Chapter 1

Introduction

1.1 Historical background

Dynamical system is one of the very useful, important and applicable branches of mathematics devoted to the study of systems governed by a consistent set of laws over time such as difference and differential equations. Beginning with the contributions of Poincaŕe and Lyapunov, theory of dynamical systems has seen significant developments in the recent years. This theory has gained considerable interest and has been found to have useful connections with many different areas of mathematics (such as number theory and topology) and science. The modern theory of dynamical systems was originated at the end of the 19th century while dealing with fundamental questions concerning the stability and evolution of the solar system. Attempts in answering such questions led to the development of this rich and powerful field with applications to physics, biology, meteorology, astronomy, economics, and other areas.

The emphasis of dynamical systems is the understanding of geometrical properties of orbits and their long term behavior. Dynamical systems can model an unbelievable range of behavior such as the motion of planets in the solar systems, the way diseases spread in a population, the shape and growth of plants, the interaction of optical pulses, or the processes that regulate electronic circuits and heart beats. For example in the study of the long term dynamics of planets around the sun, or stars in a galaxies, dynamical systems theory is useful in finding the mathematical answers to the fundamental questions like whether they exhibit regular and predictable behavior or would their motion eventually become chaotic and unpredictable or could planets in our solar system be ejected, or collide with each other etc.

Definition of a dynamical system includes three components: (i) Phase space, which is a set whose elements present possible states of the system at any moment of time. (ii) Time, which can be either discrete, whose set of values is the set of integers, $\mathbb{Z}$, or continuous, whose set of values is the set of real numbers $\mathbb{R}$ (when the integers are acting, the system is called a discrete dynamical system, and when the reals are acting, the system is called a continuous dynamical system). (iii) Law of evolution, which is the rule which allows, if one knows the state of the system at some moment of time, to determine the state of the system at any other moment of time.

Most of the natural systems in this world, be it the rhythm of day and night or the yearly seasons or weather patterns that vary from one year to another are subjected to time-dependent external forces and their modeling leads to a mathematical theory of what are called non-autonomous discrete dynamical systems. An autonomous discrete dynamical system is a dynamical system which has no external input and always evolves according to the same unchanging law. The theory of non-autonomous dynamical systems helps characterizing the behavior of various natural phenomenons which cannot be modeled by autonomous systems. The mathematical theory of non-autonomous systems is considerably more involved than the theory of autonomous systems. Over recent years, the theory of such systems has developed into a highly active field related to, yet recognizably distinct from that of classical autonomous dynamical systems. This development was motivated by problems of applied mathematics, in particular, in the life sciences where genuinely nonautonomous systems are in abundance. In general the dynamics of non-autonomous discrete dynamical system is much richer and quiet different from the dynamics of autonomous discrete dynamical systems.

The notion of expansive homeomorphism, or, homeomorphism for which two orbits cannot remain close to each other, was introduced by Utz [60] in 1950 with the term "Unstable homeomorphism". Since then an extensive literature about these homeomorphisms has been developed. One can refer [2] for basic results on such homeomorphisms. The work mainly concerns the study of properties of expansive homeomorphisms,their existence / non-existence on different metric spaces, their extension problems, their characterizations, their asymptotic properties, and so on. Especially interesting are expansive homeomorphisms of 0-dimensional spaces because they can be embedded in shifts, or, in other words, they are equivalent to subshifts. In 1955, Williams [74] first showed that there is a l-dimensional continuum admitting expansive homeomorphism. It has been proved that there are no expansive homeomorphisms of the compact interval, the circle and the compact 2-disc [30]. The same negative result was obtained independently by Hiraide [23] and Lewowicz [40] for the 2-sphere. In [43], Mané proved that a compact metric space exhibiting expansive homeomorphisms must be finite dimensional and, further, every minimal set of such homeomorphisms is zero dimensional. In 1993, Vieitez obtained that expansive homeomorphisms on closed 3-manifolds with dense topologically hyperbolic periodic points are both supported on the 3-torus and topologically conjugated to linear Anosov isomorphisms. This dynamical property on compact metric spaces has frequent applications in many fields including stability theory, symbolic dynamics, continuum theory and ergodic theory. This notion has been generalized to positive expansiveness [20], point-wise expansiveness [60] , entropy expansiveness [9], continuum-wise expansiveness [32], measure expansiveness [45] and n-expansiveness

[46].

Another important dynamical property is the pseudo-orbit tracing property (or shadowing property) which is closely related to stability and chaos of systems. The notion of pseudo-orbit goes back at least to Birkhoff [7], and plays a very significant role in the investigation of properties of discrete dynamical systems. In the study of dynamical systems, people often make computer simulations in which there are always no real trajectories of dynamical systems. Bowen [10] and Conley have independently discovered that pseudo-orbit could be used as a conceptual tool for discussing the relationship between the computer output and the underlying dynamics of the dynamical systems. It is important to find out in which cases a numerically obtained pseudo-orbit can be shadowed (traced) by a real trajectory. This problem has been well studied in the last several decades, for example, the shadowing near a hyperbolic set of a homeomorphism [65] and the shadowing in structurally stable systems [57]. One can refer [58] for detailed study of maps with this property. In recent years theory of shadowing has become a significant part of qualitative theory of dynamical systems containing a lot of interesting and deep results. It plays an important role in the investigation of the stability theory. Shadowing property has also been used to give global error estimates for numerically computed orbits of dynamical systems and to rigorously prove the existence of periodic orbits and chaotic behavior. Several problems including properties of maps possessing shadowing property and its relation with other dynamical properties have been studied in detail. Moreover, one of the basic problems studied in the theory of shadowing is finding class of maps possessing or not possessing shadowing property [14]. Various kinds of shadowing also have been defined and their equivalences have been studied in [39, 58].

In [72], Walters has introduced the concept of topological stability and proved that Anosov diffiomorphisms are topologically stable.

Expansiveness and shadowing play an important role in the study of topological stability of maps on a compact metric space [73]. In [51], Nitecki has shown that topological stability is a necessary condition to get axiom A together with strong transversality. Morse-Smale flows are topologically stable is proved by Robinson in [60]. In [28, 26, 27], Hurley has obtained necessary conditions for topological stability. Moriyasu [47] has proved that the C^1-interior of the set of all topologically stable diffiomorphisms is characterized as the set of all C^1-structurally stable diffiomorphisms. In [48], authors have proved that, if X^t is a flow in the C^1-interior of the set of topologically stable flows, then X^t satisfies the Axiom A and the strong transversality condition. In [5], authors have proved similar result for the class of incompressible flows and also for volume-preserving diffiomorphisms. In [8, 5], authors have generalized results of [48, 6] for symplectomorphisms. Recently in [18], authors have studied expansiveness, shadowing, topological stability and decomposition theorems for homeomorphisms on non-compact and non-metrizable spaces.

In the study of dynamics of a map f from a compact metric space X to itself, central role is played by the various recursive properties of points of X. One of important such properties is nonwanderingness. Nonwandering points play an important role in study of autonomous and nonautonomous discrete dynamical systems. For study of nonwandering points for autonomous discrete dynamical system, one can refer [2]. In [36], authors have studied nonwandering set from the view-point of topological entropy.

The notion of chain recurrence, introduced by Conley [16], is a way of getting at the recurrence properties of an autonomous system. Chain recurrent sets play an important role in the study of qualitative behaviors of dynamical systems. Conley discovered fundamental connection between the chain recurrent set and the collection of attractors for a deterministic dynamical system on a compact metric space: the complement of the chain recurrent set in the whole state space X is the union of the complements of attractors A in their own basins of attraction. This result is called the Conley's theorem which approaches the Fundamental Theorem of dynamical systems [52] and the Fundamental Theorem can be applied to such as the bifurcation theorem [31]. In [22], author has studied chain recurrence and asymptotic shadowing in autonomous dynamical system. For more results on chain recurrent sets in autonomous dynamical systems one can refer [2, 21, 35, 54, 53, 56]. In [29], author has studied non-compact chain recurrence and attraction in discrete dynamical systems.

The Smale's spectral decomposition theorem was first proved for Anosov diffiomorphism of compact manifolds [62]. The topological version of Smale's decomposition theorem in classical autonomous dynamical system was first proved by Conley [16, 2]. In [76], Yang extended this result to noncompact metric spaces with the additional strong requirement that the chain recurrent set be compact. In [54], P. Oprocha proved this theorem for multidimensional discrete dynamical system. In [35], authors have studied spectral decomposition theorem of k-type nonwandering sets for $\mathbb{Z}^2$-actions. In [18], T. Das et al. have proved this result for non-compact and non-metrizable spaces. In [15], authors have studied Conley's state space decomposition theorem for nonautonomous dynamical systems.

