Chapter 6

The Perturbed Photo–Gravitational Restricted Three–Body Problem: Analysis of Resonant Periodic Orbits

6.1 Introduction

The study of resonance in the solar system has received wide attention in the recent past. A resonance can arise due to a simple numerical relationship between frequencies or periods. The periods involved can be the rotational period and orbital period of a single body or orbital periods of more than two bodies. The former is termed as spin–orbit coupling and the latter as orbit–orbit coupling. For instance, the Earth–Moon spin–orbit is 1:1, while the Neptune–Pluto has a $3:2$ orbit–orbit resonance [Murray and Dermot (1999)]. A number of articles have been emerged related to the study of resonance in the solar system [Pstor et.al. (2009), Cachucho et. al. (2010), Dvorak (2010)] and resonance of orbits of planets outside solar system [Gayon et. al. (2009), Hadjidemetriou et. al.(2009), Hadjidemetriou and Voyatzis(2010), Libert and Tsiganis (2011)].

It is important to have an understanding of the dynamics of resonance and to develop analytical models that precisely reflect the true nature of resonant interactions. Since the late twentieth century until today, enormous number of researches have enriched the study of RTBP. But the influence of the various perturbing forces has not been studied in many of such interesting problems [Pushparaj and Sharma (2017)]. Two such forces are due to radiation pressure and oblateness. This is one of our motivations to study the effect of these perturbing forces on the exterior and interior resonant periodic orbits.

The aforementioned reasons motivate us to study the resonance phenomena, in the framework the model of photo–gravitational RTBP, by taking the first primary as radiating and the second primary as an oblate spheroid. The importance of the phenomena of resonance in the space dynamics was studied by [Roy and Ovenden (1954), Murray and Dermot (1999)].

[Tsiganis et. al. (2000)] have studied asteroids with autocorrelation time series function. [Ferraz–Mello et. al.(2003)] have analyzed existence of asymmetric libration and their importance for the stability of the 1 : 2 and 1 : 3 resonant motion in satellite and extra solar planetary systems. [Voyatzis and Kotoulas (2005)] have studied number of resonances associated with the dynamical features of Kuiper belt and located between 30 and 48 AU. This study is based on the computation of resonant periodic orbits and their stability. [Voyatzis and Kotoulas (2005)] have described exterior mean motion resonances 1 : 2, 1 : 3 and 1 : 4 of family of periodic orbits with Neptune and stability of family of periodic orbits.

Poincaré surface section (PSS) technique given by [Poincare (1892)] is a widely used technique for analyzing periodic and quasi–periodic orbits in a qualitative way. [Douskos et. al. (2007)] have investigated numerically using Poincaré surface of section the stability of evolution of family $\boldsymbol{f}$ for the Earth–Moon system. They demonstrated that the resonances of third order are the main cause of the reduction of the stability region of retrograde satellites. [Dutt and Sharma (2010)], [Dutt and Sharma (2011a)] analyzed periodic orbits for Earth–Moon and Sun– Mars systems using PSS. [Pathak et. al. (2016a)], [Pathak and Thomas(2016b)] have used PSS technique to analyze periodic and quasi–periodic orbits for Sun–Saturn system with actual oblateness of Saturn and solar radiation pressure as perturbation. [Pathak and Thomas(2016b)] have studied stability analysis and separatrix analysis for different solar radiation pressure using PSS technique. [Pathak and Thomas (2016c)] and [Pathak and Thomas (2016d)] have analyzed periodic orbits around both primaries for Sun–Earth and Sun–Mars systems with oblateness and solar radiation pressure as the perturbation and have studied stability analysis using PSS. The PSS can be used to identify the periodic, quasi–periodic and chaotic regions in the phase space. Further, using Jacobi integral and PSS technique we can identify the order of resonance with the help of number of islands in the PSS.

In the framework of Sitnikov problem, [Perdiou et. al. (2012)] studied the families of three–dimensional periodic orbits, which bifurcate from self–resonant orbits of the Sitnikov family at double, triple and quadruple period of the bifurcation orbit. They have shown that the branch families close upon themselves and remain 3D up to their terminations having two common members with the Sitnikov family. They also studied the evolution of some calculated families by varying the parameter of mass ratio. They have shown that these families are isolated and disappear gradually in 3D reducing to a point size.

This chapter aims to analyse periodic orbits of different orders of resonance, both interior and exterior using PSS in the photo–gravitational RTBP for two systems, namely, the Sun–Earth and the Sun–Mars systems. The effect of resonance on the dynamical structure, location and period of orbits have been studied extensively.

In the framework of the perturbed photo–gravitational RTBP, the first order exterior resonant orbits and the first, third and fifth order interior resonant periodic orbits have been analyzed. The location, eccentricity and period of the first order exterior and interior resonant orbits are investigated in the unperturbed and perturbed cases for a specified value of Jacobi constant C. It is observed that as the number of loops increases successively from single–loop to five–loops, the period of secondary body increases in such a way that the successive difference of periods is either 6 or 7 units.

The evolution of interior first order resonant orbit with three–loops have been studied for different values of Jacobi constant C. It is observed that when the value of C increases, the size of the loop decreases and degenerates finally into a circle, the eccentricity of periodic orbit decreases and location of the periodic orbit moves towards the second primary body.

6.2 Estimation of resonant ratio

The equations of motion of the second primary in the dimensionless synodic coordinates are given by equations (1.4.12) through (1.4.16). Jacobi constant C is given by equation (1.4.21). Mean motion n is given by equation (1.4.17), oblateness coefficient A_2 and solar radiation pressure q are given by equations (1.4.18) and (1.4.19) respectively. The semi–major axis a and the eccentricity e of the orbit of the second primary are given by equations (1.5.48) through (1.5.51).

The period of planet's orbit T_P is given by the relation

$$(1 + \frac{3}{2}A_2)T_P = 2\pi, \tag{6.2.1}$$

and Kepler's third law is given by,

$$\frac{T_1}{T_2} = \left(\frac{a_1}{a_2}\right)^{3/2}. \tag{6.2.2}$$

So, using semi–major axis of orbit of second primary body and semi–major axis of orbit of secondary body, we can obtain order of resonance. Which also can be determined from the number of islands in PSS. This is considered as one of the characteristic of the resonance [Murray and Dermot (1999)].

Throughout this chapter, coefficient of oblateness is taken as $A_2 = 0.0001$ for both systems. Thereby the period of Earth's orbit $T_E \approx 6.282714$ and the period of Mars's orbit $T_M \approx 6.282714$. While the semi-major axis of Earth's orbit is $a_E = 1.0000011$ and the semi-major axis of Mars's orbit $a_M = 1.0003$.

The technique of the determination of $p : q$ ratio for a potential resonant orbit from its location is investigated. This technique includes the use of Poincaré sections and two–body approximations as initial assumptions.

From equation (6.2.2), we can obtain the ratio between periods by

$$\frac{p}{q} = \frac{T_q}{T_p}. \tag{6.2.3}$$

The difference between p and q is equal to order of the resonance which can be determined using PSS of that orbits, because the number of island visible in PSS indicate the difference of p and q.

The values of a, e and T_p are used to calculate the approximate resonant orbit. In addition the period p in the $p:q$ ratio can be determined from the number of loops, thereby q can be evaluated from the value of p with the help of equation (6.2.3). Hence the values of location and velocity, which are obtained from Poincaré section and the ratio of $p:q$ from the two–body approximation are used as the initial condition in the correction scheme to evaluate the desired resonant orbit in the perturbed RTBP related to Sun–Earth and the Sun–Mars systems. Generally the above investigations summarize the strategy on estimating resonant ratio from a surface of section.

6.3 Exterior first order resonance

We have analyzed four families of periodic orbits for exterior first order resonance in the Sun–Earth and the Sun–Mars systems. These are periodic orbit with exterior resonant orbits possessing inner loops. Family–I is unperturbed by radiation pressure and oblateness coefficient (i.e. $q = 1$ and $A_2 = 0$). Perturbation due to oblateness alone is considered in Family–II (i.e. $q = 1$ and $A_2 = 0.0001$). Family–III is characterized by perturbation due to radiation pressure only (i.e. $q = 0.9845$ and $A_2 = 0$) and in Family–IV both radiation pressure and oblateness are included (i.e. $q = 0.9845$ and $A_2 = 0.0001$). For simplicity in writing the head rows of Tables, numerical estimates for relevant quantities of family such as: solar radiation pressure, oblateness parameter, number of loops, location of the periodic orbit, number of islands, resonance order, eccentricity, time period of the orbit and ratio of the orbital periods will be denoted by FA, SR, OB, NL, LO, NI, RO,EC, TP and RP, respectively.

Since the period of Earth's orbit $T_E \approx 6.282714$ units, it can be noticed that as the number of loops increases successively from one–loop to five–loops, period of the orbit of secondary body increases in such a way that the successive difference of periods differ either by 6 or 7 units as shown in Table 6.1. The period of one–loop orbit of Family–I is 13, while that of two–loops orbit is 19 with a difference of 6 units. Now the period of three–loops orbit is 26 so that the difference in the period of two–loops and three–loops orbits is 7 units. Resonance of order 1 : 2 indicates that the time taken by Earth to orbit twice around the Sun is equal to the time taken by the secondary body to orbit once around the Sun. Also period of Mar's orbit $T_M \approx 6.282714$ units. So, in a similar manner it can be noticed that as the number of loops increases successively from single–loop to five–loops, the period of the orbit of secondary body increases in such a way that the successive difference of periods differ either by 6 or 7 units as shown in Table 6.2.

