Chapter 3

Shadowing Property and Topological Stability in Nonautonomous Discrete Dynamical Systems

In this chapter, we define and study shadowing and topological stability in nonautonomous discrete dynamical systems given by a sequence of continuous maps and a sequence of homeomorphisms on a metric space. Expansive time varying homeomorphism having shadowing property is called a topologically Anosov time varying homeomorphism and we show that they are topologically stable in the class of all time varying homeomorphisms.

3.1 Shadowing in Nonautonomous Discrete Dynamical Systems

We first define δ-pseudo orbit and shadowing property in a nonautonomous discrete dynamical system.

Definition 3.1.1 *Let (X, d) be a metric space, $f_n : X \to X$, $n = 0, 1, \ldots,$ be a sequence of continuous maps on X and $F = \{f_n\}_{n=0}^{\infty}$. For $\delta > 0$, the sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be* **a δ-pseudo orbit** *of F if*

$$d(f_{n+1}(x_n), x_{n+1}) < \delta,$$

for all $n = 0, 1, 2, \ldots$. For given $\varepsilon > 0$, a δ-pseudo orbit $\{x_n\}_{n=0}^{\infty}$ is said to be **ε-traced** *by $y \in X$ if*

$$d(F_n(y), x_n) < \varepsilon,$$

for all $n = 0, 1, 2, \ldots$

The time varying map F is said to have **shadowing property or pseudo orbit tracing property** *(P.O.T.P) if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit is ε- traced by some point of X.*

Remark 3.1 *If in the above definition $f_n = f$, for all $n \geq 0$, where $f : X \to X$ is continuous map, then P.O.T.P of time varying map $F = \{f_n\}_{n=0}^{\infty}$ on X is equivalent to P.O.T.P. of f on X ([2]).*

Remark 3.2 *Note that shadowing property is independent of choice of metric if X is compact. Let d_1 and d_2 be two equivalent metrics on a compact space X. Suppose F has P.O.T.P. in (X, d_1). Let $\varepsilon > 0$ be given. Since d_1 is equivalent to d_2, there exists an $\varepsilon_1 > 0$ such that for any $x \in X$,*

$$N_{d_1}(x, \varepsilon_1) \subset N_{d_2}(x, \varepsilon).$$

Since X is compact, ε_1 only depends on ε and not on the point x. Since F has P.O.T.P. in (X, d_1), for this ε_1, we get a $\delta_1 > 0$ such that any δ_1-pseudo-orbit is ε_1-traced by some point.

Further, since d_1 and d_2 are equivalent, for this δ_1, we get a $\delta > 0$ such that for any $x \in X$,

$$N_{d_2}(x, \delta) \subset N_{d_1}(x, \delta_1).$$

Now let $\{x_n\}_{n=0}^{\infty}$ be a δ-pseudo-orbit of F in (X, d_2). Thus

$$f_{n+1}(x_n) \in N_{d_2}(x_{n+1}, \delta) \subset N_{d_1}(x_{n+1}, \delta_1).$$

Hence $\{x_n\}_{n=0}^{\infty}$ is a δ_1-pseudo orbit of F in (X, d_1), therefore there exists a $y \in X$ which ε_1-traces $\{x_n\}_{n=0}^{\infty}$. Thus

$$F_n(y) \in N_{d_1}(x_n, \varepsilon_1) \subset N_{d_2}(x_n, \varepsilon)$$

which implies $\{x_n\}_{n=0}^{\infty}$ is ε-traced by y in (X, d_2). Hence F has P.O.T.P in (X, d_2).

The following result shows that P.O.T.P. of a time varying map is a property which is preserved under uniform conjugacy.

