Chapter 2

Expansiveness in Nonautonomous Discrete Dynamical Systems

In this chapter, we define the notion of expansiveness in nonautonomous discrete dynamical systems given by a sequence of maps and also by a sequence of homeomorphisms on a metric space. We give examples, study properties and obtain a characterization of expansiveness in nonautonomous discrete dynamical systems.

2.1 Expansiveness in Nonautonomous Discrete Dynamical Systems

Let (X, d) be a metric space and

$$\{f_n : X \to X : n = 0, 1, 2, \ldots\}$$

be a sequence of continuous maps with f_0 as the identity map on X. We call $F = \{f_n\}_{n=0}^{\infty}$ to be **a time varying map** on X and (X, F) to be **a**

nonautonomous discrete dynamical system. We denote

$$F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0, \ \ for \ \ all \ \ n = 0, 1, 2, \ldots$$

and define

$$F_{[i,j]} = \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i; & 0 \leq i \leq j \\ the \ \ identity \ \ map \ \ on \ \ X; & i > j. \end{cases}$$

For any $k > 0$, k^{th}-iterate of F is defined to be a time varying map $F^k = \{g_n\}_{n=0}^{\infty}$ on X, where

$$g_n = f_{nk} \circ f_{(n-1)k+k-1} \circ \cdots \circ f_{(n-1)k+2} \circ f_{(n-1)k+1} \ \ for \ \ all \ \ n \geq 0.$$

Thus $F^k = \{F_{[(n-1)k+1,nk]}\}_{n=0}^{\infty}$.

Following are the definitions of orbit, periodic point and fixed point in a nonautonomous discrete dynamical system induced by a sequence of continuous maps.

Definition 2.1.1 *[64] Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of continuous maps, $n = 0, 1, 2, \ldots$. For a point $x_0 \in X$, define a sequence as follows :*

$$\begin{aligned} x_{n+1} &= f_{n+1}(x_n), \\ &= F_{n+1}(x_0), \qquad n = 0, 1, 2, \ldots. \end{aligned}$$

Then the sequence $O(x_0) = \{x_n\}_{n=0}^{\infty}$ is said to be **the orbit of $\mathbf{x_0}$** *under time varying map $F = \{f_n\}_{n=0}^{\infty}$.*

Definition 2.1.2 *[64] Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of continuous maps, $n = 0, 1, 2, \ldots$. A point $x_0 \in X$ is said to be* **a periodic point of time varying map F = $\{\mathbf{f_n}\}_{\mathbf{n=0}}^{\infty}$** *if $O(x_0) = \{x_n\}_{n=0}^{\infty}$ is periodic i.e. there exists an integer $k > 0$ such that*

$$x_{i+k} = x_i, \ \ for \ \ all \ \ i = 0, 1, 2, \ldots$$

Hence $x_{ik+j} = x_j$ i.e. $F_{ik+j}(x_0) = F_j(x_0)$, for every $i \geq 0$ and $0 \leq j < k$. The set of all periodic points of F is denoted by $Per(F)$.

Definition 2.1.3 *[64] Let* (X, d) *be a metric space and* $f_n : X \to X$ *be a sequence of continuous maps,* $n = 0, 1, 2, ...$ *A point* $x \in X$ *is said to be* **a fixed point of time varying map** $\mathbf{F} = \{\mathbf{f_n}\}_{\mathbf{n=0}}^{\infty}$ *if* $f_n(x) = x$*, for all* $n = 0, 1, 2, \ldots$.

Now we define expansiveness of a nonautonomous discrete dynamical system induced by a sequence of continuous maps.

Definition 2.1.4 *Let* (X, d) *be a metric space and* $f_n : X \to X$ *be a sequence of continuous maps,* $n = 0, 1, 2, ...$ *The time varying map* $F = \{f_n\}_{n=0}^{\infty}$ *is said to be* **expansive** *if there exists a constant* $c > 0$ *(called* **an expansive constant***) such that for any* $x, y \in X$, $x \neq y$,

$$d(F_n(x), F_n(y)) > c \quad for \quad some \quad n \geq 0.$$

Equivalently, if for $x, y \in X$,

$$d(F_n(x), F_n(y)) \leq c \quad for \quad all \quad n \geq 0 \quad then \quad x = y.$$

Remark 2.1 *If in the above definition* $f_n = f$*, for all* $n \geq 0$*, where* $f : X \to X$ *is continuous, then expansiveness of time varying map* $F = \{f_n\}_{n=0}^{\infty}$ *on* X *is equivalent to positive-expansiveness of* f *on* X *([20]), as*

$$F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1 = f \circ f \circ \cdots \circ f \circ f = f^n.$$

Remark 2.2 *Note that expansiveness of a time varying map* F *is independent of choice of metric if* X *is compact. For metric space* (X, d)*, let* $N_d(x, \delta) = \{y \in X : d(x, y) < \delta\}$*. Let* d_1 *and* d_2 *be two equivalent metrics on a compact metric space* X*. Suppose* F *is expansive on* (X, d_1) *with expansive constant* $\varepsilon > 0$*. Since* d_1 *is equivalent to* d_2*, there exists an* $\varepsilon_1 > 0$ *such that for any* $x \in X$,

$$N_{d_2}(x, \varepsilon_1) \subset N_{d_1}(x, \varepsilon),$$

where $N_{d_i}(z, \delta)$ *denotes the open ball centred at* z *in* X *of radius* δ *under metric* d_i, $i = 1, 2$*. Since* X *is compact,* ε_1 *depends only on* ε *and not on* x*. Let* $x \neq y$*. Since* F *is expansive in* (X, d_1) *with expansive constant* $\varepsilon > 0$,

$$F_n(y) \notin N_{d_1}(F_n(x), \varepsilon)$$

for some $n \geq 0$. *Now since*

$$N_{d_2}(F_n(x), \varepsilon_1) \subset N_{d_1}(F_n(x), \varepsilon),$$

we have

$$F_n(y) \notin N_{d_2}(F_n(x), \varepsilon_1).$$

Thus F *is expansive on* (X, d_2) *with expansive constant* ε_1.

