end{aligned}$$

This together with (2.1) and (2.2) imply that $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$. Moreover,

$$V_{\bigwedge_p}(f_k - f, \sigma, \mathbb{B}, P) = \left(\sum_i \sum_j \frac{(\|\Delta (f_k - f)(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2}\right)^{\frac{1}{p}}$$

$$= \lim_{l\to\infty} \left(\sum_i \sum_j \frac{(\|\Delta(f_k - f_l)(x_i, y_j)\|_{\mathbb{B}})^p}{\lambda_i^1 \ \lambda_j^2} \right)^{\frac{1}{p}}$$

$$\leq \lim_{l\to\infty} V_{\bigwedge_p}(f_k - f_l, \sigma, \mathbb{B}) \to 0 \ as \ k \to \infty.$$

Hence, the theorem follows from (2.1) and (2.2).

Remark 2.2.9. *The Theorem 2.2.8, with $\sigma = R^2$ and $\mathbb{B} = \mathbb{R}$, generalize the Theorem A* (p.26)*, for functions of two variables.*

2.3 New results for functions of N−variables

Generalizing the classes $\bigwedge BV^{(p)}(\sigma_1 \times \sigma_2, \mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma_1 \times \sigma_2, \mathbb{B})$ for functions of $N-$variables, we have extended the results of the Section 2.2 for functions of $N-$variables in the following way.

Throughout the present section, $\sigma_1, \ \sigma_2, \cdots, \sigma_{N-1}$ and σ_N represent non-empty compact subsets of $\mathbb{R}$ and $R^N = \prod_{i=1}^N I^i \subset \mathbb{R}^N$ is the smallest closed parallelepiped containing $\sigma = \prod_{i=1}^N \sigma_i$, where $I^i = [a_i, b_i]$ are the smallest closed intervals containing σ_i, for all $i = 1, 2, \cdots, N$.

Definition 2.3.1. *Given $\bigwedge = (\Lambda^1, \cdots, \Lambda^N)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^\infty \in \mathbb{L}$, for $k = 1, 2, \cdots, N$, and $p \geq 1$, a function $f : \sigma \to \mathbb{B}$ is said to be of $p - \bigwedge -$bounded variation* (*that is, $f \in \bigwedge BV^{(p)}(\sigma, \mathbb{B})$*) *if*

$$V_{\bigwedge_p}(f, \sigma, \mathbb{B}) = \sup_{P = P_1 \times \cdots \times P_N} \left\{ V_{\bigwedge_p}(f, \sigma, \mathbb{B}, P) \right\} < \infty,$$

where $P_i = \{x_i^{k_i}\}_{k_i=1}^{s_i} \in \Pi(\sigma_i), \ for \ all \ i = 1, 2, \cdots, N,$

and

$$V_{\bigwedge_p}(f, \sigma, \mathbb{B}, P) = \left(\sum_{k_1} \cdots \sum_{k_N} \frac{(\|\Delta f(x_1^{k_1}, \cdots, x_N^{k_N})\|_{\mathbb{B}})^p}{\lambda_{k_1}^1 \cdots \lambda_{k_N}^N} \right)^{\frac{1}{p}},$$

in which

$$\Delta f(x_1^{k_1}, \cdots, x_N^{k_N}) = f([x_1^{k_1}, x_1^{k_1+1}] \times \cdots \times [x_N^{k_N}, x_N^{k_N+1}])$$

and $\| \cdot \|_{\mathbb{B}}$ is as defined earlier in the Definition 2.1.1 (p.36).

This class is further generalized to the class $\bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ in the sense of Hardy as follows.

A function $f \in \bigwedge BV^{(p)}(\sigma,\mathbb{B})$ is said to be of $p-\bigwedge^*$ −bounded variation (that is, $f \in \bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$) if for each of its marginal functions

$$f(x_1,\cdots,x_{i-1},a_i,x_{i+1},\cdots,x_N) \in (\Lambda^1,\cdots,\Lambda^{i-1},\Lambda^{i+1},\cdots,\Lambda^N)^* BV^{(p)}(\sigma(a_i),\mathbb{B}),$$

for all $i = 1,2,\cdots,N$, where $\sigma(a_i) = \left\{(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_N) \in \prod_{\substack{j=1\\ j\neq i}}^N \sigma_j\right\}$.

Note that, for $\sigma = R^N$, the classes $\bigwedge BV^{(p)}(\sigma,\mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ reduce to the classes $\bigwedge BV^{(p)}(R^N,\mathbb{B})$ and $\bigwedge^* BV^{(p)}(R^N,\mathbb{B})$ respectively. For $\mathbb{B} = \mathbb{C}$, we omit writing $\mathbb{C}$, the classes $\bigwedge BV^{(p)}(\sigma,\mathbb{B})$ and $\bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ reduce to the classes $\bigwedge BV^{(p)}(\sigma)$ and $\bigwedge^* BV^{(p)}(\sigma)$ respectively.