Proposition 3.1.1 *Let (X, d_1) and (Y, d_2) be metric spaces. Let $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$ be time varying maps on X and Y respectively such that F is uniformly conjugate to G. Then F has P.O.T.P. if and only if G has P.O.T.P.*

Proof : Let $\varepsilon > 0$ be given. Since F is uniformly conjugate to G, there exists a uniform homeomorphism $h : X \to Y$ such that

$$h \circ f_n = g_n \circ h.$$

i.e.

$$f_n \circ h^{-1} = h^{-1} \circ g_n,$$

for all $n \geq 0$. Now h is uniformly continuous being a uniform homeomorphism, therefore there exists an $\varepsilon_0 > 0$ such that

$$d_1(x, y) < \varepsilon_0 \;\; \textit{implies} \;\; d_2(h(x), h(y)) < \varepsilon.$$

Let F has P.O.T.P. Then there exists a $\delta_0 > 0$ such that any δ_0-pseudo orbit of F is ε_0-traced by F-orbit of some point of X. Since h being a uniform homeomorphism, h^{-1} is uniformly continuous map, therefore for $\delta_0 > 0$ there exists a $\delta > 0$ such that for any $x, y \in Y$,

$$d_2(x, y) < \delta \;\; \textit{implies} \;\; d_1(h^{-1}(x), h^{-1}(y)) < \delta_0.$$

Let $\{y_n\}_{n=0}^{\infty}$ be a δ-pseudo orbit for G i.e. $d_2(g_{n+1}(y_n), y_{n+1}) < \delta$ i.e.

$$d_1(h^{-1}(g_{n+1}(y_n)), h^{-1}(y_{n+1})) < \delta_0$$

i.e.

$$d_1(f_{n+1}(h^{-1}(y_n)), h^{-1}(y_{n+1})) < \delta_0$$

which implies $\{h^{-1}(y_n)\}_{n=0}^{\infty}$ is a δ_0-pseudo orbit for F. Thus there exists an $x \in X$ such that

$$d_1(F_n(x), h^{-1}(y_n)) < \varepsilon_0$$

and hence

$$d_2(h(F_n(x)), y_n) < \varepsilon.$$

Now for all $n \geq 0$,

$$\begin{aligned} h \circ F_n &= h \circ f_n \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ f_0 \\ &= g_n \circ h \circ f_{n-1} \circ \cdots f_2 \circ f_1 \circ f_0 \\ &\quad\vdots \\ &= g_n \circ g_{n-1} \circ \cdots g_2 \circ g_1 \circ g_0 \circ h \\ &= G_n \circ h \end{aligned}$$

implies $d_2(G_n(h(x)), y_n) < \varepsilon$ i.e. $\{y_n\}_{n=0}^{\infty}$ is ε-traced by $h(x) \in Y$. Thus G has P.O.T.P.

Note that if F is conjugate to G then G is conjugate to F. Thus converse follows from above if we interchange the role of F and G.

Next, we show that every finite direct product of time varying maps having P.O.T.P., has P.O.T.P.

Theorem 3.1.1 *Let (X, d_1) and (Y, d_2) be metric spaces and $F = \{f_n\}_{n=0}^{\infty}$, $G = \{g_n\}_{n=0}^{\infty}$ be time varying maps on X and Y respectively. Define metric d on $X \times Y$ by*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}, \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Then the time varying map $F \times G = \{f_n \times g_n\}_{n=0}^{\infty}$ has P.O.T.P. if and only if both F and G have P.O.T.P.

Proof : Note that for any $n \geq 0$,

$$(F \times G)_n(x, y) = (F_n(x), G_n(y)) \;\; (x, y) \in X \times Y.$$

Let $\varepsilon > 0$ be given. Then there exists a $\delta_1 > 0$ and a $\delta_2 > 0$ such that every δ_1-pseudo orbit of F and δ_2-pseudo orbit of G can be ε-traced by some F-orbit and G-orbit respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and $\{(x_i, y_i)\}_{i=0}^{\infty}$ be a δ pseudo orbit of $F \times G$. Then

$$d((f_{i+1} \times g_{i+1})(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta$$

i.e.

$$d((f_{i+1}(x_i), g_{i+1}(y_i)), (x_{i+1}, y_{i+1})) < \delta$$

which by definition of d implies

$$d_1(f_{i+1}(x_i), x_{i+1}) < \delta \leq \delta_1$$

and

$$d_2(g_{i+1}(y_i), y_{i+1}) < \delta \leq \delta_2.$$

Hence there exist $x \in X$ and $y \in Y$ such that $d_1(F_i(x), x_i) < \varepsilon$ and

$$d_2(G_i(y), y_i) < \varepsilon.$$

Hence

$$d((F_i(x), G_i(y)), (x_i, y_i)) < \varepsilon$$

i.e.

$$d((F \times G)_i(x, y), (x_i, y_i)) < \varepsilon$$

which implies $\{(x_i, y_i)\}_{i=0}^{\infty}$ is ε-traced by $(x, y) \in X \times Y$. Hence any δ-pseudo orbit of $F \times G$ can be ε-traced by some point of $X \times Y$.