The period of single–loop orbit of Family–I is 13, while that of two–loops orbit is 19 with a difference of 6 units. Now the period of three–loops orbit is 26 so that the difference in the period of two–loops and three–loops orbits is 7 units. Resonance of order 1 : 2 indicates that the time taken by Mars to orbit twice around the Sun is equal to the time taken by the secondary body to orbit once around the Sun. The PSS of periodic orbit with number of loops one and two are shown in Fig.6.1 and

Table 6.1: Analysis of exterior first order resonance for $C = 2.93$ for perturbed Sun–Earth system.

FA	SR	OB	NL	LO	NI	RO	EC	TP	RP
I	1	0	1	0.93904	1	1:2	0.40895	13	0.49936
			2	0.88740		2:3	0.32301	19	0.66633
			3	0.85623		3:4	0.29337	26	0.74943
			4	0.83547		4:5	0.28015	32	0.79977
			5	0.82075		5:6	0.27331	38	0.83313
II	1	0.0001	1	0.93877	1	1:2	0.40904	13	0.49945
			2	0.88710		2:3	0.32319	19	0.66641
			3	0.85592		3:4	0.29358	26	0.74979
			4	0.83516		4:5	0.28039	32	0.79982
			5	0.82044		5:6	0.27355	38	0.83318
III	0.9845	0	1	0.97895	1	1:2	0.36044	13	0.52805
			2	0.93800		2:3	0.26118	19	0.69904
			3	0.91210		3:4	0.22428	26	0.78432
			4	0.89403		4:5	0.20685	32	0.83562
			5	0.88075		5:6	0.19740	38	0.86990
IV	0.9845	0.0001	1	0.97870	1	1:2	0.36081	13	0.52779
			2	0.93764		2:3	0.26136	19	0.69919
			3	0.91172		3:4	0.22453	26	0.78443
			4	0.89363		4:5	0.20713	32	0.83574
			5	0.88035		5:6	0.19771	38	0.86999

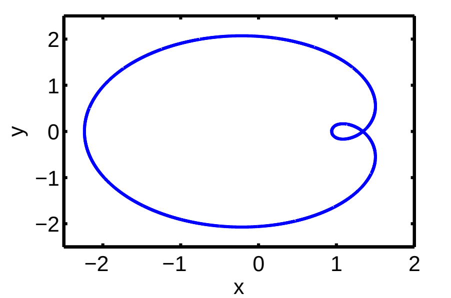
Fig.6.2.

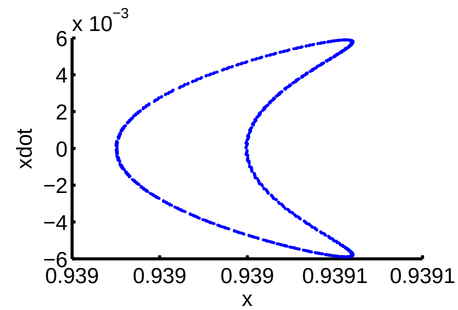
Fig.6.1(a) depicts single–loop orbit corresponding to $x_0 = 0.93904$ for the classical case with $q = 1$ and $A_2 = 0$, for the unperturbed case. The inner single–loop orbit has a period 13, and PSS at $x_0 = 0.93904$ gives single island as shown in Fig.6.1(b). Using the characteristics of resonance for PSS, number of islands indicates order of the resonance. So the resulting orbit is of first order exterior resonance, because the ratio of periods $T_1/T_2 \approx 0.49936$ which indicates that the resonance is 1 : 2 type.

In Fig.6.1(c), we have shown a single–loop periodic orbit with perturbation $q = 0.9845$ and $A_2 = 0.0001$ with period 13 corresponding to $x_0 = 0.97870$. The PSS at $x_0 = 0.97870$ gives single island as shown in Fig.6.1(d). In this case, the value of $T_1/T_2 \approx 0.52779$ which indicates that the resonance is of the order 1 : 2. It can be noticed that the size of the single–loop orbit has reduced due to the effect of perturbations due to radiation and oblateness.

Table 6.2: Analysis of exterior first order resonance for $C = 2.93$ for perturbed Sun–Mars system.

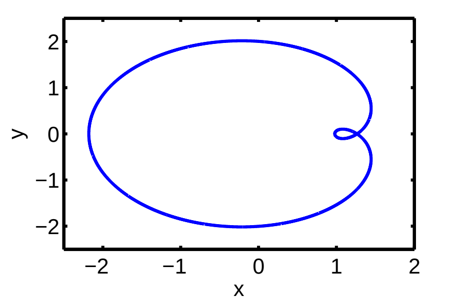
FA	SR	OB	NL	LO	NI	RO	EC	TP	RP
I	1	0	1	0.939000	1	1:2	0.40852	13	0.50015
			2	0.887370		2:3	0.32284	19	0.66692
			3	0.856190		3:4	0.29324	26	0.75031
			4	0.835433		4:5	0.28006	32	0.80033
			5	0.820715		5:6	0.27323	38	0.83368
II	1	0.0001	1	0.93875	1	1:2	0.40866	13	0.50017
			2	0.88708		2:3	0.32302	19	0.66697
			3	0.85588		3:4	0.29346	26	0.75037
			4	0.83512		4:5	0.28029	32	0.80039
			5	0.82040		5:6	0.27347	38	0.83374
III	0.9845	0	1	0.97891	1	1:2	0.35887	13	0.53027
			2	0.93795		2:3	0.26069	19	0.70019
			3	0.91204		3:4	0.22397	26	0.78521
			4	0.89397		4:5	0.20662	32	0.83643
			5	0.88069		5:6	0.19722	38	0.87067
IV	0.9845	0.0001	1	0.97861	1	1:2	0.35895	13	0.53041
			2	0.93761		2:3	0.26090	19	0.70018
			3	0.91167		3:4	0.22423	26	0.78529
			4	0.89358		4:5	0.20691	32	0.83652
			5	0.88029		5:6	0.19752	38	0.87076

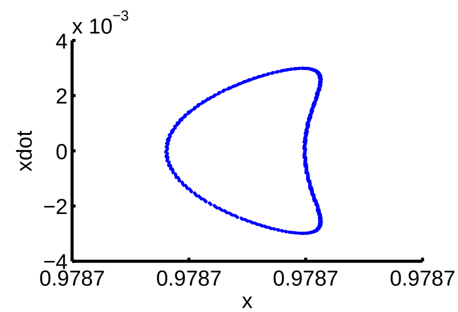




(a) Orbit at $x_0 = 0.93904$ for $q = 1$ and $A_2 = 0$.

(b) PSS at $x_0 = 0.93904$ for $q = 1$, $A_2 = 0$.

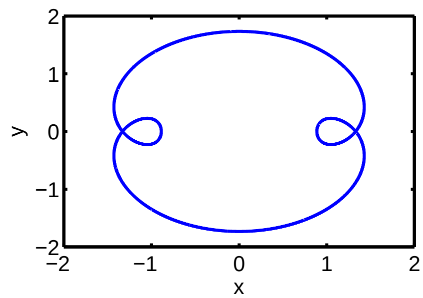




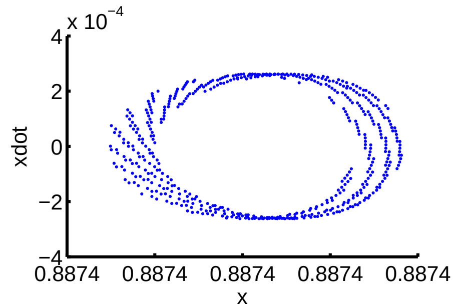
(c) Orbit at $x_0 = 0.97870$ for $q = 0.9845$, $A_2 = 0.0001$.

(d) PSS at $x_0 = 0.97870$ for $q = 0.9845$, $A_2 = 0.0001$.

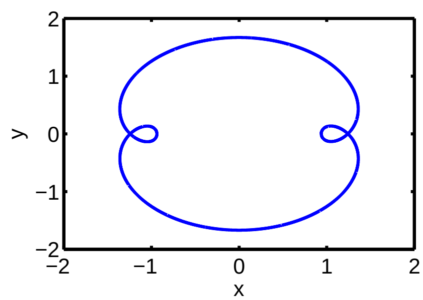
Figure 6.1: Exterior first order resonant single–loop orbit and PSS for $C = 2.93$ for Sun–Earth system.



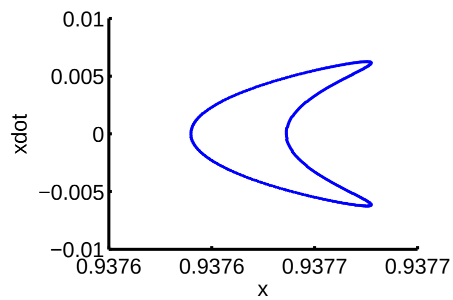
(a) Orbit at $x_0 = 0.88740$ for $q = 1$ and $A_2 = 0$.



(b) PSS at $x_0 = 0.88740$ for $q = 1$, $A_2 = 0$.



(c) Orbit at $x_0 = 0.93764$ for $q = 0.9845$, $A_2 = 0.0001$.



(d) PSS at $x_0 = 0.93764$ for $q = 0.9845$, $A_2 = 0.0001$.

Figure 6.2: Exterior first order resonant two–loops orbit and PSS for $C = 2.93$ for Sun–Earth system.

In Fig.6.2(a), we have displayed two–loops periodic orbits with period 19 corresponding to $x_0 = 0.88740$ in the unperturbed case. The eccentricity of the orbit is 0.32301. The PSS at $x_0 = 0.88740$ as shown in Fig.6.2(b) consists of a single island and hence the order of the resonance is one. Furthermore, $T_1/T_2 = 0.66633$ indicates that the ratio of resonance is 2 : 3. Fig.6.2(c) and Fig.6.2(d), respectively, show two–loops periodic orbits and the PSS corresponding to $x_0 = 0.93764$ in the perturbed case with $q = 0.9845$ and $A_2 = 0.0001$. The period is same as in the unperturbed case. The eccentricity of the orbit is 0.26136 and $T_1/T_2 = 0.69919$.

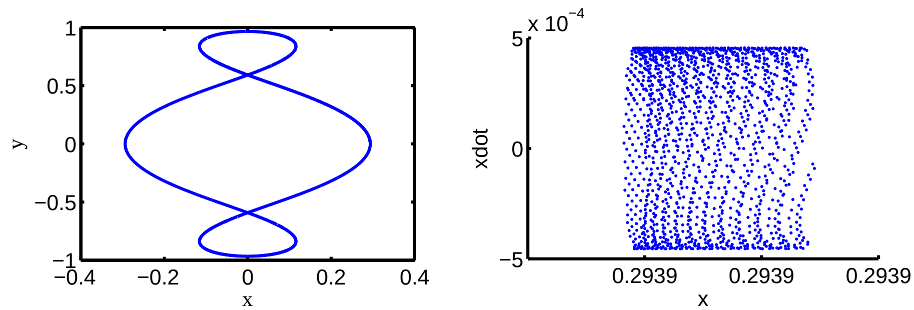
As in the single–loop case, the size of the loops in the orbit and the eccentricity have reduced due to perturbations. The order of the resonance is one and its ratio is 2 : 3. Similar analysis have been conducted for periodic orbits with number of loops ranging from 3 to 5 for Family–I and Family–IV in the unperturbed as well as perturbed cases and arrived at similar conclusions as in single–loop and two–loops orbit cases.

6.4 Interior first order resonance

In Table 6.3 and Table 6.4, we have displayed relevant quantities for first order interior resonance for number of loops varying from 2 to 8 for the Sun–Earth and the Sun–Mars systems. It should be noted that for internal first order resonance single–loop orbit does not exist. For $C = 2.93$ we have divided the table into four families in which Family–I is without perturbation and other families incorporating perturbation.

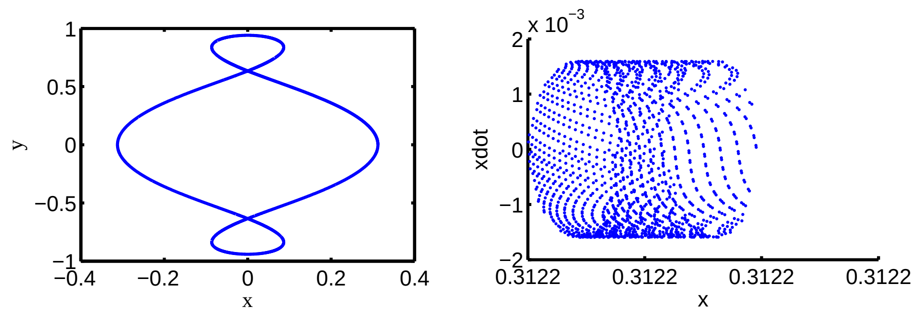
Fig. 6.3(a) depicts two–loops orbit at $x_0 = 0.29385$ without perturbation and corresponding PSS is given in Fig. 6.3(b) for Sun–Earth system. The two–loops orbit at $x_0 = 0.31222$ for the Sun–Earth with perturbation and its PSS are given in Fig. 6.3(c) and Fig. 6.3(d). In all the cases, it can be seen that no loop is formed around each of the primaries and the orbit is around the larger primary, namely the Sun. Further, the size of the loop reduces due to perturbation.

In the case of interior resonance for Sun–Earth system, Figure 6.4 shows three–loops orbits for different Jacobi constant values, when the mass reduction value



(a) *Orbit at* $x_0 = 0.29385$ *for* $q = 1$ *and* $A_2 = 0$.

(b) *PSS at* $x_0 = 0.29385$ *for* $q = 1$, $A_2 = 0$.



(c) *Orbit at* $x_0 = 0.31222$ *for* $q = 0.9845$, $A_2 = 0.0001$.

(d) *PSS at* $x_0 = 0.31222$ *for* $q = 0.9845$, $A_2 = 0.0001$.

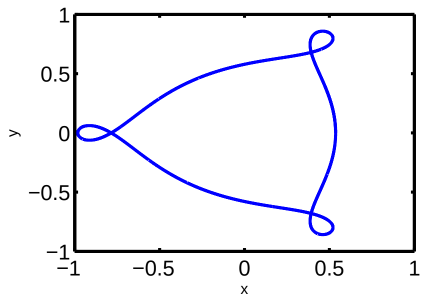
Figure 6.3: Interior first order resonant two–loops orbit and PSS for $C = 2.93$ *for Sun–Earth system*

$q = 0.9845$ and the parameter of oblateness $A_2 = 0.0001$. It is clear that the size of loop decreases with increasing value of Jacobi constant C. These observations are shown in Figures 6.4(a) – (f), when $C = 2.95, 2.97, 2.99, 3.01, 3.02$ and 3.03. Figure 6.5 presents the variation in PSS of three–loops periodic orbits, for different values of C, the oblateness parameter A_2 and mass reduction factor q. It is observed that the shape of the PSS changes and finally degenerates in to circle, as shown in Figure 6.5(a) – (f). In addition, Figure 6.4 and Figure 6.5 show the evolution of three–loops orbits, with perturbation for Family–IV of the Sun–Earth system together with its PSS for different values of C. The loops eventually degenerates into a circle.

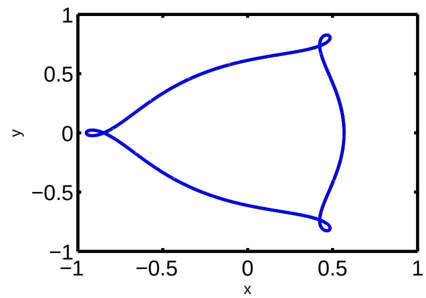
Figure 6.6 shows variation in the location of the periodic orbits of first order interior

Table 6.3: Analysis of interior first order resonance for $C = 2.93$ for perturbed Sun–Earth system.

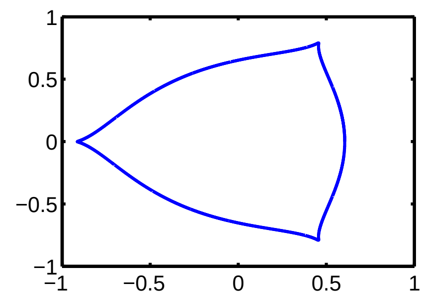
FA	SR	OB	NL	LO	NI	RO	EC	TP	RP
			2	0.29385		2:1	0.53353	07	2.00000
			3	0.47692		3:2	0.37506	13	1.50000
I	1	0	4	0.55735	1	4:3	0.32483	19	1.33330
			5	0.60105		5:4	0.30258	26	1.24991
			6	0.62815		6:5	0.29074	32	1.19981
			7	0.64650		7:6	0.28366	38	1.16633
			8	0.66000		8:7	0.27897	44	1.14185
			2	0.29375		2:1	0.53367	07	2.00015
			3	0.47675		3:2	0.37525	13	1.50009
II	1	0.0001	4	0.55713	1	4:3	0.32506	19	1.33339
			5	0.60080		5:4	0.30283	26	1.25001
			6	0.62788		6:5	0.29100	32	1.19992
			7	0.64627		7:6	0.28391	38	1.16635
			8	0.65980		8:7	0.27921	44	1.14180
			2	0.31234		2:1	0.47832	07	2.15851
			3	0.50990		3:2	0.30776	13	1.58182
III	0.9845	0	4	0.59888	1	4:3	0.25104	19	1.39854
			5	0.64770		5:4	0.22523	26	1.30827
			6	0.67801		6:5	0.21135	32	1.25452
			7	0.69851		7:6	0.20301	38	1.21879
			8	0.71327		8:7	0.19758	44	1.19323
			2	0.31222		2:1	0.47848	07	2.15875
			3	0.50970		3:2	0.30799	13	1.58195
IV	0.9845	0.0001	4	0.59861	1	4:3	0.25133	19	1.39869
			5	0.64740		5:4	0.22554	26	1.30839
			6	0.67768		6:5	0.21168	32	1.25465
			7	0.69819		7:6	0.20333	38	1.21887
			8	0.71290		8:7	0.19793	44	1.19338



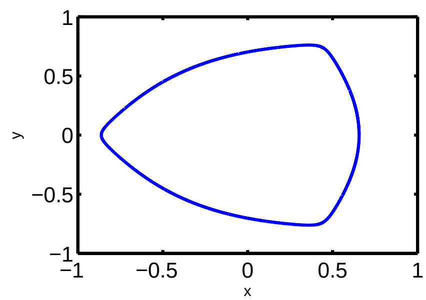
(a) Orbit at $x_0 = 0.53653$ *and* $C = 2.95$.



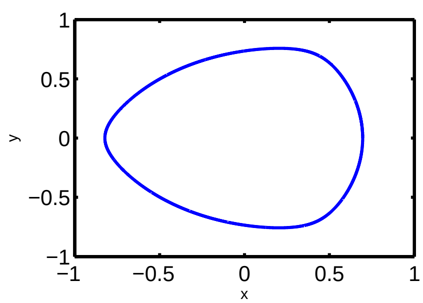
(b) Orbit at $x_0 = 0.56750$ *and* $C = 2.97$.



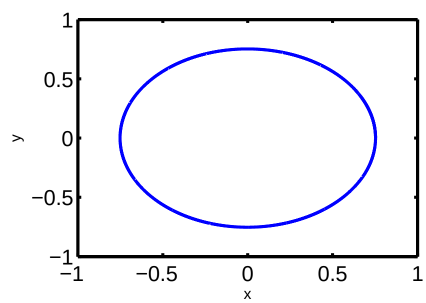
(c) Orbit at $x_0 = 0.60501$ *and* $C = 2.99$.



(d) Orbit at $x_0 = 0.65610$ *and* $C = 3.01$.

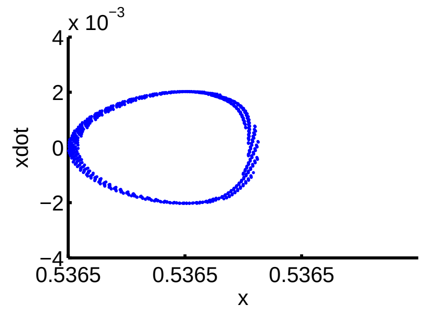


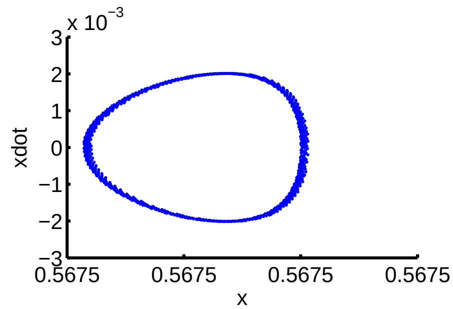
(e) Orbit at $x_0 = 0.69590$ *and* $C = 3.02$.



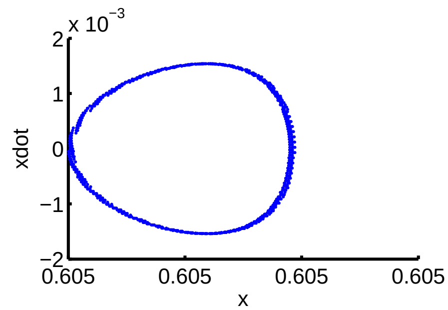
(f) Orbit at $x_0 = 0.75300$ *and* $C = 3.03$.

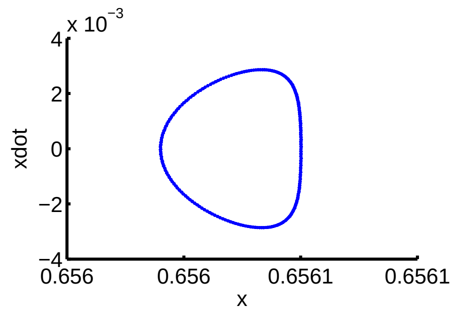
Figure 6.4: Variation in three–loops interior first order resonant orbit when $q = 0.9845$ *and* $A_2 = 0.0001$ *in the Sun–Earth system.*



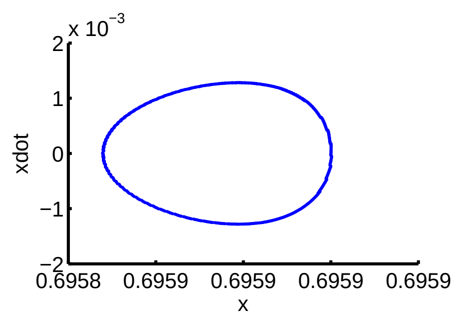


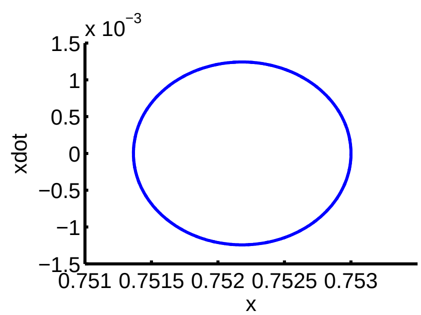
(a) PSS at $x_0 = 0.53653$ and $C = 2.95$. *(b) PSS at $x_0 = 0.56750$ and $C = 2.97$.*





(c) PSS at $x_0 = 0.60501$ and $C = 2.99$. *(d) PSS at $x_0 = 0.65610$ and $C = 3.01$.*





(e) PSS at $x_0 = 0.69590$ and $C = 3.02$. *(f) PSS at $x_0 = 0.75300$ and $C = 3.03$.*

Figure 6.5: Variation in PSS of three–loops orbit due to interior first order resonance when $q = 0.9845$, $A_2 = 0.0001$ in the Sun–Earth system.

Table 6.4: Analysis of interior first order resonance $C = 2.93$ for perturbed Sun–Mars system.

FA	SR	OB	NL	LO	NI	RO	EC	TP	RP
			2	0.29386		2:1	0.53352	07	2.00090
			3	0.47693		3:2	0.37504	13	1.50067
I	1	0	4	0.55734	1	4:3	0.32482	19	1.33393
			5	0.60102		5:4	0.30258	26	1.25055
			6	0.62808		6:5	0.29075	32	1.20005
			7	0.64637		7:6	0.28369	38	1.16792
			8	0.65954		8:7	0.27911	44	1.14323
			2	0.293750		2:1	0.533670	07	2.00105
			3	0.476745		3:2	0.375250	13	1.50079
II	1	0.0001	4	0.557125	1	4:3	0.325050	19	1.33402
			5	0.600770		5:4	0.302830	26	1.25065
			6	0.627830		6:5	0.291011	32	1.20058
			7	0.646100		7:6	0.283950	38	1.16722
			8	0.659270		8:7	0.279370	44	1.14331
			2	0.31235		2:1	0.47831	07	2.15947
			3	0.50991		3:2	0.30774	13	1.58251
III	0.9845	0	4	0.59887	1	4:3	0.25103	19	1.39922
			5	0.64768		5:4	0.22522	26	1.30892
			6	0.67797		6:5	0.21134	32	1.25519
			7	0.69843		7:6	0.20301	38	1.21952
			8	0.71309		8:7	0.19761	44	1.19412
			2	0.31223		2:1	0.478470	07	2.15970
			3	0.50971		3:2	0.307980	13	1.58266
IV	0.9845	0.0001	4	0.59860	1	4:3	0.251320	19	1.39936
			5	0.64738		5:4	0.225530	26	1.30905
			6	0.67764		6:5	0.211567	32	1.25532
			7	0.69809		7:6	0.203340	38	1.21963
			8	0.71273		8:7	0.197960	44	1.19426

Table 6.5: Variation in three–loops orbit due to variation in C for $q = 0.9845$ and $A_2 = 0.0001$ for Sun–Earth system

JC	LO	NI	RO	EC	TP	RP
2.93	0.50970			0.30799		1.58195
2.95	0.53653			0.27320		1.57665
2.97	0.56750			0.23304		1.57112
2.99	0.60501	1	3:2	0.18439	13	1.56524
3.01	0.65610			0.11817		1.55825
3.02	0.69590			0.06656		1.55357
3.03	0.75300			0.01658		1.56821

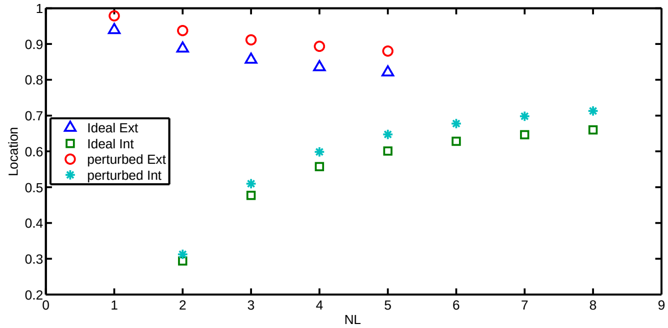


Figure 6.6: Variation in location of the first order interior and exterior resonant periodic orbit for $C = 2.93$ in perturbed case ($q = 0.9845$, $A_2 = 0.0001$) and ideal case ($q = 1$, $A_2 = 0$) for the Sun–Earth system.

and exterior resonance in ideal case (i.e. $q = 1$ and $A_2 = 0$) and in perturbed case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) in the Sun–Earth system. It is clearly seen that for the exterior resonance as the number of loops increases, location of the periodic orbit moves towards the Sun whereas for the interior resonance as the number of loops increases, location of the periodic orbit moves away from the Sun.

In this context the location of exterior or interior first order resonant orbits moves away from the Sun whenever perturbation is included. From locations of orbits it can be seen that exterior resonant orbits with and without perturbation are nearer to the Earth whereas interior resonant orbits are nearer to the Sun. So, for the orbit having same number of loops, location of interior resonant orbit is nearer to the Sun in comparison with exterior resonant orbit.

Figure 6.7 shows variation in the eccentricity of periodic orbits of first order interior and exterior resonance in classical case (i.e. $q = 1$ and $A_2 = 0$) and in perturbed

Table 6.6: Variation in three–loops orbit due to variation in C for $q = 0.9845$ and $A_2 = 0.0001$ in the Sun–Mars system

JC	LO	NI	RO	EC	TP	RP
2.93	0.50971	1	3:2	0.30798	13	1.58266
2.97	0.56750			0.23303		1.57184
3.01	0.65608			0.11815		1.55901
3.02	0.69590			0.06651		1.55431
3.03	0.75200			0.01514		1.56908

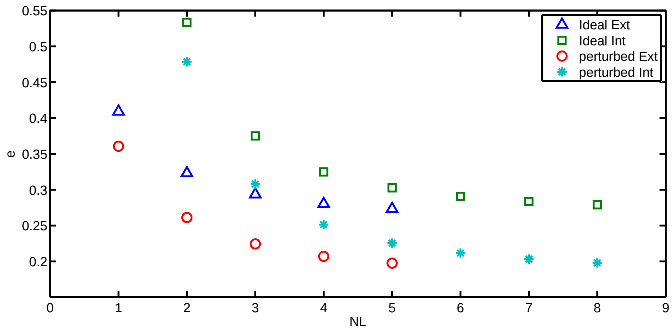


Figure 6.7: Variation in eccentricity of the first order interior and exterior resonant periodic orbit for $C = 2.93$ in perturbed case ($q = 0.9845$, $A_2 = 0.0001$) and ideal case ($q = 1$, $A_2 = 0$) for the Sun–Earth system

case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) for the Sun–Earth system. It is clearly seen that for all four cases eccentricity of the periodic orbit decreases as number of loops increases. Eccentricity of interior resonant periodic orbit in ideal case is highest among all the four cases.