Following is an example of an expansive time varying map.

Example 2.1 *Consider the time varying map* $F = \{f_n\}_{n=0}^{\infty}$ *on the real line* $\mathbb{R}$ *defined by* $f_n(x) = (n+1)x$, *for* $x \in \mathbb{R}$ *and* $n \geq 0$.

Choose $c > 0$. Then for $x, y \in \mathbb{R},\ x \neq y$, there exists $n \geq 0$ such that

$$|F_n(x) - F_n(y)| = (n+1)!|x - y| > c.$$

Thus F is expansive with expansive constant c.

Definition 2.1.5 *If* $h\colon X \to Y$ *is a homeomorphism,* h *is uniformly continuous on* X *and* h^{-1} *is uniformly continuous on* Y, *then* h *is said to be* **a uniform homeomorphism**.

Now we define uniform conjugacy between two nonautonomous discrete dynamical systems induced by a sequence of maps.

Definition 2.1.6 *Let* (X, d_1) *and* (Y, d_2) *be two metric spaces. Let* $F = \{f_n\}_{n=0}^{\infty}$ *and* $G = \{g_n\}_{n=0}^{\infty}$ *be time varying maps on* X *and* Y *respectively. If there is a homeomorphism* $h : X \to Y$ *such that*

$$h \circ f_n = g_n \circ h,$$

for all $n = 0, 1, 2, \ldots$ *then* F *and* G *are said to be* **conjugate** *with respect to the map* h *or* **h-conjugate**. *In particular, if* $h\colon X \to Y$ *is a uniform homeomorphism then* F *and* G *are said to be* **uniformly conjugate** *or* **uniformly h-conjugate**.

For example, if $F = \{x^{n+1}\}_{n=0}^{\infty}$ on $[0,1]$, $G = \{2((x+1)/2)^{n+1} - 1\}_{n=0}^{\infty}$ on $[-1,1]$ then F is uniformly h-conjugate to G, where $h : [0,1] \to [-1,1]$ is defined by $h(x) = 2x - 1$.

Next we show that expansiveness is a property in a nonautonomous discrete dynamical system which is preserved under uniform conjugacy.

Theorem 2.1.1 *Let (X, d_1) and (Y, d_2) be metric spaces. Let $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$ be time varying maps on X and Y respectively such that F is uniformly conjugate to G. Then F is expansive on X if and only if G is expansive on Y.*

Proof : Since F is uniformly conjugate to G, there exists a uniform homeomorphism $h : X \to Y$ such that

$$h \circ f_n = g_n \circ h,$$

for all $n \geq 0$ i.e.

$$f_n \circ h^{-1} = h^{-1} \circ g_n,$$

for all $n \geq 0$ which implies

$$\begin{aligned}
F_n \circ h^{-1} &= f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ f_0 \circ h^{-1} \\
&= f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ h^{-1} \circ g_0 \\
&\ \ \vdots \\
&= h^{-1} \circ g_n \circ g_{n-1} \circ \cdots g_2 \circ g_1 \circ g_0 \\
&= h^{-1} \circ G_n,
\end{aligned}$$

for all $n \geq 0$. Similarly $h \circ f_n = g_n \circ h$, $\forall n \geq 0$ will imply that for all $n \geq 0$,

$$h \circ F_n = G_n \circ h.$$

Let F be expansive with an expansive constant $\varepsilon > 0$. Now, h being a uniform homeomorphism, h^{-1} is uniformly continuous therefore for $\varepsilon > 0$ there exists a $\delta > 0$ such that for $y_1, y_2 \in Y$,

$$d_2(y_1, y_2) < \delta \ \ \textit{implies} \ \ d_1(h^{-1}(y_1), h^{-1}(y_2)) < \varepsilon.$$

Let $y_1, y_2 \in Y$. Suppose that for all $n \geq 0$, $d_2(G_n(y_1), G_n(y_2)) < \delta$. Then

$$d_1(h^{-1}(G_n(y_1)), h^{-1}(G_n(y_2))) < \varepsilon,$$

for all $n \geq 0$ i.e.

$$d_1(F_n(h^{-1}(y_1)), F_n(h^{-1}(y_2))) < \varepsilon,$$

for all $n \geq 0$. Since F is expansive with expansive constant ε, we get $h^{-1}(y_1) = h^{-1}(y_2)$ which implies $y_1 = y_2$. Thus G is expansive with expansive constant δ.
Conversely, suppose G is expansive with expansive constant $\varepsilon_1 > 0$. Since h is continuous, there exists $\delta_1 > 0$ such that for any $x_1, x_2 \in X$,

$$d_1(x_1, x_2) < \delta_1 \quad \textit{implies} \quad d_2(h(x_1), h(x_2)) < \varepsilon_1.$$

For any $x_1, x_2 \in X$ with $x_1 \neq x_2$, $h(x_1) \neq h(x_2)$, it follows that there exists $n \in \mathbb{Z}$ such that

$$d_2(h(F_n(x_1)), h(F_n(x_2))) = d_2(G_n(h(x_1)), G_n(h(x_1))) > \epsilon_1.$$

which implies $d_1(F_n(x_1), F_n(x_2)) \geq \delta_1$. Thus F is expansive on X.

Corollary 2.1.1 *Let (X, d_1) be a compact metric space and (Y, d_2) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on X and $h \colon X \to Y$ is a homeomorphism. If F is expansive on X then $G = h \circ F \circ h^{-1} = \{g_n\}_{n=0}^{\infty}$, where $g_n = h \circ f_n \circ h^{-1}$; $n = 0, 1, 2, \ldots$ is expansive on Y.*

Definition 2.1.7 *Let (X, d_1) and (Y, d_2) be metric spaces. A family of functions $\{f_n : X \to Y\}_{n=0}^{\infty}$ is said to be equicontinuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on the point x_0) such that $d_2(f_n(x_0), f_n(x)) < \varepsilon$ for each $n = 0, 1, 2, \ldots$ and for each $x \in X$ satisfying $d_1(x_0, x) < \delta$. The family $\{f_n\}_{n=0}^{\infty}$ is called equicontinuous if it is equicontinuous at each point $x_0 \in X$.*

Definition 2.1.8 *Let (X, d_1) and (Y, d_2) be metric spaces. A family of functions $\{f_n : X \to Y\}_{n=0}^{\infty}$ is said to be uniformly equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ (depending on ε only) such that $d_2(f_n(x), f_n(y)) < \varepsilon$ for each $n = 0, 1, 2, \ldots$ and for all $x, y \in X$ satisfying $d_1(x, y) < \delta$.*

Now, we show that a time varying map $F = \{f_n\}_{n=0}^{\infty}$, where the family $\{f_n\}_{n=0}^{\infty}$ is equicontinuous, is expansive if and only if its k^{th} iterate is expansive, where k is a positive integer.

Theorem 2.1.2 *Let (X, d) be a compact metric space, $\{f_n\}_{n=0}^{\infty}$ be an equicontinuous family of self maps on X and k be a positive integer. Then time varying map $F = \{f_n\}_{n=0}^{\infty}$ is expansive if and only if F^k is expansive.*

Proof : Let $e > 0$ be an expansive constant for F. Since X is compact and $\{f_n\}_{n=0}^{\infty}$ is equicontinuous family, for any $n > 0$ and $nk + 1 \leq j \leq (n+1)k$, $F_{[nk+1,j]}$ is uniformly continuous on X and therefore there exists a $\delta_i > 0$ ($i = j - (nk+1) \in \{0, 1, 2, \ldots, k-1\}$) such that

$$d(x, y) < \delta_i \Rightarrow d(F_{[nk+1,j]}(x), F_{[nk+1,j]}(y)) < e.$$

Note that due to equicontinuity of $\{f_n\}_{n=0}^{\infty}$, δ_j does not depend on n. Take $\delta = \min\{\delta_i : 0 \leq i \leq k-1\}$. Then for any $n \geq 0$,

$$d(x, y) < \delta \Rightarrow d(F_{[nk+1,j]}(x), F_{[nk+1,j]}(y)) < e.$$

Now $F^k = \{g_n\}_{n=0}^{\infty}$, where $g_n = F_{[(n-1)k+1,nk]}$ and $G_n = g_n \circ \cdots \circ g_1 \circ g_0$. It is easy to see that $G_n = F_{nk}$. Note that for any $j \geq 0$, there exists $n \geq 0$ such that $nk \leq j \leq (n+1)k$. Now for any $n \geq 0$ and $nk \leq j \leq (n+1)k$,

$$\begin{aligned} d(G_n(x), G_n(y)) < \delta &\Rightarrow d(F_{nk}(x), F_{nk}(y)) < \delta \\ &\Rightarrow d(F_{[nk+1,j]}(F_{nk}(x)), F_{[nk+1,j]}(F_{nk}(y))) < e \\ &\Rightarrow d(F_j(x), F_j(y)) < e. \end{aligned}$$

Since e is an expansive constant for F, $x = y$ and hence δ is an expansive constant for F^k.

Conversely, if F^k is expansive with an expansive constant ε then for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ such that

$$d(G_n(x), G_n(y)) > \varepsilon$$

which implies
$$d(F_{nk}(x), F_{nk}(y)) > \varepsilon$$
proving that ε is an expansive constant for F.

The following example shows that 'equicontinuity' in the hypothesis of Theorem 2.1.2 is necessary.

Example 2.2 *Consider the sequence of maps $F = \{f_n\}_{n=0}^{\infty}$ on the unit circle S^1 defined by*
$$f_n(z) = \begin{cases} z^{\frac{n}{2}+1} & if \;\; n \;\; is \;\; even; \\ z^{\frac{2}{n+1}} & if \;\; n \;\; is \;\; odd, \end{cases}$$
for any $z \in S^1$, where $z \in S^1$, $z^{\frac{1}{m}} = \exp\{\frac{i}{m} Arg(z)\}$, in which $Arg(z)$ is the principle argument of z.

Note that
$$F_n(z) = \begin{cases} z^{\frac{n}{2}+1} & if \;\; n \;\; is \;\; even; \\ z & if \;\; n \;\; is \;\; odd, \end{cases}$$
for any $z \in S^1$ and therefore F is expansive. Observe that $F^2 = \{g_n\}_{n=0}^{\infty}$, where each $g_n = F_{[2n-1,2n]}$ is the identity map, therefore F^2 is not expansive. Note that $\{f_n\}$ is not an equicontinuous family on S^1.

Using Theorem 2.1.2, we have the following example of a time varying map which is not expansive.

Example 2.3 *Let N be any positive integer. Consider the time varying map $F = \{f_n\}_{n=0}^{\infty}$ on the unit circle S^1 defined by*
$$f_n(z) = \begin{cases} z^{k+1} & 0 \le n = 2k \le 2N; \\ z^{\frac{1}{k+2}} & 1 \le n = 2k+1 < 2N; \\ z & n > 2N. \end{cases}$$
for any $z \in S^1$.

Note that $\{f_n\}_{n=0}^{\infty}$ is equicontinuous family of maps on compact space S^1 and $F^2 = \{g_n\}_{n=0}^{\infty}$, where each $g_n = F_{[2n-1,2n]}$ is the identity map. Since F^2 is not expansive, by Theorem 2.1.2, F is not expansive.

Now we define invariant subset for a time varying map and show that if a time varying map is expansive on a metric space then it is also expansive on any invariant subset of it.

Definition 2.1.9 *Let (X, d) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on X and Y be a subset of X. Then Y is said to be* **invariant under F** *if*

$$f_n(Y) \subset Y,$$

for all $n \geq 0$, equivalently $F_n(Y) \subset Y$, for all $n \geq 0$.

Lemma 2.1.1 *Let (X, d) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map which is expansive on X and Y be an invariant subset of X, then restriction of F to Y, defined by $F|Y = \{f_n|Y\}$ is expansive.*

Proof: Let $\varepsilon > 0$ be an expansive constant for F on X. Let $x \neq y,\ x, y \in Y$ then $x, y \in X$ also, therefore there exists $n \geq 0$ such that

$$d(F_n(x), F_n(y)) > \varepsilon.$$

Since Y is invariant under F, $F_n(x), F_n(y) \in Y$. Hence $F|Y$ is also expansive with expansive constant ε.

Now we show that every finite direct product of expansive time varying maps is expansive.

Theorem 2.1.3 *Let (X, d_1) and (Y, d_2) be metric spaces and $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ be time varying maps on X and Y respectively. Define a metric d on $X \times Y$ by*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}; \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Also for any $f : X \to X$ and $g : Y \to Y$ define $f \times g : X \times Y \to X \times Y$ by

$$(f \times g)(x, y) = (f(x), g(y)), \quad (x, y) \in X \times Y.$$

Then time varying map $F \times G = \{f_n \times g_n\}_{n=0}^{\infty}$ is expansive on $X \times Y$ if and only if F and G are expansive on X and Y respectively.

Proof : Let $F \times G$ is expansive on $X \times Y$ with expansive constant $\varepsilon > 0$. For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have $(x_1, y_1), (x_2, y_2) \in X \times Y$. Suppose for any $n > 0, d_1(F_n(x_1), f_n(x_2)) < \varepsilon$ and $d_2(G_n(y_1), G_n(y_2)) < \varepsilon$ then for any $n > 0$,

$$\begin{aligned} &d((F \times G)_n(x_1, y_1), (F \times G)_n(x_2, y_2)) \\ &= \max\{d_1(F_n(x_1), f_n(x_2)), d_2(G_n(y_1), G_n(y_2))\} \\ &< \varepsilon. \end{aligned}$$

Since $F \times G$ is expansive with expansive constant ε, we have $(x_1, y_1) = (x_2, y_2)$ i.e. $x_1 = x_2$ and $y_1 = y_2$ which implies that F and G both are expansive with expansive constant ε. Conversely suppose F and G are expansive on X and Y respectively. Note that for any $n \geq 0$,

$$(F \times G)_n(x, y) = (F_n(x), G_n(y)), \quad (x, y) \in X \times Y.$$

Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be expansive constants for F and G respectively. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $(x_1, y_1), (x_2, y_2) \in X \times Y$.

If for all $n \geq 0$,

$$d((F \times G)_n(x_1, y_1), (F \times G)_n(x_2, y_2)) < \varepsilon$$

then

$$d((F_n(x_1), G_n(y_1)), (F_n(x_2), G_n(y_2))) < \varepsilon$$

which implies

$$\max\{d_1(F_n(x_1), F_n(x_2)), d_2(G_n(y_1), G_n(y_2))\} < \varepsilon.$$

Hence

$$d_2(F_n(x_1), F_n(x_2)) < \varepsilon \leq \varepsilon_1$$

and

$$d_1(G_n(y_1), G_n(y_2)) < \varepsilon \leq \varepsilon_2$$

which by expansiveness of F and G implies $x_1 = x_2$ and $y_1 = y_2$ i.e. $(x_1, y_1) = (x_2, y_2)$. Hence $F \times G$ is expansive with expansive constant ε.

2.2 Invertible Nonautonomous Discrete Dynamical Systems and Expansiveness

Let (X, d) to be a metric space and $f_n : X \to X$ to be a sequence of homeomorphisms, $n = 0, 1, 2, \ldots$, where we always consider f_0 to be the identity map on X. We call $F = \{f_n\}_{n=0}^{\infty}$ to be **a time varying homeomorphism** on X and (X, F), **an invertible nonautonomous discrete dynamical system.** We denote

$$F_n = \begin{cases} f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0, & for \;\; n \geq 0 \\ f_{-n}^{-1} \circ f_{-(n-1)}^{-1} \circ \cdots \circ f_1^{-1} \circ f_0^{-1}, & for \;\; n \leq -1. \end{cases}$$

For any $0 \leq i \leq j$, we define

$$F_{[i,j]} = \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i, & 0 \leq i \leq j \\ the \;\; identity \;\; map \;\; on \;\; X, & i > j. \end{cases}$$

For time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X, its inverse map is given by $F^{-1} = \{f_n^{-1}\}_{n=0}^{\infty}$. Thus

$$F_{[i,j]}^{-1} = \begin{cases} f_j^{-1} \circ f_{j-1}^{-1} \circ \cdots \circ f_{i+1}^{-1} \circ f_i^{-1}, & 0 \leq i \leq j \\ the \;\; identity \;\; map \;\; on \;\; X, & i > j. \end{cases}$$

For any $k > 0$, we define a time varying homeomorphism (k^{th}-iterate of F) $F^k = \{g_n\}_{n=0}^{\infty}$ on X, where

$$g_n = f_{nk} \circ f_{(n-1)k+k-1} \circ \cdots \circ f_{(n-1)k+2} \circ f_{(n-1)k+1} \;\; for \;\; all \;\; n \geq 0.$$

Thus $F^k = \{F_{[(n-1)k+1,nk]}\}_{n=0}^{\infty}$, for $k > 0$ and for $k = -m < 0$, $F^k = (F^{-1})^m$. Also, for $k = 0$, $F^k = \{f_n\}_{n=0}^{\infty}$, where each f_n is the identity map on X. Thus $F^k = \{g_n\}_{n=0}^{\infty}$, where

$$g_n = \begin{cases} F_{[(n-1)k+1,nk]} & if \;\; k > 0; \\ F_{[(n-1)k+1,nk]}^{-1} & if \;\; k < 0; \\ the \;\; identity \;\; map \;\; on \;\; X & if \;\; k = 0. \end{cases}$$

Let us define orbit, periodic point and fixed point for a time varying homeomorphism.