Theorem 2.3.2. *If $f,g \in \bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ then the following hold:*
(i) f and g are bounded.
(ii) $V_{\bigwedge_p}(f+g,\sigma,\mathbb{B}) \leq V_{\bigwedge_p}(f,\sigma,\mathbb{B}) + V_{\bigwedge_p}(g,\sigma,\mathbb{B})$.
(iii) $V_{\bigwedge_p}(\alpha f,\sigma,\mathbb{B}) = |\alpha| V_{\bigwedge_p}(f,\sigma,\mathbb{B})$, for any $\alpha \in \mathbb{C}$.
(iv) $V_{\bigwedge_p}(f,\sigma',\mathbb{B}) \leq V_{\bigwedge_p}(f,\sigma,\mathbb{B})$, if $\sigma' = \prod_{i=1}^N \sigma_i' \subset \sigma$, in which σ_i' are non-empty compact subsets of $\mathbb{R}$, for all $i = 1,2,\cdots,N$.

Corollary 2.3.3. *Let σ_i and τ_i be non-empty compact subsets of $\mathbb{R}$, for all $i = 1,2,\cdots,N$, such that $\sigma = \prod_{i=1}^N \sigma_i \subset \tau = \prod_{i=1}^N \tau_i$. If $f \in \bigwedge^* BV^{(p)}(\tau,\mathbb{B})$ then $f|\sigma \in \bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ and $\|f|\sigma\|_{\bigwedge_p(\sigma,\mathbb{B})} \leq \|f\|_{\bigwedge_p(\tau,\mathbb{B})}$, where*

$$\|f\|_{\bigwedge_p(\sigma,\mathbb{B})} = \|f\|_\infty + V_{\bigwedge_p}(f,\sigma,\mathbb{B}) + \sum_{i=1}^N V_{(\Lambda^1,\cdots,\Lambda^{i-1},\Lambda^{i+1},\cdots,\Lambda^N)_p}\left(f(\cdots,a_i,\cdots), \prod_{\substack{j=1\\ j\neq i}}^N \sigma_j, \mathbb{B}\right),$$

in which

$$\|f\|_\infty = \sup_{(x_1,\cdots,x_N)\in\sigma} \|f(x_1,\cdots,x_N)\|_{\mathbb{B}} < \infty.$$

For many of the properties of the class $\bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$, it is easy to embed the class $\bigwedge^* BV^{(p)}(\sigma,\mathbb{B})$ into the class $\bigwedge^* BV^{(p)}(R^N,\mathbb{B})$ and then use the classical theory.

Definition 2.3.4. *Given a function* $f : \sigma \to \mathbb{B}$, *define the function* $E_f : R^N \to \mathbb{B}$ *by* $E_f(x_1, \cdots, x_N) = f(\alpha(x_1), \cdots, \alpha(x_N))$, *where*

$$\alpha(x_i) = \begin{cases} x_i, & \text{if } \; x_i \in \sigma_i, \\ \sup \; \{t \; : \; [x_i, t] \subset [a_i, b_i] \setminus \sigma_i\}, & \text{otherwise}, \end{cases}$$

for all $i = 1, 2, \cdots, N$.

Theorem 2.3.5. *If* $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ *then* $V_{\bigwedge_p}(f, \sigma, \mathbb{B}) = V_{\bigwedge_p}(E_f, R^N, \mathbb{B})$, *where* $R^N = \prod_{i=1}^N [a_i, b_i]$ *is the smallest closed parallelepiped containing* $\sigma = \prod_{i=1}^N \sigma_i$.

Corollary 2.3.6. *Given a function* $f : \sigma \to \mathbb{B}$, $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$ *if and only if* $E_f \in \bigwedge^* BV^{(p)}(R^N, \mathbb{B})$.

Corollary 2.3.7. *The map* $F : \bigwedge^* BV^{(p)}(\sigma, \mathbb{B}) \to \bigwedge^* BV^{(p)}(R^N, \mathbb{B})$, *defined as* $F(f) = E_f$ *for all* $f \in \bigwedge^* BV^{(p)}(\sigma, \mathbb{B})$, *is a linear isometry.*

Theorem 2.3.8. $\left(\bigwedge^* BV^{(p)}(\sigma, \mathbb{B}), \| \cdot \|_{\bigwedge_p(\sigma, \mathbb{B})}\right)$ *is a Banach space with respect to the pointwise operations.*

All extended results of this section can be proved in the same way as the results in the Section 2.2.