Thus $F \times G$ also has P.O.T.P. By induction, we get that every finite direct product of time varying maps having P.O.T.P. has P.O.T.P.

The converse can be proved by similar arguments.

Now we show that if a time varying map has P.O.T.P. if and only if its k^{th} iterate has P.O.T.P.

Theorem 3.1.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ be the time varying map on a metric space (X, d). If F has P.O.T.P. then F^k has P.O.T.P. for every $k > 0$.*

Proof : If $k = 1$ nothing to prove. Suppose $k \geq 2$. Let $\varepsilon > 0$ be given. Since F has P.O.T.P., therefore there exists a $\delta > 0$ such that every δ-pseudo orbit of F is ε-traced by some point of X. Let $\{y_i\}_{i=0}^{\infty}$ be a δ-pseudo orbit of F^k. Then

$$d(g_{n+1}(y_n), y_{n+1}) < \delta,$$

for all $n \geq 0$, where $g_n = F_{[(n-1)k+1, nk]}$, i.e

$$d(F_{[nk+1,(n+1)k]}(y_n), y_{n+1}) < \delta,$$

for all $n \geq 0$.
For $0 \leq j < k$ and $n \geq 0$ put $x_{nk+j} = F_{[nk+1,nk+j]}(y_n)$.

Claim : $\{x_i\}_{n=0}^{\infty}$ is a δ-pseudo orbit for F
i.e. to show :

$$d(f_{nk+j+1}(x_{nk+j}), x_{nk+j+1}) < \delta,$$

for all $n \geq 0$ and for all j, $0 \leq j < k$.

Choose any $n \geq 0$. Now for any j, $0 \leq j \leq k - 2$,

$$f_{nk+j+1}(x_{nk+j}) = f_{nk+j+1}(F_{[nk+1,nk+j]}(y_n)) = F_{[nk+1,nk+j+1]}(y_n) = x_{nk+j+1}.$$

Thus

$$d(f_{nk+j+1}(x_{nk+j}), x_{nk+j+1}) = 0 < \delta$$

for all j, $0 \leq j \leq k - 2$.

Now for $j = k - 1$,

$$\begin{aligned} d(f_{nk+k}(x_{nk+k-1}), x_{nk+k}) &= d(f_{nk+k}(F_{[nk+1,nk+k-1]}(y_n)), x_{(n+1)k}) \\ &= d(F_{[nk+1,(n+1)k]}(y_n), y_{n+1}) \\ &< \delta \end{aligned}$$

Hence the claim.

By P.O.T.P. of F, $\{x_i\}_{i=0}^{\infty}$ is ε-traced by some $y \in X$ i.e.

$$d(F_i(y), x_i) < \varepsilon,$$

for all $i \geq 0$. In particular for $i = kn$,

$$d(F_{kn}(y), x_{kn}) < \varepsilon.$$

Thus

$$d(G_n(y), y_n) < \varepsilon,$$

where

$$G_n = g_n \circ \cdots \circ g_1 \circ g_0 = F_{nk}.$$

Thus $F^k = \{g_n\}_{n=0}^{\infty}$ has P.O.T.P.

Remark 3.3 *For time varying map given in Example 2.3, $F^2 = \{g_n\}_{n=0}^{\infty}$, where each g_n is the identity map. Now since F^2 does not have P.O.T.P., by above theorem, F does not have P.O.T.P.*

For the converse we first prove the following result :

Lemma 3.1.1 *Let (X, d) be a compact metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on X (where each f_n is continuous on X) and N be a natural number. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that each δ-finite pseudo orbit $\{x_i \colon 0 \leq i \leq N\}$ satisfies*

$$d(F_i(x_0), x_i) < \varepsilon,$$

for $0 \leq i \leq N$.