Definition 2.2.1 *Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphism, $n = 0, 1, 2, \ldots$. For a point $x_0 \in X$, let*

$$x_n = \begin{cases} f_n(x_{n-1}) & n \geq 1; \\ f_{-n}^{-1}(x_{n+1}) & n \leq -1 \end{cases}$$

then the sequence $\{x_n\}_{n=-\infty}^{\infty}$, denoted by $O(x_0)$, is said to be **the orbit of $\mathbf{x_0}$ under time varying homeomorphism $\mathbf{F} = \{\mathbf{f_n}\}_{\mathbf{n=0}}^{\infty}$**.

Definition 2.2.2 *Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0, 1, 2, \ldots$. A point $x_0 \in X$ is said to be* **a periodic point** *of $F = \{f_n\}_{n=0}^{\infty}$ if orbit of x_0 ($O(x_0) = \{x_n\}_{n=-\infty}^{\infty}$) is periodic i.e. there exists an integer $k > 0$ such that*

$$x_{n+k} = x_n,$$

for all $n \in \mathbb{Z}$, where

$$x_n = \begin{cases} f_n(x_{n-1}) & if \;\; n \geq 0; \\ f_{-n}^{-1}(x_{n+1}) & if \;\; n < 0. \end{cases}$$

The set of all periodic points of F is denoted by $Per(F)$.

Definition 2.2.3 *Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0, 1, 2, \ldots$. A point $x \in X$ is said to be* **a fixed point** *of time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ if*

$$f_n(x) = x$$

for all $n = 0, 1, 2, \ldots$.

Note 2.1 *If $f_n(x) = x$ then f_n being homeomorphism $f_n^{-1}(x) = x$. Hence orbit of fixed point x is $\{x\}$.*

Now we define expansiveness of a time varying homeomorphism i.e. expansiveness of an invertible nonautonomous discrete dynamical system and study them in detail.

Definition 2.2.4 *Let (X, d) be a metric space and $f_n : X \to X$ be a sequence of homeomorphisms, $n = 0, 1, 2, \dots$. The time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ is said to be* **expansive** *if there exists a constant $e > 0$ (called* **an expansive constant***) such that for any $x, y \in X$, $x \neq y$,*

$$d(F_n(x), F_n(y)) > e$$

for some $n \in \mathbb{Z}$. Equivalently, if for $x, y \in X$,

$$d(F_n(x), F_n(y)) \leq e \quad for \;\; all \;\; n \in \mathbb{Z} \;\; then \;\; x = y.$$

Remark 2.3 *If in the above definition $f_n = f$ for all $n \geq 0$, where $f : X \to X$ is homeomorphism, then expansiveness of time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X is equivalent to expansiveness of f on X ([2]).*

Remark 2.4 *Note that expansiveness of a time varying homeomorphism F is independent of the choice of metric for X if X is compact.*

Let us define conjugacy between two time varying homeomorphisms.

Definition 2.2.5 *Let (X, d_1) and (Y, d_2) be two metric spaces. Let $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$ be time varying homeomorphisms on X and Y respectively. If there exists a homeomorphism $h : X \to Y$ such that*

$$h \circ f_n = g_n \circ h,$$

for all $n = 0, 1, 2, \dots$ then F and G are said to be **conjugate** *with respect to the map h or* **h-conjugate***.*

In particular, if $h : X \to Y$ is a uniform homeomorphism then F and G are said to be **uniformly conjugate** *or* **uniformly h-conjugate***.*

The following theorem shows that expansiveness of a time varying homeomorphism is a property which is preserved under uniform conjugacy.

Theorem 2.2.1 *Let (X, d_1) and (Y, d_2) be metric spaces. Let $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$ be time varying homeomorphisms on X and Y respectively such that F is uniformly conjugate to G. Then F is expansive on X if and only if G is expansive on Y.*

Proof : Since F is uniformly conjugate to G therefore there exists a uniform homeomorphism $h\colon X \to Y$ such that

$$h \circ f_n = g_n \circ h,$$

for all $n \geq 0$, which implies

$$f_n \circ h^{-1} = h^{-1} \circ g_n,$$

for all $n \geq 0$ and

$$f_n^{-1} \circ h^{-1} = h^{-1} \circ g_n^{-1},$$

for all $n \geq 0$. Now for all $n \geq 0$,

$$\begin{aligned} F_n \circ h^{-1} &= f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ f_0 \circ h^{-1} \\ &= f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ h^{-1} \circ g_0 \\ &\;\;\vdots \\ &= h^{-1} \circ g_n \circ g_{n-1} \circ \cdots g_2 \circ g_1 \circ g_0 \\ &= h^{-1} \circ G_n \end{aligned}$$

and similarly for all $n \leq 0$, we also have

$$\begin{aligned} F_n \circ h^{-1} &= f_{-n}^{-1} \circ f_{-n+1}^{-1} \circ \cdots f_2^{-1} \circ f_1^{-1} \circ f_0^{-1} \circ h^{-1} \\ &= f_{-n}^{-1} \circ f_{-n+1}^{-1} \circ \cdots f_2^{-1} \circ f_1^{-1} \circ h^{-1} \circ g_0^{-1} \\ &\;\;\vdots \\ &= h^{-1} \circ g_n^{-1} \circ g_{-n+1}^{-1} \circ \cdots g_2^{-1} \circ g_1^{-1} \circ g_0^{-1} \\ &= h^{-1} \circ G_n. \end{aligned}$$