As perturbation due to solar radiation pressure of Sun and oblateness of Earth increases eccentricity of the periodic orbit decreases. Eccentricity of exterior resonant periodic orbit in the perturbed case is lowest among all the four cases. Among the orbits having same number of loops, eccentricity of interior resonant orbit is more in comparison to exterior resonant orbit.

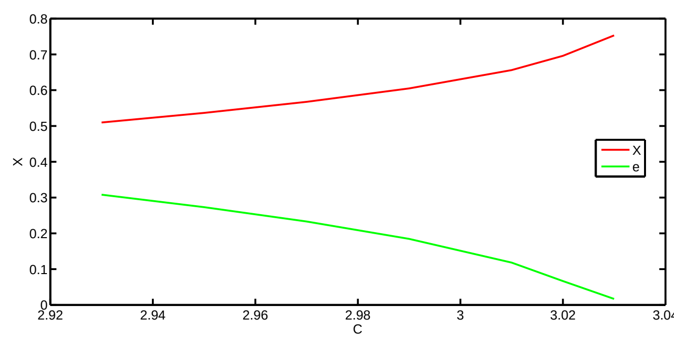


Figure 6.8: Variation in location and eccentricity of first order interior three–loops orbit when $q = 0.9845$ and $A_2 = 0.0001$ for Sun–Earth system.

Fig. 6.8 shows variation in location and eccentricity of three–loops interior resonant periodic orbit due to variation in C in perturbed case (i.e. $q = 0.9845$ and $A_2 = 0.0001$) for the Sun–Earth system. As C increases, location of periodic orbit moves away from the Sun and eccentricity decreases. Figures (6.9 – 6.11) show similar

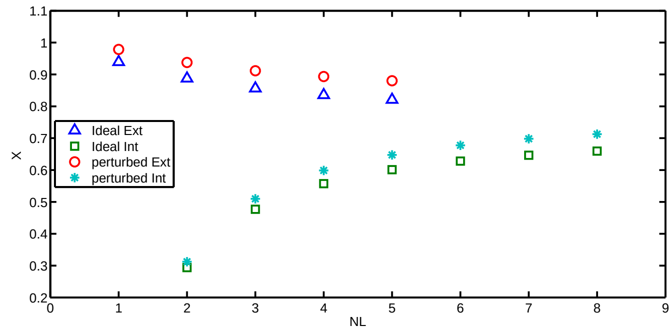


Figure 6.9: Variation in location of the first order interior and exterior resonant periodic orbit for C = 2.93 in perturbed case ($q = 0.9845$, $A_2 = 0.0001$) and ideal case($q = 1$, $A_2 = 0$) in the Sun–Mars system.

results for the Sun–Mars system.

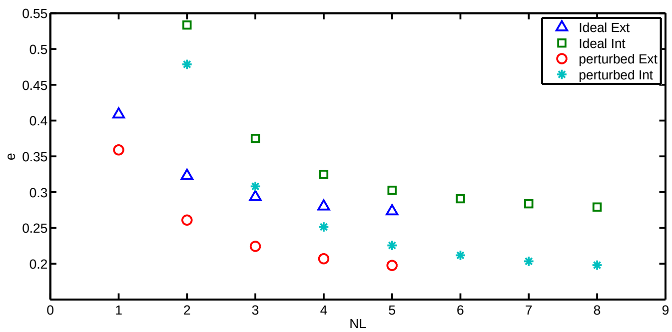


Figure 6.10: Variation in eccentricity of the first order interior and exterior resonant periodic orbit for C = 2.93 in perturbed case ($q = 0.9845$, $A_2 = 0.0001$) and ideal case ($q = 1$, $A_2 = 0$) for the Sun–Mars system.

6.5 Interior resonance of third order

In this section we have analyzed the third order resonant orbits with different number of loops for different parameters of the orbit. The variations of position and eccentricity for different values of the Jacobi constant C for seven–loops orbit is given in Table 6.7 for the Sun–Earth system and in Table 6.8 for the Sun–Mars system.

Figures 6.12(a) – (c) are seven–loops orbits for $C = 2.93$, 2.96 and 2.98, respectively, for $q = 0.9845$ and $A_2 = 0.0001$. It can be noticed that the size of the periodic orbit as well as size of the loops decrease with increase in the value of C when q and A_2

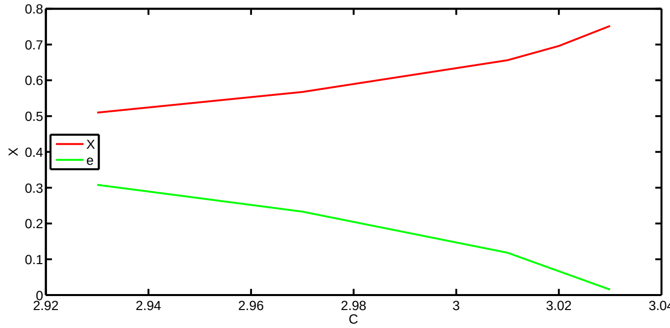


Figure 6.11: Variation in location and eccentricity of interior first order three–loops orbit for $q = 0.9845$, $A_2 = 0.0001$ and C = 2.93 for the Sun–Mars system.

are fixed. However, the period of the orbits remain same as 26 indicating that an increase in C decreases the orbital velocity of the particle.

Similar observation have been made in the Sun–Mars system too. The PSS at $x = 0.39923$ for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ for the Sun–Earth system given in Figure 6.13(a) shows three islands indicating third order of resonance. Figure 6.13(b) is the magnified version of one of the islands of Figure 6.13(a).

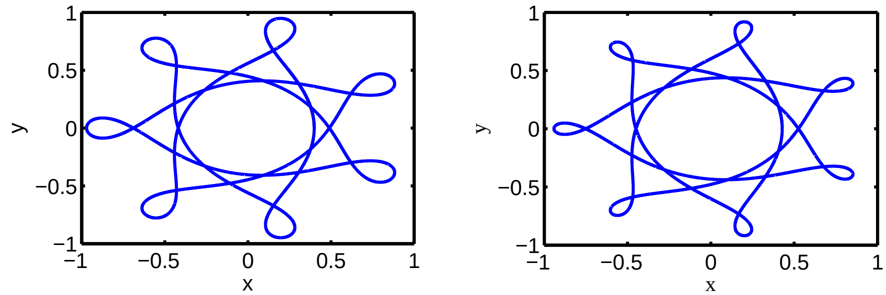
Table 6.7: Variation in third order interior resonant seven loops orbit due to variation in C for the Sun–Earth system

JC	LO	NI	RS	EC	TP	RP
2.93	0.39923			0.39522		1.86449
2.96	0.42824	3	7:4	0.35353	26	1.85476
2.98	0.44991			0.32238		1.84836

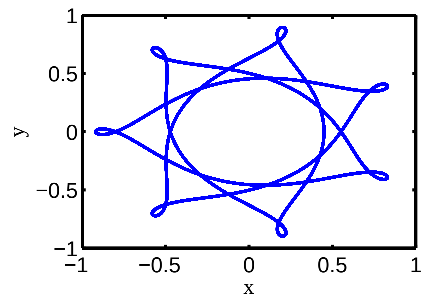
Table 6.8: Variation in third order interior resonant seven– loops orbit due to variation in C for the Sun–Mars system

JC	LO	NI	RS	EC	TP	RP
2.93	0.39923			0.39521		1.86534
2.96	0.42823	3	7:4	0.35353	26	1.85564
2.98	0.44991			0.32232		1.84921

Numerical estimates of position, eccentricity, period and other relevant quantities

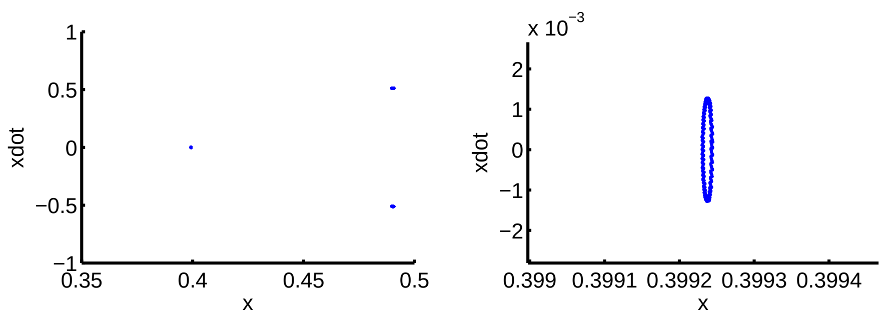


(a) Orbit at $x_0 = 0.39923$ *and* $C =$ 2.93. *(b) Orbit at* $x_0 = 0.42824$ *and* $C =$ 2.96.



(c) Orbit at $x_0 = 0.44991$ *and* $C =$ 2.98.

Figure 6.12: Variation in interior third order resonant seven–loops orbit for $q = 0.9845$, $A_2 = 0.0001$ *and* C = *2.93 for Sun – Earth system.*



(a) PSS at $x_0 = 0.39923$ *and* $C = 2.93$. *(b) Enlarged PSS at* $x_0 = 0.39923$ *for* $C = 2.93$.

Figure 6.13: PSS of interior third order resonant seven–loops orbit of Family–I for $q = 0.9845$, $A_2 = 0.0001$ *and* C = *2.93 for Sun–Earth system*

for third order interior resonance, with number of loops varying from 7 to 14 for Sun–Earth and Sun–Mars systems are shown in Table 6.9 and Table 6.10. They are divided in to two families; Family–I and Family–II. Seven–loops orbit from Family–I has period 26 while seven–loops orbit in Family–II has period 32. These orbits are 7 : 4 resonant orbits. For a given number of loops, and the given resonance, period of Family–II orbit is more than period of Family–I orbit.

Orbits of Family–I are around the first primary only, whereas, orbits of Family–II are around both primaries in which one of the loops of the orbit is around the second primary body, namely, Earth or Mars. Periodic orbits with loops 7, 8, 10, 11, 13 and 14 of Family–I with third order interior resonance for the Sun–Earth system for

Table 6.9: Third order interior resonance $C = 2.93$ in the Sun–Earth system.

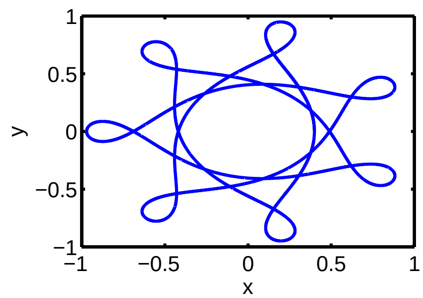
FA	NL	LO	NI	RO	EC	TP	RP
I	7	0.39923	3	7:4	0.39522	26	1.86449
	8	0.46231		8:5	0.34307	32	1.69384
	10	0.54635		10:7	0.28318	44	1.50281
	11	0.57515		11:8	0.26511	51	1.44432
	13	0.61796		13:10	0.24063	63	1.36221
	14	0.63358		14:11	0.23244	70	1.33343
II	7	0.56160	3	7:4	0.27346	32	1.47147
	9	0.62620		9:6	0.23625	44	1.34696
	11	0.66415		11:8	0.21766	57	1.27850
	13	0.68886		13:10	0.20702	70	1.23510

Table 6.10: Third order interior resonance $C = 2.93$ for Sun–Mars system.

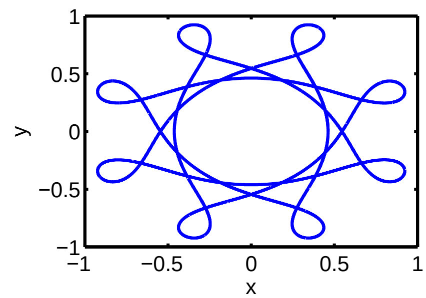
FA	NL	LO	NI	RO	EC	TP	RP
I	7	0.39923	3	7:4	0.39521	26	1.86534
	8	0.46235		8:5	0.34303	32	1.69452
	10	0.54630		10:7	0.28320	44	1.50361
	11	0.57521		11:8	0.26506	51	1.44487
	13	0.61783		13:10	0.24068	63	1.36310
	14	0.63380		14:11	0.23231	70	1.33366
II	7	0.56158	3	7:4	0.27346	32	1.47220
	9	0.62616		9:6	0.23626	44	1.34767
	11	0.66413		11:8	0.21764	57	1.2794
	13	0.68878		13:10	0.20702	70	1.23584

$q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ are shown in Fig. 6.14(a)–(f), respectively.

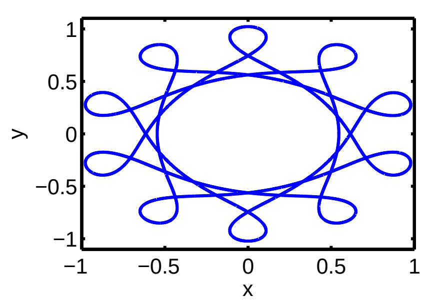
The PSS at $x_0 = 0.5616$ for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun–Earth system given in Fig.6.15(a) contains three islands indicating third order of resonance. Fig.6.15(b) is the magnified version of one of the islands of Fig. 6.15(a). Periodic orbits with number of loops 7, 9, 11 and 13 of Family–II for the Sun–Earth system are shown in Fig. 6.16 (a)–(d), respectively.



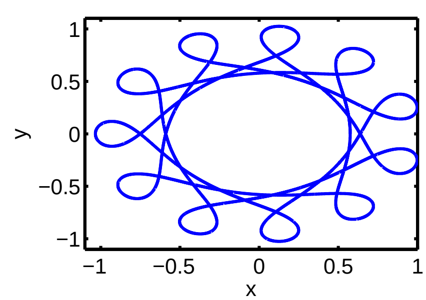
(a) Seven–loops orbit at $x_0 = 0.39923$.



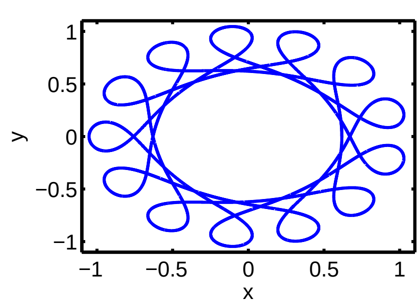
(b) Eight–loops periodic orbit at $x_0 = 0.46231$.



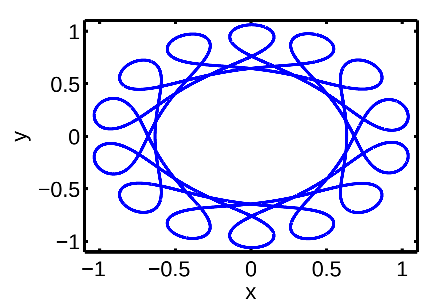
(c) Ten–loops periodic orbit at $x_0 = 0.54635$.



(d) Eleven–loops periodic orbit at $x_0 = 0.57515$.

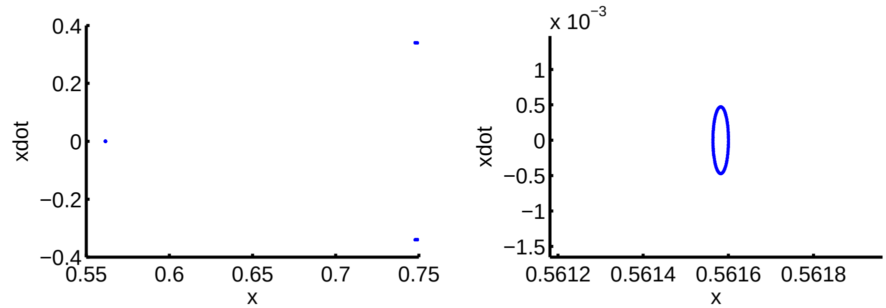


(e) Thirteen–loops periodic orbit at $x_0 = 0.61796$.



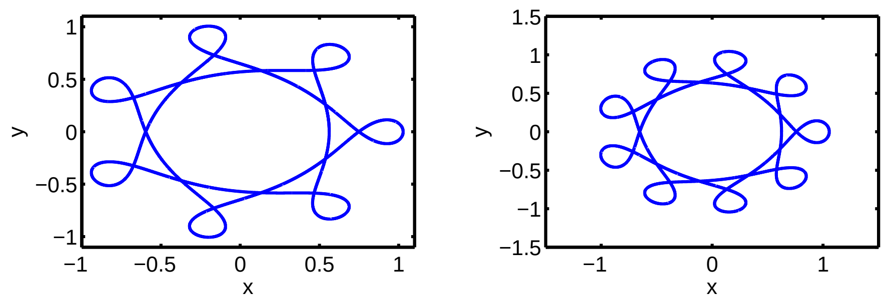
(f) Fourteen–loops periodic orbit at $x_0 = 0.63358$.

Figure 6.14: Family–I interior third order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Earth system.

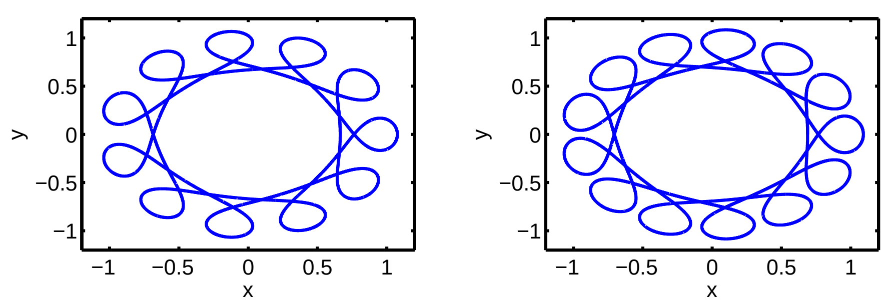


(a) PSS at $x = 0.56160$. *(b) Enlarged PSS at $x = 0.56160$.*

Figure 6.15: PSS of interior third order resonant seven loops orbits from Family–II for $q = 0.9845$, $A_2 = 0.0001$ and C = 2.93 for Sun–Earth system.



(a) Seven–loops orbit at $x_0 = 0.56160$. *(b) Nine–loops periodic orbit at $x_0 = 0.62620$.*



(c) Eleven–loops periodic orbit at $x_0 = 0.66415$. *(d) Thirteen–loops periodic orbit at $x_0 = 0.68886$.*

Figure 6.16: Family–II interior third order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ and C = 2.93 for Sun–Earth system

6.6 Interior resonance of fifth order

Table 6.11: Fifth order interior resonance $C = 2.93$ for Sun–Earth system.