In discrete dynamical system (X, f), where X is a metric space and $f\colon X \to X$ is a continuous map, we consider the iterates of points of X under the action of f with discrete ticks of time. Let us consider the case when the function f is changing with the ticks of time i.e. we consider the action of sequence of functions $\{f_n\}_{n=0}^{\infty}$, with f_0 to be the identity map. The action by $F = \{f_n\}_{n=0}^{\infty}$ is called a time varying map and (X, F) is said to be nonautonomous discrete dynamical system. For example, any moving picture on a television screen is an example of nonautonomous discrete dynamical system. In fact television screen is divided into pixels each of a single color red,blue or green. Also if (X, σ) is a shift-space and $\{t_n\}$ is a sequence of integers then $\{\sigma^{t_n}\}$ is a time varying map on X. The notion of nonautonomous discrete dynamical system was introduced by Kolyada and Snoha in [36]. Since then lots of study has been done regarding dynamical properties in nonautonomous discrete dynamical systems. In [36], authors have defined and studied the notion of topological entropy, in [37], authors have discussed minimality, in [33, 11], authors have studied ω-limit sets, in [38], author has discussed stability and controllability, in [24, 25], authors have studied topological pressure and pre-image entropy, in [4, 13, 12, 19, 55, 61, 63, 64, 75], authors have studied chaos in non-autonomous discrete dynamical systems.

In [41], authors have studied ω-limit sets and attraction, in [4, 49], authors have studied weak mixing and chaos, in [77], author has studied recurrence properties of a class of nonautonomous discrete dynamical systems, in [12, 77], authors have studied structures of nonwandering sets in nonautonomous discrete dynamical systems. Similar kind of study related to random perturbations of dynamical systems has been done by Araúјo in [3]. In [17], author has studied G-chaos of a sequence of maps in a metric G-space. For more recent results in nonautonomous discrete dynamical systems one can refer to [42, 44].

1.2 Preliminaries

By **a discrete dynamical system** we mean a pair (X, f), where X is a compact metric space and f is a continuous a self map (or a self homeomorphism) of X. For x in X, the **orbit** of x is the set $\{f^n(x) : n \geq 0\}$ (and in case of homeomorphism $\{f^n(x) : n \in \mathbb{Z}\}$.)

we recall following definitions :

Definition 1.2.1 *Let (X, d) be a metric space and $f : X \to X$ be a continuous function. A point $x \in X$ is said to be* ***a periodic point*** *if its orbit is* *finite i.e. for some $m > 0$, $f^m(x) = x$. A point $x \in X$ is said to be a* ***fixed point*** *if $f(x) = x$. The set of all periodic points for f is denoted by $Per(f)$ and set of all fixed points for f is denoted by $Fix(f)$.*

Following is the definition of expansiveness for a homeomorphism in a discrete dynamical system.

Definition 1.2.2 *Let (X, d) be a metric space and $f : X \to X$ be a homeomorphism. Then f is said to be* ***expansive*** *if there exists a constant $c > 0$ (called* ***an expansive constant****) such that for any $x, y \in X$, $x \neq y$, there exists an $n \in \mathbb{Z}$ (depending on the pair (x, y)) satisfying*

$$d(f^n(x), f^n(y)) > c.$$

Conjugacy or Uniform conjugacy between two continuous maps is given below :

Definition 1.2.3 *Let (X, d_1) and (Y, d_2) be two metric spaces, f and g be continuous self maps on X and Y respectively. If there is a homeomorphism $h : X \to Y$ such that*

$$h \circ f = g \circ h$$

then f and g are said to be **conjugate with respect to the map h** *or* **h-conjugate**. *In particular, if $h\colon X \to Y$ is a uniform homeomorphism then f and g are said to be* **uniformly conjugate with respect to the map h** *or* **uniformly h-conjugate**. *(Note that a homeomorphism $h\colon X \to Y$ such that h and h^{-1} are uniformly continuous, is called a uniform homeomorphism. If X is compact then conjugacy and uniform conjugacy are same.)*

A dynamical property of a continuous map is that property which is invariant under topological conjugacy.

Theorem 1.2.1 *Let X be a compact metric space and let f be an expansive homeomorphism on X. If f is topologically conjugate to g on a metric space Y then g is also expansive.*

The composition of two expansive homeomorphisms need not be expansive even if the underlying space is a compact metric space. However, the following result for compact metric spaces concerning the composition of an expansive homeomorphism with itself is proved by Utz in [71].

Theorem 1.2.2 *Let X be a compact metric space and let f be an expansive homeomorphism on X. Then for each integer $m \neq 0$, f^m is expansive on X.*

Concerning the restrictions and product of expansive homeomorphisms, it is easy to see that the restriction of an expansive homeomorphism h on a metric space X to a subspace Y of X is expansive if $h(Y) = Y$; and, if h and g are expansive homeomorphisms on metric spaces X and Y respectively then so is the homeomorphism $h \times g$ on the product space $X \times Y$. The later property extends to any finite product but not to infinite product [2].

If X is a compact metric space then the set of all expansive constants for an expansive homeomorphism h on X is a bounded subset of real numbers and hence has a least upper bound. The question whether this least upper bound is an expansive constant for h was answered in negation by Bryant in [8]. His result follows.

Theorem 1.2.3 *If X is a compact metric space and θ is the least upper bound of the expansive constants for an expansive homeomorphism h on X then θ is not an expansive constant for h.*

As a consequence of the facts that the real line $\mathbb{R}$ does carry expansive homeomorphisms but there does not exist an expansive homeomorphism on the open unit interval $(0, 1)$, one observes that possessing an expansive homeomorphism is not a topological property for metric spaces. In this connection, Bryant [8] proves the following theorem giving sufficiency condition for preserving expansiveness under a homeomorphism.

Theorem 1.2.4 *Let f be an expansive homeomorphism on a metric space X and let g be a homeomorphism from X onto a metric space Y. If g^{-1} is uniformly continuous, then gfg^{-1} is an expansive homeomorphism on Y.*

One can also see from the same facts that an expansive homeomorphism on a metric space need not remain expansive under an equivalent metric; however, for a compact metric space expansiveness of a homeomorphism is independent of the choice of a metric as far as metric generates the same topology.

A characterization of expansive homeomorphisms on a compact metric space is obtained by Keynes and Robertson [34] in terms of topological analogue of generators of measure preserving transformations - the concept defined by them in the same paper. The definition of this concept of topological generators is as follows.

Definition 1.2.4 *Given a compact Hausdorff space X and a homeomorphism h on X, a finite open cover α of X is called a generator (respectively weak generator)for (X, f) if for each bisequence $\{A_i\}_{i=-\infty}^{\infty}$ of members of α, $\bigcap_{i=-\infty}^{\infty} h^{-i}\bar{A}_i$ (respectively $\bigcap_{i=-\infty}^{\infty} h^{-i}A_i$) contains at most one point.*

The Keynes-Robertson characterization of expansive homeomorphism on a compact metric space is as follows.

Theorem 1.2.5 *Let $f$be a homeomorphism on a compact metric space$X$. Then $f$is expansive if and only if (X, f) has a generator if and only if (X, f) has a weak generator.*

Following is the definition of shadowing or pseudo orbit tracing property (P.O.T.P.) for a continuous self map on a metric space:

Definition 1.2.5 *Let (X, d) be a metric space and f be a continuous self map of X. For $\delta > 0$, the sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be* **a δ-pseudo orbit** *of f if*

$$d(f^{n+1}(x_n), x_{n+1}) < \delta,$$

for all $n = 0, 1, 2, \ldots$. *For given* $\varepsilon > 0$, *a* δ*-pseudo orbit* $\{x_n\}_{n=0}^{\infty}$ *is said to be* ε**-traced** *by* $y \in X$ *if*

$$d(f^n(y), x_n) < \varepsilon,$$

for all $n = 0, 1, 2, \ldots$.

The map f *is said to have* **shadowing property or pseudo orbit tracing property** *(P.O.T.P) if for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that every* δ*-pseudo orbit is* ε*- traced by some point of X.*

If X is a compact metric space then the shadowing property of $f : X \to X$ is independent of the choice of metric d compatible with the topology of X.

In case of a homeomorphism, we have the following:

Theorem 1.2.6 *Let X be a compact metric space. If* $f : X \to X$ *is a homeomorphism with the shadowing property then* f^{-1} *also has the shadowing property.*

Following theorem gives the condition under which the shadowing property is preserved under the conjugacy.

Theorem 1.2.7 *Let* X, Y *be compact metric spaces,* $f : X \to X$ *be a continuous map and if* $g : Y \to Y$ *is conjugate to* f *then* g *also has the shadowing property.*

The following result shows that the shadowing property is preserved by product of two maps having shadowing property and vice versa.

Theorem 1.2.8 *Let* (X, d_1) *and* (Y, d_2) *be metric spaces and* $X \times Y$ *be the product space with metric* $d((x, y), (x', y')) = max\{d_1(x, x'), d_2(y, y')\}$. *Let* $f : X \to X$ *and* $g : Y \to Y$ *be continuous maps and let* $f \times g$ *be the map defined by*

$$(f \times g)(x, y) = (f(x), g(y)), (x, y) \in X \times Y.$$

Then $f \times g$ *has the shadowing property if and only if both* f *and* g *have the shadowing property.*

Topological stability in (autonomous)discrete dynamical system is defined as follows :

Definition 1.2.6 *Let X be a compact metric space with metric d. A homeomorphism $f : X \to X$ is said to be* **topologically stable in the class of homeomorphisms** *if for each $\epsilon > 0$, there exists $\delta > 0$ such that for a homeomorphism g with $d(f(x), g(x)) < \delta$ for all x in X, there exists a continuous map $h : X \to X$ such that $h \circ g = f \circ h$ and $d(h(x), x) < \epsilon$ for all x in X.*

Nonwandering point in an autonomous discrete dynamical system is defined as follows :

Definition 1.2.7 *Let (X, d) be a metric space and f be a self homeomorphism on X. A point $x \in X$ is said to be* **a nonwandering point for f** *if for any neighborhood U of x there is an $n \in \mathbb{Z}$ such that*

$$f^n(U) \cap U \neq \emptyset.$$

The set of all nonwandering points for f is called **nonwandering set of f** *and is denoted by $\Omega(f)$.*

Following is the definition of transitivity for a continuous self map on a space X.

Definition 1.2.8 *Let X be a topological space. A continuous self map f on X is said to be* **transitive on X** *if for any two nonempty open sets U and V of X there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$.*

Note 1.1 *If f is transitive on X then every point of X is a nonwandering point for f.*

Definition 1.2.9 *Let (X, d) be a metric space and f be a self homeomorphism on X. A finite δ-pseudo orbit from x to x is called* **a δ-chain for x**. *A point x is said to be* **chain recurrent** *if for every $\delta > 0$, there is a δ-chain for x.*

Let $f : X \to X$ be a homeomorphism where (X, ρ) be a compact metric space. Consider the set $\Omega(f)$ of all nonwandering points, the set $\mathrm{Per}(f)$ of all periodic points, the set $CR(f)$ of all chain recurrent point. We recall following results proved in [2].

1. $\Omega(f)$ is closed and invariant set.
2. $\mathrm{Per}(f) \subseteq \Omega(f)$.
3. If f is transitive then $\Omega(f) = X$.
4. $CR(f)$ is closed and invariant.
5. $\mathrm{Per}(f) \subseteq CR(f)$.
6. $CR(f|_{CR(f)}) = CR(f)$.
7. If f is expansive and $f|_{CR(f)}$ has P.O.T.P. then $CR(f)$ is isolated.

One of the fundamental results in the theory of dynamical systems is decomposition theorem proved by S. Smale in [62]. The Topological version of Smale's Spectral decomposition theorem is as follows:

Theorem 1.2.9 *Let f be a self homeomorphism of a compact metric space X such that $f|_{CR(f)}$ is expansive and has P.O.T.P. then $CR(f)$ contains a finite sequence $B_i (1 \leq i \leq l)$ of closed invariant sets such that $CR(f)$ is a disjoint union of $B_1, B_2, ..., B_l$ and $f|_{B_i}$ is topologically transitive.*

1.3 Overview of the work done in the thesis

Let (X, d) be a metric space and

$$\{f_n : X \to X : n = 0, 1, 2, \dots\}$$

be a sequence of maps. If each f_n is continuous (respectively a homeomorphism), we call $F = \{f_n\}_{n=0}^{\infty}$ to be a **time varying map** (respectively a **time varying homeomorphism**) on X and (X, F) a

nonautonomous discrete dynamical system (respectively **invertible nonautonomous discrete dynamical system**). If each $f_n = f$, where f is a self homeomorphism of X, then (X, F) is equivalent to autonomous discrete dynamical system (X, f).

There are six chapters in this thesis. The present chapter gives historical background, preliminaries and overview of the work done in the thesis.

In the second chapter, we extend the notion of expansiveness in nonautonomous discrete dynamical systems. In the first section, we define expansiveness of a time varying map and in the second section, we define expansiveness of a time varying homeomorphism on a metric space. For both the cases, we give examples and study results related to their conjugacy invariance, composition, product and other related properties. In the third section, we define notions of generator and weak generator for a time varying homeomorphism on a compact metric space and obtain a characterization of expansiveness in terms of generator and weak generator. Results of this chapter are published in [67] and [68].

In the third chapter, we define and study shadowing property and topological stability in nonautonomous discrete dynamical systems. In the first section, we define shadowing property for a time varying map and in the second section for a time varying homeomorphism on a metric space. We prove that on a compact metric space shadowing property is independent of the choice of metric (for both the cases). We have shown that shadowing property is a property which is preserved under uniform conjugacy. We have also obtained results regarding product, composition etc. In the third section, we study topological stability in nonautonomous discrete dynamical systems given by sequence of maps as well as given by a sequence of homeomorphisms. We have proved that on a compact metric space a time varying map(respectively homeomorphism) which is expansive and has shadowing property is topologically stable in the class of continuous self maps (respectively homeomorphisms). Results of this chapter are published in [67] and [68].

In the fourth chapter, we define and study nonwandering set, α-limit set of a point, ω-limit set of a point and recurrent set for an invertible nonautonomous discrete dynamical system. Our definition of nonwandering point is different from those given in [36] and [77]. In the first section, we study properties of nonwandering sets. We prove that the set of all nonwandering points for an invertible nonautonomous discrete dynamical system is a nonempty, closed set containing the set of all periodic points. In the second section, we study transitivity and strong transitivity in invertible nonautonomous discrete dynamical systems. We prove that strong transitivity implies transitivity and justify that converse is not true by providing an example. We also prove that for a transitive system, the nonwandering set is the whole space. In the third section, we define α-limit set, ω-limit set of a point $x \in X$ and recurrent point in an invertible nonautonomous system. We prove that for a time varying homeomorphism F on a compact metric space X, the set of α-limit points of x and the set of ω-limit points of x are contained in the set of nonwandering points of F. We give an example justifying that the set of all periodic points may be a proper subset of the set of recurrent points of F. Results of this chapter are published in [66].

In the fifth chapter, we define, give examples and study chain recurrent set in an invertible nonautonomous discrete dynamical system. In the first section, we prove that for an invertible nonautonomous discrete dynamical system on a compact metric space under certain conditions, the set of all chain recurrent points is a closed set containing the set of all nonwandering points. We give an example justifying that the set of nowandering points may be a proper subset of the set of all chain recurrent points of F. In the second section, we deal with the problem of reverse containment.

We prove that for a time varying homeomorphism F, having shadowing property, the set of all chain recurrent points is contained in the set of all nonwandering points of F. In the third section, we define and study the notion of a weak isolated set for an invertible nonautonomous discrete dynamical system. We obtain sufficient condition under which the set of all chain recurrent points of F is a weak isolated set. Results of this chapter are published in [69].

In the sixth chapter, we prove a decomposition theorem similar to Smale's spectral decomposition theorem in an equicontinuous invertible nonautonomous discrete dynamical system F. In the first section, we define weak chain recurrence and study properties of weak chain recurrent sets. We prove that the set of all chain recurrent points is contained in the set of all weak chain recurrent points of F. In the second section, we define the weak pseudo orbit extending property and show that for an invertible nonautonomous discrete dynamical system having this property, the set of all chain recurrent points coincides with the set of all weak chain recurrent points. We define an equivalence relation on the set of all weak chain recurrent points and show that if X is compact and F is equicontinuous then each equivalence class is a clopen set. Finally in the third section, we prove a decomposition theorem similar to Smale's spectral decomposition theorem in an equicontinuous invertible nonautonomous discrete dynamical system. Results of above three sections are published in [70]. In the fourth section, we discuss some open problems which give us a scope of further research in this area.