So, we get $F_n \circ h^{-1} = h^{-1} \circ G_n$, for all $n \in \mathbb{Z}$. Similarly,

$$h \circ F_n = G_n \circ h,$$

for all $n \in \mathbb{Z}$. Suppose F is expansive on X with expansive constant $\varepsilon > 0$. Since h^{-1} is uniformly continuous therefore there exists a $\delta > 0$ such that for any $y_1, y_2 \in Y$ with $d_2(y_1, y_2) < \delta$,

$$d_1(h^{-1}(y_1), h^{-1}(y_2)) < \varepsilon.$$

Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ then $h^{-1}(y_1) \neq h^{-1}(y_2)$ therefore F being expansive on X, there exists $n \in \mathbb{Z}$ such that

$$d_1(h^{-1}(G_n(y_1)), h^{-1}(G_n(y_2))) = d_1(F_n(h^{-1}(y_1)), F_n(h^{-1}(y_2))) > \varepsilon$$

which implies

$$d_2(G_n(y_1), G_n(y_2)) \geq \delta.$$

Hence G is expansive on Y.
Conversely, suppose G is expansive on Y with expansive constant $\varepsilon > 0$. Since h is uniformly continuous, there exists $\delta > 0$ such that for any $x_1, x_2 \in X$ with $d_1(x_1, x_2) < \delta$,

$$d_2(h(x_1), h(x_2)) < \varepsilon.$$

For any $x_1, x_2 \in X$ with $x_1 \neq x_2$, observing that $h(x_1) \neq h(x_2)$, it follows that there exists $n \in \mathbb{Z}$ such that

$$d_2(h(F_n(x_1)), h(F_n(x_2))) = d_2(G_n(h(x_1)), G_n(h(x_1))) > \epsilon$$

which implies $d_1(F_n(x_1), F_n(x_2)) \geq \delta$. Thus F is expansive on X.

Corollary 2.2.1 *Let (X, d_1) be a compact metric space, (Y, d_2) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X and $h\colon X \to Y$ is a homeomorphism. If F is expansive on X then $G = h \circ F \circ h^{-1} = \{g_n\}_{n=0}^{\infty}$, where $g_n = h \circ f_n \circ h^{-1}$; $n = 0, 1, 2, \ldots$ is expansive on Y.*

The following theorem shows that a time varying homeomorphism is expansive if and only if its inverse is expansive.

Theorem 2.2.2 *Let* (X, d) *be a compact metric space,* $\{f_n\}_{n=0}^{\infty}$ *be a family of self homeomorphisms on X. Then time varying homeomorphism* $F = \{f_n\}_{n=0}^{\infty}$ *is expansive if and only if* F^{-1} *is expansive.*

Proof : Let F be expansive with an expansive constant $e > 0$. It is easy to verify that

$$F_{(-n)} = (F^{-1})_n,$$

for all $n \in \mathbb{Z}$. Equivalently $(F^{-1})_{(-n)} = F_n$, for all $n \in \mathbb{Z}$. Let $x \neq y$, $x, y \in X$ then there is some $n \in \mathbb{Z}$ such that

$$d(F_n(x), F_n(y)) > e$$

i.e.

$$d((F^{-1})_{(-n)}(x), (F^{-1})_{(-n)}(y)) > e$$

for some $(-n) \in \mathbb{Z}$, which implies F^{-1} is also expansive. Consequently, F^{-1} expansive implies $(F^{-1})^{-1} = F$ is expansive.

By above result and analogous to the Theorem 2.1.2, we have the following result.

Theorem 2.2.3 *Let* (X, d) *be a compact metric space,* $\{f_n\}_{n=0}^{\infty}$ *be an equicontinuous family of self maps on X and k be an integer. Then time varying homeomorphism* $F = \{f_n\}_{n=0}^{\infty}$ *is expansive if and only if* F^k *is expansive for any* $k \in \mathbb{Z} - \{0\}$.

Now we define invariant set under time varying homeomorphism and show that if a time varying homeomorphism is expansive on a metric space then the restricted map to an invariant subset is also expansive.

Definition 2.2.6 *Let* (X, d) *be a metric space,* $F = \{f_n\}_{n=0}^{\infty}$ *be a time varying homeomorphism on X and Y be a subset of X. Then Y is said to be* **invariant** *under F if* $f_n(Y) = Y$, *(and therefore* $f_n^{-1}(Y) = Y$*) for all* $n \geq 0$, *equivalently* $F_n(Y) = Y$, *for all* $n \in \mathbb{Z}$.

Analogous to Theorem 2.1.1, one can prove the following result.

Theorem 2.2.4 *Let (X, d) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism which is expansive on X and Y be an invariant subset of X, then restriction of F to Y, defined by $F|Y = \{f_n|Y\}$ is expansive.*

Analogous to Theorem 2.1.3, one can prove the following result.

Theorem 2.2.5 *Let (X, d_1) and (Y, d_2) be metric spaces and $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ be time varying homeomorphisms on X and Y respectively. Consider the metric d on $X \times Y$ defined by*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}; \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Then the time varying homeomorphism $F \times G = \{f_n \times g_n\}_{n=0}^{\infty}$ is expansive on $X \times Y$ if and only if F and G are expansive on X and Y respectively. Hence every finite direct product of expansive time varying homeomorphisms is expansive.

We have following result for time varying homeomorphism similar to that for an expansive homeomorphism on a compact metric space ([6]).

Theorem 2.2.6 *Let (X, d) be a compact metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism such that for any given pair of integers r and s, there is an integer t such that $F_r(F_s(x)) = F_t(x)$, for all $x \in X$. If F is expansive on X and θ is the least upper bound of the set of expansive constants for F then θ is not an expansive constant for F.*

Proof: Let e be an expansive constant for F, θ be the least upper bound of the set of expansive constants for F and $\varepsilon_i = \frac{1}{i}$ for $i = 1, 2, 3, \cdots$. Since $\theta + \varepsilon_i$ is not an expansive constant for F therefore for each i there exist $x'_i \neq y'_i$ such that

$$d(F_n(x'_i), F_n(y'_i)) \leq \theta + \varepsilon_i \tag{2.1}$$

for each integer n. Also, for each i, there exists an integer k_i such that

$$d(F_{k_i}(x'_i), F_{k_i}(y'_i)) > e.$$

(Since e is an expansive constant for F and $x'_i \neq y'_i$, for each i.)

Let $x_i = F_{k_i}(x'_i)$ and $y_i = F_{k_i}(y'_i)$. Since X is a compact metric space therefore passing to a subsequence if necessary, without loss of generality, we can assume that there exist $x, y \in X$ such that $x_j \to x$ and $y_j \to y$. Note that $x \neq y$. (As for each i, $d(x_i, y_i) > e$.)

Let m be an arbitrary integer and α be an arbitrary positive real number. Choose p, q and η with the following properties:

(a) $\varepsilon_p < \frac{\alpha}{3}$,
(Such p exists as $\varepsilon_i = \frac{1}{i}$ converges to zero.)

(b) $d(u, v) < \eta$ implies $d(F_m(u), F_m(v)) < \frac{\alpha}{3}$,
(Such η exists as X being compact, F_m is uniformly continuous on X.)

(c) $n > p$ implies $d(x, x_n) < \eta$ and $n > q$ implies $d(y, y_n) < \eta$.
(As the sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ converge to x and y respectively.)

Let $i > \max\{p, q\}$, then

$$d(F_m(x), F_m(y)) \leq d(F_m(x), F_m(x_i)) + d(F_m(x_i), F_m(y_i)) + d(F_m(y_i), F_m(y))$$
$$< \frac{\alpha}{3} + \left(\theta + \frac{\alpha}{3}\right) + \frac{\alpha}{3} = \alpha + \theta$$

(Since $i > p$ therefore from (c), $d(x, x_i) < \eta$ and hence form (b),

$$d(F_m(x), F_m(x_i)) < \frac{\alpha}{3}.$$

Similarly $i > q$ implies

$$d(F_m(y_i), F_m(y)) < \frac{\alpha}{3}.$$

Further since $i > p$ therefore using (a), $\varepsilon_i < \varepsilon_p < \frac{\alpha}{3}$ and

$$\begin{aligned} d(F_m(x_i), F_m(y_i)) &= d(F_m(F_{k_i}(x'_i)), F_m(F_{k_i}(y'_i))) \\ &= d(F_t(x'_i), F_t(y'_i)) \;\; for \;\; some \;\; integer \;\; t \\ &\leq \theta + \varepsilon_i \;\; (from \;\; equation \;\; (2.1)) \\ &< \theta + \frac{\alpha}{3}. \qquad) \end{aligned}$$

Thus $d(F_m(x), F_m(y)) \leq \theta$ implying θ is not an expansive constant for F.

Definition 2.2.7 *Let (X, d) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X and $A \subseteq X$. Then F is said to be expansive on A with expansive constant $e > 0$ if for any $x, y \in A$, $x \neq y$, there exists an integer $n \geq 0$ (depending upon pair (x, y)) such that $d(F_n(x), F_n(y)) > e$ or equivalently if for $x, y \in A$*

$$d(F_n(x), F_n(y)) \leq e, \;\; for \;\; all \;\; n \geq 0 \;\; then \;\; x = y.$$

Now we show that a if a time varying homeomorphism is expansive on a subset whose complement is finite then it is expansive on the entire metric space.

Theorem 2.2.7 *Let (X, d) be a metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X and $A \subseteq X$ such that $X - A$ is finite. If F is expansive on A then it is expansive on X.*

Proof : Let F be expansive on A with expansive constant e. Since $X - A$ is finite, it is sufficient to show that F is expansive on $A \cup \{x\}$, where $x \in X - A$. Then one can show expansiveness of F on X using induction for finitely many steps. Note that there is at most one point $p \in A$ such that

$$d(F_n(x), F_n(p)) \leq \frac{e}{2}, \;\; \forall n \in \mathbb{Z}.$$

If p and q are two such points in A, then

$$\begin{aligned} d(F_n(p), F_n(q)) &\leq d(F_n(p), F_n(x)) + d(F_n(x), F_n(q)) \\ &\leq \frac{e}{2} + \frac{e}{2} \\ &\leq e, \end{aligned}$$

for all $n \in \mathbb{Z}$, which contradicts the expansiveness of F on A. If a point p as described above exists, let $0 < c < d(x, p)$; otherwise let $c = \frac{e}{2}$. Now for any $a, b \in A \cup \{x\},\ a \neq b$, if $a, b \in A$ then expansiveness of F on A implies that there exist $m \in \mathbb{Z}$ such that

$$d(F_m(a), F_m(b)) > e > c.$$

If $a = x$ and $b \in A$ with $b \neq p$ then there exists $m \in \mathbb{Z}$ such that

$$\begin{aligned} d(F_m(a), F_m(b)) &= d(F_m(x), F_m(b)) \\ &> \frac{e}{2} \geq c. \end{aligned}$$

If $a = x$ and $b \in A$ with $b = p$ then for $m = 0$ we have

$$\begin{aligned} d(F_m(a), F_m(b)) &= d(F_m(x), F_m(p)) \\ &= d(x, p) \qquad\qquad (as \ \ m = 0) \\ &> c. \end{aligned}$$

Thus in any case there exists $m \in \mathbb{Z}$ such that

$$d(F_m(a), F_m(b)) > c.$$

Thus F is expansive on $A \cup \{x\}$ with expansive constant c.

2.3 Generator and Weak Generator in Nonautonomous Discrete Dynamical Systems

The topological analogue of generator was defined and studied by Keynes and Robertson [34]. We define and study this notion for an invertible non-autonomous discrete dynamical system and obtain a characterization of expansiveness in terms of generator and weak generator.

Definition 2.3.1 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A finite open cover α of X is said to be* **a generator for F** *if for every bisequence $\{A_n\}$ of members of α,*

$$\bigcap_{n=-\infty}^{\infty} (F_n)^{-1}(\overline{A_n})$$

is at most one point, where $\overline{A_n}$ denotes the closure of set A_n.

Definition 2.3.2 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. A finite open cover α of X is said to be* **a weak generator for F** *if for every bisequence $\{A_n\}$ of members of α,*

$$\bigcap_{n=-\infty}^{\infty} (F_n)^{-1}(A_n)$$

is at most one point .

The following result gives a characterization of expansiveness. We show that a time varying homeomorphism on a compact metric space is expansive if and only if it has a generator or a weak generator.

Theorem 2.3.1 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. Then following are equivalent :*
(1) *F is expansive,*
(2) *F has a generator,*
(3) *F has a weak generator.*

Proof : We first show that (2) ⇒ (3).
(2) ⇒ (3) : Let α be a finite open cover of X and $\{A_n\}$ be bisequence of members of α. Since $A_n \subseteq \overline{A_n}$, we have

$$\bigcap_{n=-\infty}^{\infty} (F_n)^{-1}(A_n) \subseteq \bigcap_{n=-\infty}^{\infty} (F_n)^{-1}(\overline{A_n}).$$

If α is generator for F then

$$\bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(\overline{A_n})$$

contains at most one point and therefore

$$\bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(A_n)$$

also contains at most one point. Hence α is also a weak generator for F.

(3) $\Rightarrow$ (2) : Let $\beta = \{B_1, B_2, \ldots, B_n\}$ be a weak generator for F and $\delta > 0$ be a Lebesgue number for β. Let α be a finite open cover by sets A_i with $diam(\overline{A_i}) \leq \delta$. If $\{A_{i_n}\}$ is a bisequence of members of α then for every n, there is j_n such that $\overline{A_{i_n}} \subset B_{j_n}$, and so

$$\bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(\overline{A_{i_n}}) \subseteq \bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(B_{j_n}).$$

Since

$$\bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(B_{j_n})$$

contains almost one point therefore

$$\bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(\overline{A_{i_n}})$$

also contains at most one point and hence α is a generator.

Next we prove that (1) $\Rightarrow$ (2) : Let $\delta > 0$ be an expansive constant for F and α be a finite open cover of X by open balls of radius $\frac{\delta}{2}$. Suppose

$$x, y \in \bigcap_{n=-\infty}^{\infty}(F_n)^{-1}(\overline{A_{i_n}}),$$

where $A_{i_n} \in \alpha$ then $d(F_n(x), F_n(y)) \leq \delta$ for every n, and since F is expansive with expansive constant δ, we have $x = y$.

(3) $\Rightarrow$ (1) : Suppose α is a weak generator. Let $\delta > 0$ be a Lebesgue number for α. If $d(F_n(x), F_n(y)) < \delta$, for all $n \in \mathbb{Z}$ then for every n, there is $A_n \in \alpha$ such that $F_n(x), F_n(y) \in A_n$ and so

$$x, y \in \bigcap_{-\infty}^{\infty} (F_n)^{-1}(A_n)$$

which is at most one point implying $x = y$.

We use the above result in the following example.

Example 2.4 *Let $F = \{f_n\}_{n=0}^{\infty}$, where $f_n \colon [0,1] \to [0,1]$ is defined by $f_n(x) = x^{n+1}$ for n=0,1,2... and $x \in [0,1]$, be a time varying homeomorphism on $[0,1]$. Now note that*

$$F_n(x) = x^{n!} \;\; and \;\; F_{-n} = x^{\frac{1}{n!}},$$

for all $n \geq 0$. Let α be a finite open cover of $[0,1]$ with Lebesgue number $0 < \delta < \frac{1}{2}$. Note that

$$\lim_{n\to\infty} F_n = 0 \;\; and \;\; \lim_{n\to\infty} F_{-n} = 1$$

uniformly on $[\delta, 1-\delta]$. So there exists $N > 0$ such that $n > N$ implies $F_n(x) \in [0,\delta)$ and $F_{-n}(x) \in (1-\delta, 1]$, for any $x \in [\delta, 1-\delta]$. Since δ is Lebesgue number of α, there are A_0 and A_1 in α such that $[0,\delta) \subseteq A_0$ and $(1-\delta, 1] \subseteq A_1$. Now since $\{F_{-N}, F_{-N+1}, \cdots, F_N\}$ is uniformly equicontinuous family, there exists $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $d(F_n(x), F_n(y)) < \delta$, for any $|n| \leq N$. Let $x, y \in [\delta, 1-\delta]$, $x \neq y$ such that $d(x,y) < \varepsilon$. Then for any n, $|n| \leq N$ there exists $A_n \in \alpha$ such that $F_n(x), F_n(y) \in A_n$. Thus $x, y \in (F_n)^{-1}(A_n)$, $|n| \leq N$. Now put

$$A_n = \begin{cases} A_0, & n \geq N+1; \\ A_1, & n \leq -(N+1). \end{cases}$$

Now note that

$$x, y \in \bigcap_{n=-\infty}^{\infty} (F_n)^{-1}(A_n).$$

Thus α can not be a weak generator for F. Therefore F has no weak generator and hence by above result F is a time-varying homeomorphism which is not-expansive.