FA	NL	LO	NI	RO	EC	TP	RP
I	11	0.36792	5	11:6	0.42357	38	1.96103
	12	0.41350		12:7	0.38286	44	1.82332
	13	0.45107		13:8	0.35190	51	1.72225
	14	0.48277		14:9	0.32751	57	1.64406
	16	0.53281		16:11	0.29211	70	1.53139
	17	0.55295		17:12	0.27893	76	1.48915
II	15	0.58150	5	15:10	0.26122	70	1.43180
	17	0.61344		17:12	0.24307	82	1.37065
	19	0.63733		19:14	0.23053	95	1.32660
	21	0.65640		21:16	0.22124	107	1.29228
	23	0.67095		23:18	0.21460	120	1.26649

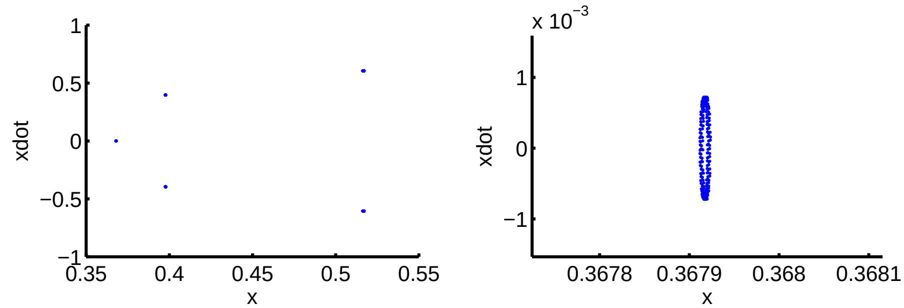
Table 6.12: Fifth order interior resonance $C = 2.93$ for Sun–Mars system.

FA	NL	LO	NI	RO	EC	TP	RP
I	11	0.36790	5	11:6	0.42359	38	1.96199
	12	0.41345		12:7	0.38289	44	1.82430
	13	0.45118		13:8	0.35180	51	1.72276
	14	0.48285		14:9	0.32744	57	1.64463
	16	0.53273		16:11	0.29216	70	1.53227
	17	0.55264		17:12	0.27912	76	1.49047
II	15	0.58152	5	15:10	0.26127	70	1.43243
	17	0.61335		17:12	0.24310	82	1.37146
	19	0.63740		19:14	0.23048	95	1.32710
	21	0.65623		21:16	0.22130	107	1.29319
	23	0.67119		23:18	0.21447	120	1.26667

Numerical estimates of different orbital elements of fifth order interior resonance with number of loops varying from 11 to 23 for the Sun–Earth and the Sun–Mars systems are shown in Table 6.11 and Table 6.12. They are divided in to two families; Family–I and Family–II. Seventeen–loops orbit from Family–I has period 76 where as Family–II has period 82. These orbits are 17 : 12 resonance orbits. So, for the given number of loops, and the given order of resonance, period of Family–II orbit is more than period of Family–I orbit. Orbits of Family–I are around the first primary only, whereas, orbits of Family–II are around the second primary, namely, Earth or Mars.

The PSS at $x_0 = 0.36792$ for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun–Earth system given in Fig. 6.17(a) contains five islands indicating fifth order of resonance. Fig. 6.17(b) is the magnified version of one of the islands of Fig. 6.17(a). Orbits with loops varying from 11, 12, 13, 14, 16 and 17 of Family–I with fifth order interior resonance for the Sun–Earth system for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ are shown in Fig. 6.18(a)–(f), respectively.

Periodic orbits with loops 15, 17, 19, 21 and 23 of Family–II for the Sun–Earth system are shown in Fig. 6.19(a)–(d), respectively.

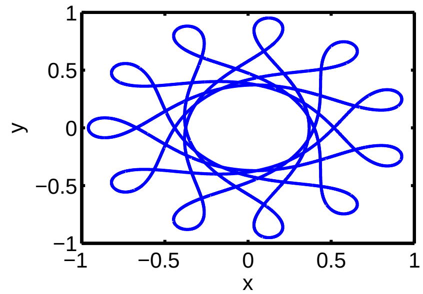


(a) PSS at $x_0 = 0.36792$. *(b) Enlarged PSS at $x_0 = 0.36792$.*

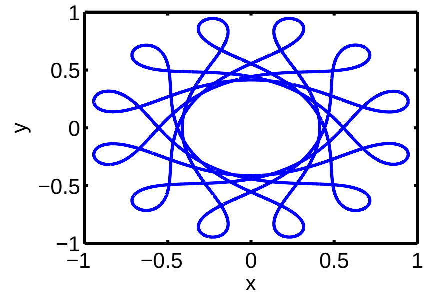
Figure 6.17: PSS of interior fifth order resonant eleven– loops orbits from Family–II for $q = 0.9845$, $A_2 = 0.0001$ and C = 2.93 in the Sun–Earth system.

The variation in the location of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun–Earth system for Family–I and Family–II is shown in Fig. 6.20 against the number of loops of the periodic orbits. It can be noticed that the location of the orbit shifts towards the second primary body as the number of loops increases. Family–I and II of third order resonance and Family–I of fifth order resonance contain periodic orbit having 11 and 13–loops. From location of these orbits as shown in Fig. 6.20, it is clear that for the given number of loops, as order of resonance increases location of periodic orbits moves towards the Sun.

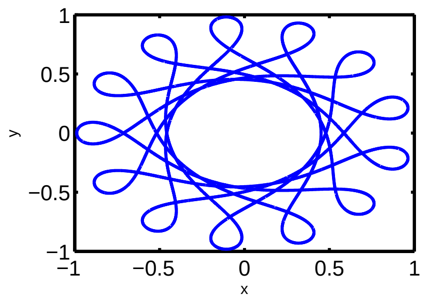
The variation in the eccentricity of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ for Sun–Earth system for Family–I and Family–II are shown in Fig. 6.21 against the number of loops of the periodic orbits. It can be noticed that the eccentricity of the orbit decreases as the number of loops increases. Also, eccentricity of the Family–I orbit is higher than the Family–II orbit for the given order of resonance. From eccentricity of 11 and 13–loops orbits as shown in



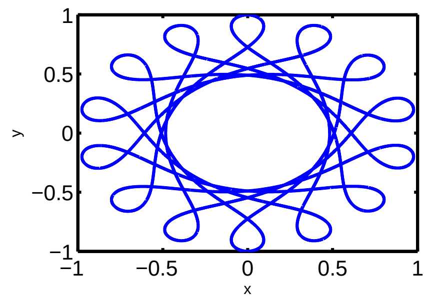
(a) Eleven–loops orbit at $x_0 = 0.36792$.



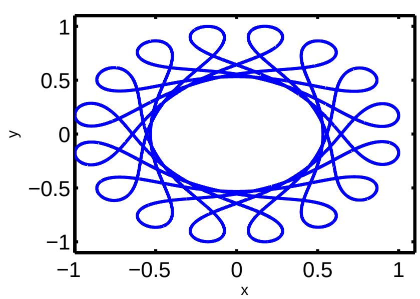
(b) Twelve–loops periodic orbit at $x_0 = 0.41350$.



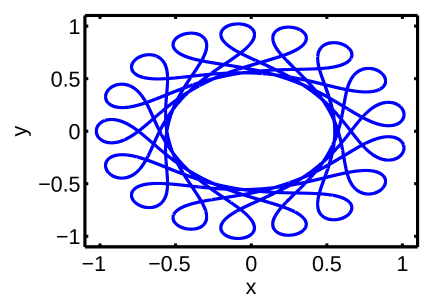
(c) Thirteen–loops periodic orbit at $x_0 = 0.45107$.



(d) Fourteen–loops periodic orbit at $x_0 = 0.48277$.

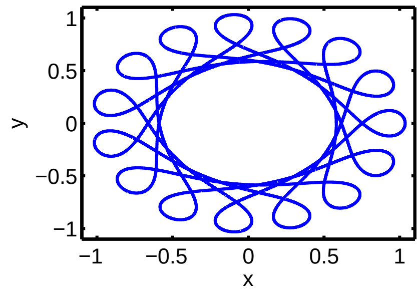


(e) Sixteen–loops periodic orbit at $x_0 = 0.53281$.

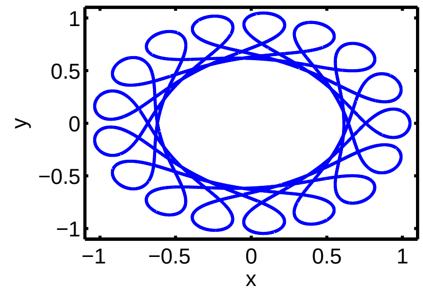


(f) Seventeen–loops periodic orbit at $x_0 = 0.55295$.

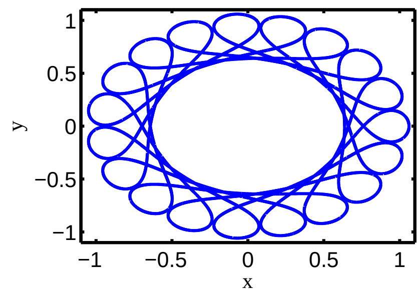
Figure 6.18: Family-I interior fifth order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Earth system.



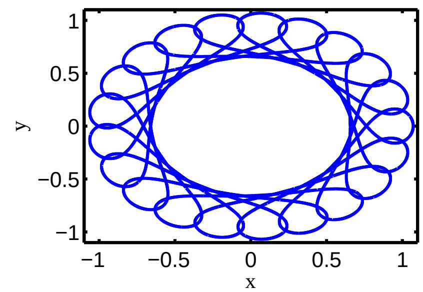
(a) Fifteen–loops orbit at $x_0 = 0.58150$.



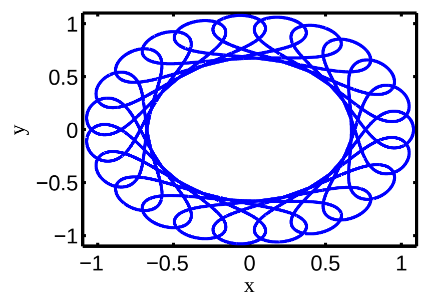
(b) Seventeen–loops periodic orbit at $x_0 = 0.61344$.



(c) Nineteen–loops periodic orbit at $x_0 = 0.63733$.



(d) Twenty one–loops periodic orbit at $x_0 = 0.65640$.



(e) Twenty three–loops periodic orbit at $x_0 = 0.67095$.

Figure 6.19: Family–II interior fifth order resonant orbits for $q = 0.9845$, $A_2 = 0.0001$ *and* C = *2.93 for Sun–Earth system.*

Fig. 6.21, it is clear that for a given number of loops, as order of resonance increases eccentricity increases.

The variation in the period of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ in the Sun–Earth system for Family–I and Family–II is shown in Fig. 6.22 against the number of loops of the periodic orbits. It can be noticed that the period of the orbit increases as the number of loops increases. Also, period of the Family–II orbit is higher than the Family–I orbit for the given order of resonance. From period of 11 and 13–loops orbits shown in Fig. 6.22, it is clear that, for the given number of loops, as order of resonance increases, the period decreases, which is obvious. In addition, Figs. (6.23 – 6.25) show similar results for the Sun–Mars system.

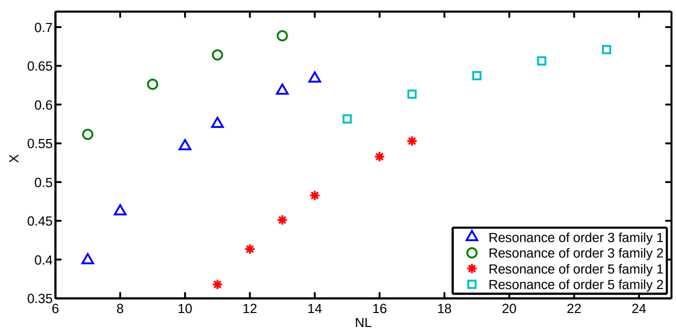


Figure 6.20: Variation in location of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Earth system.

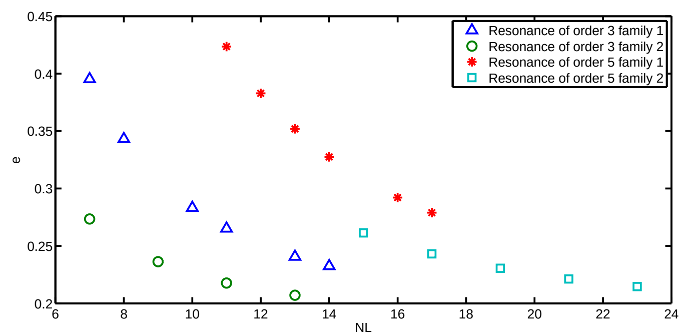


Figure 6.21: Variation in eccentricity of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Earth system.

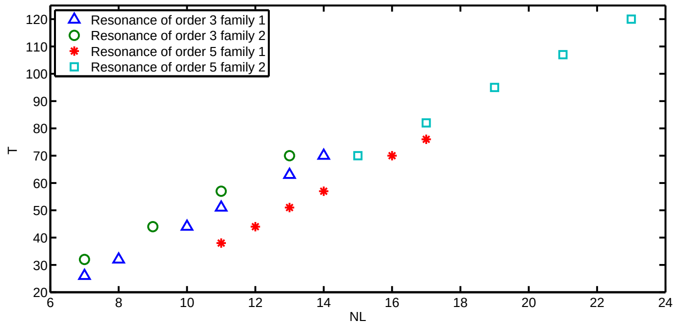


Figure 6.22: Variation in period of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Earth system.

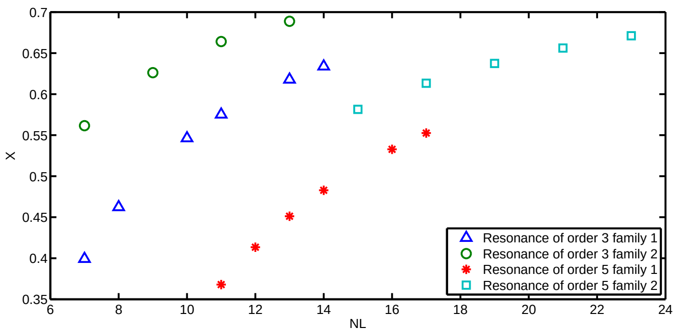


Figure 6.23: Variation in location of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Mars system.

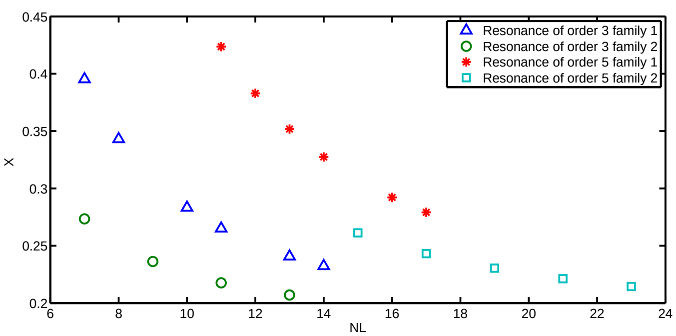


Figure 6.24: Variation in eccentricity of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ and $C = 2.93$ for Sun–Mars system.

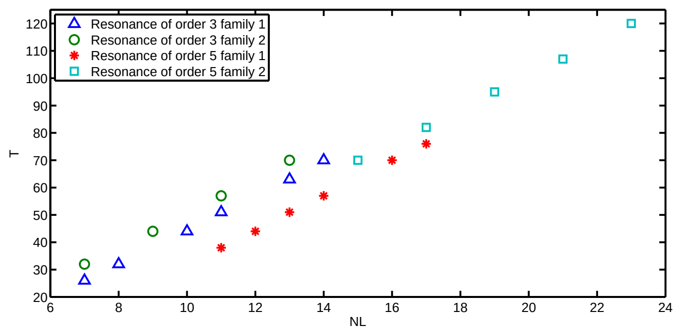


Figure 6.25: Variation in period of the interior third and interior fifth order resonant periodic orbits for $q = 0.9845$, $A_2 = 0.0001$ *and* C *= 2.93 Sun–Mars system.*

6.7 Conclusion

We have studied exterior and interior first, third and fifth order resonances in the photo–gravitational restricted three–body problem, by numerical methods for the Sun–Earth and the Sun–Mars systems considering the Sun as a radiating body and Earth and Mars as oblate spheroids. In this context, the first order exterior and interior resonant orbits, location, eccentricity and period of the orbits are analyzed with and without perturbation for $C = 2.93$. It is observed that for the given order of resonance, period of the orbit is increased by exactly 6 or 7 units as number of loops is increased by 1 because period of the Earth's orbit is 6.282714 and period of Mars's orbit is 6.282714 units.

It is concluded that for the external resonance as the number of loops increases location of the periodic orbit moves towards the Sun whereas for the internal resonance, as the number of loops increases, location of the periodic orbit moves away from the Sun. Also, location of exterior or interior first order resonant orbits moves away from the Sun whenever perturbation is included. While from location of orbits, we can notice that exterior resonant orbits with and without perturbation are nearer to the Earth whereas interior resonant orbits are nearer to the Sun. So, for the orbit having same number of loops, location of interior resonant orbit is nearer to the Sun in comparison to the exterior resonant orbit.

Eccentricity of the periodic orbit decreases as number of loops increases for both interior and exterior resonance in both perturbed and unperturbed cases. Also, for the orbit having same number of loops, eccentricity of interior resonant orbit is more in comparison to exterior resonant orbit. We also observe that for the given order of resonance as perturbation increases eccentricity of the periodic orbit decreases.

Furthermore, we study the evolution of three loops orbit for interior first order resonance by changing value of Jacobi constant C. As value of C increases, size of the loop reduces, and hence the shape of the orbit changes and finally it becomes circle. Thus, as C increases, eccentricity of the periodic orbit decreases and location of the periodic orbit moves towards the second primary body, namely, Earth or Mars.

As the number of loops increases, the location of third and fifth order resonant orbits for $C = 2.93$, $q = 0.9845$ and $A_2 = 0.0001$ shifts towards the second primary. Third and fifth order resonant orbits are divided in to two families. Orbits of Family–I are around the first primary only, whereas, orbits of Family–II are around both the primaries in which one of the loops of the orbit is around the second primary body, namely, Earth or Mars. It is concluded that for the given number of loops, as order of resonance increases location of periodic orbits moves towards the Sun. Also, eccentricity of the orbit decreases as the number of loops increases, and eccentricity of Family–I orbit is more than Family–II orbit for a given order of resonance.

It can be observed that for the given number of loops, as order of resonance increases eccentricity increases. Period of the first, third and fifth order resonant orbit increases as the number of loops increases. Also, period of Family–II orbit is more than the Family–I orbit for the given order of resonance. Further, we notice that for the given number of loops, as order of resonance increases period decreases, which is obvious.