Proof : Let $\varepsilon > 0$ be given. For $N = 1$, take $\delta = \varepsilon$. Here

$$d(F_0(x_0), x_0) = d(x_0, x_0) = 0 < \varepsilon$$

and since $\{x_i \colon 0 \leq i \leq 1\}$ is an ε- pseudo orbit, we have

$$d(f_1(x_0), x_1) < \varepsilon.$$

i.e.

$$d(F_1(x_0), x_1) < \varepsilon.$$

Thus result holds for $N = 1$. Suppose result holds for $N-1$. Since f_N is uniformly continuous, for every $\varepsilon > 0$, there exists ε_1, $0 < \varepsilon_1 < \varepsilon$ such that $d(x, y) < \varepsilon_1$ implies

$$d(f_N(x), f_N(y)) < \frac{\varepsilon}{2};$$

$x, y \in X$. By our assumption there exists a δ_1, $0 < \delta_1 < \varepsilon$ such that each δ_1-pseudo orbit $\{y_i \colon 0 \le i \le N-1\}$ is ε_1-traced by $y_0 \in X$. We show that each $\frac{\delta_1}{2}$-pseudo orbit $\{x_i \colon 0 \le i \le N\}$ is ε-traced by $x_0 \in X$.

Since the $\frac{\delta_1}{2}$-pseudo orbit is a δ_1-pseudo-orbit, the finite pseudo-orbit $\{x_i \colon 0 \le i \le N-1\}$ is ε_1-traced by the point $x_0 \in X$. Hence

$$d(F_i(x_0), x_i) < \varepsilon_1 < \varepsilon, \quad for \;\; 0 \le i \le N-1.$$

In particular,

$$d(F_{N-1}(x_0), x_{N-1}) < \varepsilon_1$$

and so

$$d(f_N(F_{N-1}(x_0)), f_N(x_{N-1})) < \frac{\varepsilon}{2}$$

i.e

$$d(F_N(x_0), f_N(x_{N-1})) < \frac{\varepsilon}{2}.$$

Since $\{x_i \colon 0 \le i \le N\}$ is a $\frac{\delta_1}{2}$-pseudo orbit, we have

$$d(f_N(x_{N-1}), x_N) < \frac{\delta_1}{2} < \frac{\varepsilon}{2}$$

and therefore

$$\begin{aligned} d(F_N(x_0), x_N) &\le d(F_N(x_0), f_N(x_{N-1})) + d(f_N(x_{N-1}), x_N) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence result is true for N. Thus by principle of mathematical induction the result is true for any natural number N.

For any $m > 0$, replacing f_i by f_{m+i} and x_i by x_{m+i} in Lemma 3.1.1 we get the following result :

Theorem 3.1.3 *Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on X, where family $\{f_n\}$ is equicontinuous on X and N is a natural number. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $m \geq 0$ and each δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$, the finite δ-pseudo orbit $\{x_{m+i}\}_{i=0}^{N}$ satisfies*

$$d(F_{[m+1,m+i]}(x_m), x_{m+i}) < \varepsilon,$$

for all i, $0 \leq i \leq N$.

Note that since $\{f_n\}$ is equicontinuous, δ does not depend upon m.

Theorem 3.1.4 *Let (X, d) be a compact metric space, $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on X, where $\{f_n\}_{n=0}^{\infty}$ is equicontinuous on X and $k > 0$. If $F^k = \{g_n\}_{n=0}^{\infty}$, where $g_n = F_{[(n-1)k+1,nk]}$, for all $n \geq 0$, has P.O.T.P. then so does $F = \{f_n\}_{n=0}^{\infty}$.*

Proof : Let us first observe the following :

(I) Let $\varepsilon > 0$ be given. Then for each $i \geq 0$, $\{F_{[ik+1,ik+j]} : 0 \leq j \leq k\}$ is equicontinuous being finite family of continuous functions on compact metric space X therefore there exists an ε_1, $0 < \varepsilon_1 < \varepsilon$ such that

$$d(x, y) < \varepsilon_1 \quad \textit{implies} \quad \max_{0 \leq j \leq k}\{d(F_{[ik+1,ik+j]}(x), F_{[ik+1,ik+j]}(y))\} < \frac{\varepsilon}{2}.$$

Note that ε_1 does not depend on i as family $\{f_n\}$ is equicontinuous.

(II) Let ε and ε_1 be as above. Then there exists a δ_0, $0 < \delta_0 < \varepsilon_1$ such that each finite δ_0-pseudo orbit $\{x_{m+j}\}_{j=1}^{k}$ satisfies

$$d(F_{[m+1,m+j]}(x_m), x_{m+j}) < \frac{\varepsilon}{2},$$

for all j, $0 \leq j \leq k$ and for each $m \geq 0$ (by Theorem 3.1.3).

(III) Let ε_1 and δ_0 be as above. Since F^k has P.O.T.P., there exists a δ_1, $0 < \delta_1 < \delta_0$ such that any δ_1-pseudo-orbit for F^k is ε_1-traced by some point of X.

(IV) Let ε_1 and δ_1 be as above. Then there exists a δ, $0 < \delta < \delta_1$ such that each δ-finite pseudo-orbit $\{z_{m+i}\}_{i=1}^{k}$ satisfies

$$d(F_{[m+1,m+i]}(z_m), z_{m+i}) < \delta_1,$$

for all i, $0 \le i \le k$ and for each $m \ge 0$ (by Theorem 3.1.3).

With these properties we prove that each δ-pseudo orbit $\{y_i\}_{i=0}^{\infty}$ of F is ε-traced by some point. Write $x_i = y_{ik}$ for $i = 0, 1, 2, \dots$ and fix i. Since $\{y_{ik+j} : 0 \le j \le k\}$ is a δ-finite pseudo-orbit for F, by (IV) we have

$$d(F_{[ik+1,ik+j]}(y_{ik}), y_{ik+j}) < \delta_1 \ (0 \le j \le k)$$

and particularly if $j = k$ then

$$d(F_{[ik+1,ik+k]}(y_{ik}), y_{ik+k}) < \delta_1$$

i.e.

$$d(g_{i+1}(x_i), x_{i+1}) < \delta_1$$

i.e. $\{x_i\}_{i=0}^{\infty}$ is δ_1-pseudo orbit for F^k. Hence by (III), there exists $y \in X$ with

$$d(G_i(y), x_i) < \varepsilon_1,$$

for $i = 0, 1, 2, \dots$, where $G_i = g_i \circ g_{i-1} \circ \cdots \circ g_1 \circ g_0 = F_{ik}$ i.e.

$$d(F_{ik}(y), y_{ik}) < \varepsilon_1,$$

for $i = 0, 1, 2, \dots$ and hence from (I)

$$d(F_{[ik+1,ik+j]}(F_{ik}(y)), F_{[ik+1,ik+j]}(y_{ik})) < \frac{\varepsilon}{2}$$

i.e.

$$d(F_{ik+j}(y), F_{[ik+1,ik+j]}(y_{ik})) < \frac{\varepsilon}{2}, \quad o \le j \le k, \quad i \ge 0. \tag{3.1}$$

On the other hand, since $\{y_{ik+j} : 0 \leq j \leq k\}$ is a finite δ_0-pseudo orbit of F (as $\{y_i\}_{i=0}^{\infty}$ is a δ-pseudo orbit and $\delta < \delta_1 < \delta_0$, so $\{y_i\}_{i=0}^{\infty}$ is also δ_0-pseudo orbit of F), from (II) it follows that for any $i \geq 0$, taking $m = ik$,

$$d(F_{[ik+1,ik+j]}(y_{ik}), y_{ik+j}) < \frac{\varepsilon}{2}, \tag{3.2}$$

for $0 \leq j \leq k$. Therefore using equations (3.1) and (3.2), we get

$$\begin{aligned} d(F_{ik+j}(y), y_{ik+j}) &\leq d(F_{ik+j}(y), F_{[ik+1,ik+j]}(y_{ik})) + d(F_{[ik+1,ik+j]}(y_{ik}), y_{ik+j}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

for $0 \leq j \leq k$. Since i is arbitrary, we have

$$d(F_n(y), y_n) < \varepsilon \quad for \quad n = 0, 1, 2, \ldots$$

and hence the δ-pseudo orbit $\{y_n\}_{n=0}^{\infty}$ is ε-traced by y.

We use above result in the following example.

Example 3.1 *Let $M > 0$ and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying map on $[0, 1]$ defined by*

$$f_n = \begin{cases} x^{2(k+1)} & n = 2k,\ k = 0, 1, \ldots, M \\ x^{\frac{1}{k+2}} & n = 2k+1,\ k = 0, 1, \ldots, M-1 \\ x^2 & n > 2M. \end{cases}$$

Here $F^2 = \{g_n\}_{n=0}^{\infty}$, where $g_n(x) = x^2$, for all $n \geq 0$. Since map x^2 on $[0, 1]$ has P.O.T.P. (by Lemma 4.1 in [14]), F^2 has P.O.T.P. By above theorem we get that F has P.O.T.P.

3.2 Shadowing Property for an Invertible Nonautonomous Discrete Dynamical System

We define and study δ-pseudo orbit and shadowing property for an invertible nonautonomous discrete dynamical system.

Definition 3.2.1 *Let (X, d) be a metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X. For $\delta > 0$, the sequence $\{x_n\}_{n=-\infty}^{\infty}$ in X is said to be* **a δ-pseudo orbit** *of F if*

$$d(f_{n+1}(x_n), x_{n+1}) < \delta, \quad for \;\; n \geq 0$$

and

$$d(f_{-n}^{-1}(x_{n+1}), x_n) < \delta, \quad for \;\; n \leq -1$$

*For given $\varepsilon > 0$, a δ-pseudo orbit $\{x_n\}_{n=-\infty}^{\infty}$ is said to be ε-***traced** *by $y \in X$ if*

$$d(F_n(y), x_n) < \varepsilon \;\; for \;\; all \;\; n \in \mathbb{Z}.$$

The time varying homeomorphism F is said to have **shadowing property or pseudo orbit tracing property** *(P.O.T.P) if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit is ε- traced by some point of X.*

Remark 3.4 *Note that $\{x_n\}_{n=-\infty}^{\infty}$ in X is a δ-pseudo orbit of F if for $n \geq 1$ we have $d(f_n(x_{n-1}), x_n) < \delta$ and for $n \leq -1$ we have $d(f_n(x_{n-1}), x_n) < \delta$.*

Remark 3.5 *If in the above definition $f_n = f$, for all $n \geq 0$, where $f : X \to X$ is homeomorphism then P.O.T.P of time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on X is equivalent to P.O.T.P. of f on X* [2].

Remark 3.6 *Note that shadowing property of time varying homeomorphism F is independent of the choice of metric if X is compact.*

Now we show that shadowing is a property for a time varying homeomorphism which is preserved under uniform conjugacy.

Theorem 3.2.1 *Let a time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ on a metric space (X, d_1) be uniformly conjugate to time varying homeomorphism $G = \{g_n\}_{n=0}^{\infty}$ on metric space (Y, d_2). Then F has shadowing property if and only if G has shadowing property.*

Proof Since F is uniformly continuous on G, there exists a uniform homeomorphism $h : X \to Y$ such that $h \circ f_n = g_n \circ h$ for all $n \geq 0$ i.e. $f_n \circ h^{-1} = h^{-1} \circ g_n$ for all $n \geq 0$. Therefore $h \circ F_n = G_n \circ h$ for all $n \geq 0$.

Given any $\varepsilon > 0$, applying the uniform continuity of h, there exists $0 < \varepsilon_1 < \varepsilon$ such that for any $x_1, x_2 \in X$ with $d_1(x_1, x_2) < \varepsilon_1$,

$$d_2(h(x_1), h(x_2)) < \varepsilon.$$

As F has P.O.T.P., there exists $0 < \delta_1 < \varepsilon_1$ such that every δ_1-pseudo orbit of F is ε_1-traced by some point of X. Noting the fact that h^{-1} is uniformly continuous, there exists $0 < \delta < \delta_1$ such that for any $y_1, y_2 \in Y$ with $d_2(y_1, y_2) < \delta$,

$$d_1(h^{-1}(y_1), h^{-1}(y_2)) < \delta_1.$$

Now, we assert that every δ-pseudo orbit of G is ε-traced by some point of Y. In fact, for any δ-pseudo orbit $\{y_n\}$ of G, applying

$$d_1(f_n(h^{-1}(y_n)), h^{-1}(y_{n+1})) = d_1(h^{-1}(g_n(y_n)), h^{-1}(y_{n+1})) < \delta_1,$$

it follows that $\{h^{-1}(y_n)\}$ is a δ_1-pseudo orbit of F. Then there exists $x \in X$ such that $\{h^{-1}(y_n)\}$ is ε_1-traced by x. This implies that for any $n \in \mathbb{Z}$,

$$d_2(G_n(h(x)), y_n) = d_2(h(F_n(x)), h(h^{-1}(y_n))) < \varepsilon.$$

By similar arguments using uniform continuity of h, one can prove the converse.

Let (X, d_1) and (Y, d_2) be metric spaces. Define metric d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}, \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

By similar arguments given in the Theorem 3.1.1, one can prove the following result:

Theorem 3.2.2 *Let $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$ be time varying homeomorphisms on X and Y respectively. Then F and G have shadowing property if and only if the time varying homeomorphism $F \times G = \{f_n \times g_n\}_{n=0}^{\infty}$*

has shadowing property on $X \times Y$. Hence every finite direct product of time varying homeomorphisms having shadowing property, has shadowing property.

Now we show that if a time varying map has P.O.T.P. then so its inverse map has.

Theorem 3.2.3 *Let $F = \{f_n\}_{n=0}^{\infty}$ be the time varying homeomorphism on a metric space (X, d). Then F has P.O.T.P. if and only if F^{-1} has P.O.T.P.*

Proof : The proof follows observing that $\{x_n\}$ is a δ-pseudo orbit of F if and only if $\{y_n\}$ (where $y_n = x_{-n},\ n \in \mathbb{Z}$) is a δ-pseudo orbit of F^{-1}. In fact, for $\delta > 0$, the sequence $\{x_n\}_{n=-\infty}^{\infty}$ in X is a δ-pseudo orbit of F if

$$d(f_{n+1}(x_n), x_{n+1}) < \delta, \quad for \ \ n \geq 0$$

and

$$d(f_{-n}^{-1}(x_{n+1}), x_n) < \delta, \quad for \ \ n \leq -1$$

i.e.

$$d(f_n(x_{n-1}), x_n) < \delta, \quad for \ \ n \geq 1$$

and

$$d(f_{-n+1}^{-1}(x_n), x_{n-1}) < \delta, \quad for \ \ n \leq 0$$

Putting $n = -m$ we have

$$d(f_{-m}(x_{-(m+1)}), x_{-m}) < \delta, \quad for \ \ -m \geq 1$$

and

$$d(f_{m+1}^{-1}(x_{-m}), x_{-(m+1)}) < \delta, \quad for \ \ -m \leq 0$$

i.e.

$$d(f_{-m}(y_{m+1}), y_m) < \delta, \quad for \ \ m \leq -1$$

and

$$d(f_{m+1}^{-1}(y_m), y_{m+1}) < \delta, \quad for \ \ m \geq 0$$

Let $F^{-1} = \{g_n\}$, where $g_n = f_n^{-1},\ n = 0, 1, 2, ...$; then we have

$$d(g_{-m}^{-1}(y_{m+1}), y_m) < \delta, \quad for \ \ m \leq -1$$

and

$$d(g_{m+1}^{-1}(y_m), y_{m+1}) < \delta, \quad for \;\; m \geq 0$$

Thus $\{y_n\}$ is δ-pseudo orbit for F^{-1}. If $y \in X$ ε-traces $\{x_n\}_{n=-\infty}^{\infty}$ then

$$d(F_n(y), x_n) < \varepsilon \quad for \;\; all \;\; n \in \mathbb{Z}.$$

Equivalently

$$d(F_{-n}(y), x_{-n}) < \varepsilon \quad for \;\; all \;\; n \in \mathbb{Z}.$$

i.e.

$$d(G_n(y), y_n) < \varepsilon \quad for \;\; all \;\; n \in \mathbb{Z}.$$

where

$$\begin{aligned} G_n(y) &= g_n \circ g_{n-1} \circ \cdots \circ g_2 \circ g_1 \\ &= f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_2^{-1} \circ f_1^{-1}(y) \\ &= F_{-n}(y) \end{aligned}$$

Thus $y \in X$ ε-traces $\{y_n\}_{n=-\infty}^{\infty}$.

Using the above result and similar arguments given in Theorem 3.1.2 and Theorem 3.1.4 we get the following result.

Theorem 3.2.4 *Let (X, d) be a compact metric space, $\{f_n\}_{n=0}^{\infty}$ be an equicontinuous family of self homeomorphisms on X. Then time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$ has P.O.T.P. if and only if F^k has P.O.T.P., for any $k \in \mathbb{Z} - \{0\}$.*

3.3 Topological stability in Nonautonomous Discrete Dynamical Systems

Let (X, d) be a compact metric space, define standard bounded metric d_1 on X by

$$d_1(x, y) = \max\{d(x, y), \; 1\}, \quad x, y \in X.$$

and $(C(X), \eta)$ be the space of all continuous self maps on X, where metric η is defined by

$$\eta(f, g) = \sup_{x \in X} d_1(f(x), g(x)), \quad f, g \in C(X)$$

A time varying map, $F = \{f_n\}_{n=0}^{\infty}$, where $f_n : X \to X$ is a sequence of continuous maps, is a countable subset of $C(X)$. Let $S(X)$ be the collection of all time varying continuous maps on X. We define a metric ρ on $S(X)$ as follows:
For $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$,

$$\rho(F, G) = \sup_{n \geq 0} \eta(f_n, g_n).$$

Now we define topological stability in a nonautonomous discrete dynamical system.

Definition 3.3.1 *A time varying map $F = \{f_n\}_{n=0}^{\infty}$, where $f_n : X \to X$ is a sequence of continuous maps, is said to be* **topologically stable** *in $S(X)$, if for every $\varepsilon > 0$, there exists a δ, $0 < \delta < 1$ such that for a time varying map G with $\rho(F, G) < \delta$, there is a continuous map h so that for all $x \in X$, $d(h(x), x) < \varepsilon$ and $d(F_n(h(x)), G_n(x)) < \varepsilon$, for all $n \geq 0$.*

The following result shows that expansivity and shadowing imply topological stability for a nonautonomous discrete dynamical system.

Theorem 3.3.1 *If a time varying map $F = \{f_n\}_{n=0}^{\infty}$, where $f_n : X \to X$ is a sequence of continuous maps, on a compact metric space X is expansive and has P.O.T.P. then it is topologically stable in $S(X)$.*

Proof Let $e > 0$ be an expansive constant for time varying map $F = \{f_n\}_{n=0}^{\infty}$ and fix $0 < \varepsilon < \frac{e}{3}$. Let $0 < \delta < \min\{\frac{\varepsilon}{3}, 1\}$ be such that every δ-pseudo orbit of F can be ε-traced by some F-orbit.

By expansiveness of F, it follows that there exists a unique $x \in X$ which ε-traces a given δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$. Indeed, let $y \in X$ be another ε-tracing point of $\{x_i\}_{i=0}^{\infty}$. Then we have

$$d(F_n(x), x_n) < \varepsilon \;\; and \;\; d(F_n(y), x_n) < \varepsilon,$$

for all $n \geq 0$. Now

$$d(F_n(x), F_n(y)) \leq d(F_n(x), x_n) + d(x_n, F_n(y)) \leq 2\varepsilon < e,$$

for all $n \geq 0$ and hence $x = y$.

Let $G = \{g_n\}_{n=0}^{\infty}$, where $g_n : X \to X$ is continuous map, be such that $\rho(F, G) < \delta$. Since $\delta < 1$,

$$d(f_n(x), g_n(x)) < \delta,$$

for all $x \in X$ and for all $n \geq 0$. