Chapter 3

Basic properties of functions of $p-$bounded variation in the mean

3.1 New results for functions of two variables

While studying the convergence of a Fourier series, C. Jordan [30] introduced the class $BV(\overline{\mathbb{T}})$ and D. Waterman [72] introduced the class $\Lambda BV(\overline{\mathbb{T}})$, where $\mathbb{T} = [0, 2\pi)$. The classical Dirichlet-Jordan test [8, V.I, p.114] asserts that the Fourier series of a $2\pi-$periodic function $f \in BV(\overline{\mathbb{T}})$ converges at each point. Dirichlet-Jordan test is further generalized in several directions. Some of the results on convergence of Fourier series of one variable functions of generalized bounded variations are listed in the paper of M. Avdispahić [6]. In 1996, F. Móricz and A. H. Siddiqi [38] obtained a version of Dirichlet-Jordan test for the convergence of Fourier series in $L^1(\overline{\mathbb{T}})-$norm. Generalizing the class $BV(\overline{\mathbb{T}})$ to the class $BVM(\overline{\mathbb{T}})$ of functions of bounded variation in the mean, F. Móricz and A. H. Siddiqi proved that the n^{th} partial sum, S_n, of the Fourier series of $f \in BVM(\overline{\mathbb{T}})$ converges to f in $L^1(\overline{\mathbb{T}})-$norm and also estimated $\|S_n - f\|_1$. In 2000, P. B. Pierce and D. Waterman [45, p.2593] observed that the class $BVM(\overline{\mathbb{T}})$ is a Banach space, with respect to the pointwise operations and the variation norm in mean as defined earlier in (1.6) (p.28), and there exists a continuous function $f \notin BVM(\overline{\mathbb{T}})$. R. E. Castillo [11] in 2005 extended the class $BVM(\overline{\mathbb{T}})$ to the class $BV^{(p)}M(\overline{\mathbb{T}})$. In 2011, Castillo [12] proved that the class $BV^{(p)}M(\overline{\mathbb{T}})$ is a Banach space, with respect to the poitwise operations and the suitable generalized variation norm in mean, (Theorem D, p.28). Moreover, for any $f \in BV^{(p)}M(\overline{\mathbb{T}}) \cap \mathbf{C}^1(\overline{\mathbb{T}})$, $\|f'\|^p_{L^p(\overline{\mathbb{T}})} = \frac{1}{2\pi} V_p^m(f, \overline{\mathbb{T}})$ (Theorem E, p.28). We have extended Castillo's results (Theorem D and E, p.28) for the class $BV^{(p)}M(\overline{\mathbb{T}}^2)$ of two variables functions of $p-$bounded variation in the mean, which are $2\pi-$periodic in each variable, as follow.

Theorem 3.1.1. *The class $BV^{(p)}M(\overline{\mathbb{T}}^2)$ is a Banach space with respect to the pointwise operations and the variation norm*

$$\|f\| = \|f\|_{L^p(\overline{\mathbb{T}}^2)} + (V_p^m(f, \overline{\mathbb{T}}^2))^{\frac{1}{p}} + (V_p^m(f(.,0), \overline{\mathbb{T}}))^{\frac{1}{p}}$$

$$+ (V_p^m(f(0,.), \overline{\mathbb{T}}))^{\frac{1}{p}}, \quad f \in BV^{(p)}M(\overline{\mathbb{T}}^2).$$

Proof of Theorem 3.1.1. Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $BV^{(p)}M(\overline{\mathbb{T}}^2)$. Therefore, it converges to some function say f in $L^p(\overline{\mathbb{T}}^2)$. In view of the Theorem D (p.28), we have

$$\lim_{k\to\infty} (V_p^m(f_k(.,0) - f(.,0), \overline{\mathbb{T}}))^{\frac{1}{p}} = 0 \tag{3.1}$$

and

$$\lim_{k\to\infty} (V_p^m(f_k(0,.) - f(0,.), \overline{\mathbb{T}}))^{\frac{1}{p}} = 0. \tag{3.2}$$

Now,

$$(V_p^m(f_k, \overline{\mathbb{T}}^2))^{\frac{1}{p}} \leq (V_p^m(f_k - f_l, \overline{\mathbb{T}}^2))^{\frac{1}{p}} + (V_p^m(f_l, \overline{\mathbb{T}}^2))^{\frac{1}{p}}$$

and

$$|(V_p^m(f_k, \overline{\mathbb{T}}^2))^{\frac{1}{p}} - (V_p^m(f_l, \overline{\mathbb{T}}^2))^{\frac{1}{p}}| \leq (V_p^m(f_k - f_l, \overline{\mathbb{T}}^2))^{\frac{1}{p}} \to 0 \ as \ k, l \to \infty.$$

Hence, $\{(V_p^m(f_k, \overline{\mathbb{T}}^2))^{\frac{1}{p}}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it is bounded by some constant say $M > 0$. Therefore,

$$\left(\sum_i \sum_j \int\int_{\overline{\mathbb{T}}^2} \frac{|f(I_{ix} \times I_{jy})|^p}{|I_{ix}|^{p-1}\ |I_{jy}|^{p-1}}\ dx\ dy\right)^{\frac{1}{p}}$$

$$= \lim_{k\to\infty} \left(\sum_i \sum_j \int\int_{\overline{\mathbb{T}}^2} \frac{|f_k(I_{ix} \times I_{jy})|^p}{|I_{ix}|^{p-1}\ |I_{jy}|^{p-1}}\ dx\ dy\right)^{\frac{1}{p}}$$

$$\leq \lim_{k\to\infty} (V_p^m(f_k, \overline{\mathbb{T}}^2))^{\frac{1}{p}} \leq M < \infty.$$

This together with (3.1) and (3.2) imply that $f \in BV^{(p)}M(\overline{\mathbb{T}}^2)$. Moreover,

$$\left(\sum_i \sum_j \int\int_{\overline{\mathbb{T}}^2} \frac{|(f_k - f)(I_{ix} \times I_{jy})|^p}{|I_{ix}|^{p-1}\ |I_{jy}|^{p-1}}\ dx\ dy \right)^{\frac{1}{p}}$$

$$= \lim_{l\to\infty} \left(\sum_i \sum_j \int\int_{\overline{\mathbb{T}}^2} \frac{|(f_k - f_l)(I_{ix} \times I_{jy})|^p}{|I_{ix}|^{p-1}\ |I_{jy}|^{p-1}}\ dx\ dy \right)^{\frac{1}{p}}$$

$$\leq \lim_{l\to\infty} (V_p^m(f_k - f_l, \overline{\mathbb{T}}^2))^{\frac{1}{p}} \to 0 \, as \, k \to \infty.$$

Hence, the theorem follows from (3.1) and (3.2).

Theorem 3.1.2. *Let* $f \in BV^{(p)}M(\overline{\mathbb{T}}^2) \cap \mathbf{C}^2(\overline{\mathbb{T}}^2)$. *Then* $f_x(., s) \in L^p(\overline{\mathbb{T}})$, $f_y(t, .) \in L^p(\overline{\mathbb{T}})$, $f_{xy} \in L^p(\overline{\mathbb{T}}^2)$ *and*

$$V_p^m(f, \overline{\mathbb{T}}^2) + V_p^m(f(., s), \overline{\mathbb{T}}) + V_p^m(f(t, .), \overline{\mathbb{T}})$$

$$= 4\pi^2 \|f_{xy}\|^p_{L^p(\overline{\mathbb{T}}^2)} + 2\pi \|f_x(., s)\|^p_{L^p(\overline{\mathbb{T}})} + 2\pi \|f_y(t, .)\|^p_{L^p(\overline{\mathbb{T}})},$$

where $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$.

We need the following lemma to prove this theorem.

Lemma 3.1.3. ([23, Proposition 3.11, p.93]) *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *satisfy the following.*

(i) For each fixed $y_0 \in [c, d]$, *the function given by* $x \mapsto f(x, y_0)$ *is continuous on* $[a, b]$ *and differentiable on* (a, b).

(ii) For each fixed $x_0 \in (a, b)$, *the function given by* $y \mapsto f_x(x_0, y)$ *is continuous on* $[c, d]$ *and differentiable on* (c, d).

Then there is $(x_0, y_0) \in (a, b) \times (c, d)$ *such that*

$$f(b, d) - f(a, d) - f(b, c) + f(a, c) = f_{xy}(x_0, y_0)(b - a)(d - c).$$

Proof of Theorem 3.1.2. In view of the Theorem E (p.28), we have $f_x(., s) \in L^p(\overline{\mathbb{T}})$, $f_y(t, .) \in L^p(\overline{\mathbb{T}})$,

$$V_p^m(f(.,s),\overline{\mathbb{T}}) = 2\pi \|f_x(.,s)\|^p_{L^p(\overline{\mathbb{T}})} \tag{3.3}$$

and

$$V_p^m(f(t,.),\overline{\mathbb{T}}) = 2\pi \|f_y(t,.)\|^p_{L^p(\overline{\mathbb{T}})}. \tag{3.4}$$

Now, in view of the above Lemma 3.1.3, for any $(x,y) \in \overline{\mathbb{T}}^2$ there exists $(\epsilon_i^1, \epsilon_j^2) \in (x+x_i, x+x_{i+1}) \times (y+y_j, y+y_{j+1})$ such that

$$\frac{|A_1 - B_1 - C_1 + D_1|^p}{|x_{i+1}-x_i|^{p-1}|y_{j+1}-y_j|^{p-1}} = |f_{xy}(\epsilon_i^1, \epsilon_j^2)|^p \ (x_{i+1}-x_i)(y_{j+1}-y_j),$$

where $A_1 = f(x+x_{i+1}, y+y_{j+1})$, $B_1 = f(x+x_i, y+y_{j+1})$, $C_1 = f(x+x_{i+1}, y+y_j)$ and $D_1 = f(x+x_i, y+y_j)$.

Integrating the above equality over $\overline{\mathbb{T}}^2$ and then summing it for $i = 0 \ to \ n-1$ and $j = 0 \ to \ r-1$, we get

$$\begin{aligned} &4\pi^2 \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} |f_{xy}(\epsilon_i^1, \epsilon_j^2)|^p \ (x_{i+1}-x_i)(y_{j+1}-y_j) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} \int\int_{\overline{\mathbb{T}}^2} \frac{|A_1 - B_1 - C_1 + D_1|^p}{|x_{i+1}-x_i|^{p-1}|y_{j+1}-y_j|^{p-1}} \ dx \ dy \\ &\le V_p^m(f, \overline{\mathbb{T}}^2). \end{aligned}$$

Therefore,

$$4\pi^2 \int\int_{\overline{\mathbb{T}}^2} |f_{xy}(x,y)|^p \ dx \ dy \le V_p^m(f, \overline{\mathbb{T}}^2) < \infty.$$

Thus, $f_{xy} \in L^p(\overline{\mathbb{T}}^2)$ and

$$4\pi^2 \|f_{xy}\|^p_{L^p(\overline{\mathbb{T}}^2)} \le V_p^m(f, \overline{\mathbb{T}}^2). \tag{3.5}$$

On the other hand, by Hölder's inequality

$$A_1 - B_1 - C_1 + D_1 = \int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} f_{uv}(u,v)\, du\, dv$$

$$\leq \left(\int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} |f_{uv}(u,v)|^p \, du \, dv \right)^{\frac{1}{p}} \left(\int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} du \, dv \right)^{\frac{p-1}{p}}.$$

Therefore,

$$\frac{|A_1 - B_1 - C_1 + D_1|^p}{|x_{i+1} - x_i|^{p-1} |y_{j+1} - y_j|^{p-1}} \leq \int_{y+y_j}^{y+y_{j+1}} \int_{x+x_i}^{x+x_{i+1}} |f_{uv}(u,v)|^p \, du \, dv.$$

Integrating the above inequality over $\overline{\mathbb{T}}^2$ and then summing it for $i = 0 \; to \; n-1$ and $j = 0 \; to \; r-1$, we get

$$V_p^m(f, \overline{\mathbb{T}}^2) \leq 4\pi^2 \|f_{xy}\|^p_{L^p(\overline{\mathbb{T}}^2)}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$4\pi^2 \|f_{xy}\|^p_{L^p(\overline{\mathbb{T}}^2)} = V_p^m(f, \overline{\mathbb{T}}^2). \tag{3.7}$$

Hence, the theorem follows from (3.3), (3.4) and (3.7).

Theorem 3.1.4. If $f \in L^p(\overline{\mathbb{T}}^2)$ satisfies the condition

$$|f([x_1, x_2] \times [y_1, y_2])| \leq M \, |x_2 - x_1| \, |y_2 - y_1|, \tag{3.8}$$

for all $(x_1, y_1) \leq (x_2, y_2) \in \overline{\mathbb{T}}^2$, where M is constant, then $V_p^m(f, \overline{\mathbb{T}}^2) < \infty$.

Proof of Theorem 3.1.4. Consider any finite collections of non-overlapping subintervals $\{[x_i, x_{i+1}]\}$, for all $i = 0, 1, \cdots, n-1$, and $\{[y_j, y_{j+1}]\}$, for all $j = 0, 1, \cdots, r-1$, in $\overline{\mathbb{T}}$. In view of (3.8), for any $(x, y) \in \overline{\mathbb{T}}^2$, we have

$$|f([x + x_i, x + x_{i+1}] \times [y + y_j, y + y_{j+1}])| \leq M \, |x_{i+1} - x_i| \, |y_{j+1} - y_j|,$$

for all $i = 0, 1, \cdots, n-1$ and for all $j = 0, 1, \cdots, r-1$.

Thus,

$$\frac{|f([x + x_i, x + x_{i+1}] \times [y + y_j, y + y_{j+1}])|^p}{|x_{i+1} - x_i|^{p-1} |y_{j+1} - y_j|^{p-1}} \leq M^p \, |x_{i+1} - x_i| \, |y_{j+1} - y_j|,$$

for all $i = 0, 1, \cdots, n-1$ and for all $j = 0, 1, \cdots, r-1$.

Integrating the above inequality over $\overline{\mathbb{T}}^2$ and then summing it for $i = 0$ to $n-1$

and $j = 0$ to $r - 1$, we get

$$\sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\int\int_{\overline{\mathbb{T}}^2} \frac{|f([x+x_i, x+x_{i+1}] \times [y+y_j, y+y_{j+1}])|^p}{|x_{i+1}-x_i|^{p-1}|y_{j+1}-y_j|^{p-1}}\, dx\, dy$$

$$\leq 4\pi^2 M^p \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} |x_{i+1}-x_i|\ |y_{j+1}-y_j|$$

$$= 4\pi^2 M^p \sum_{i=0}^{n-1} |x_{i+1}-x_i| \left(\sum_{j=0}^{r-1} |y_{j+1}-y_j|\right)$$

$$\leq 8\pi^3 M^p \sum_{i=0}^{n-1} |x_{i+1}-x_i|$$

$$\leq 16\pi^4 M^p < \infty.$$

Thus, $V_p^m(f, \overline{\mathbb{T}}^2) < \infty$.

This completes the proof of the theorem.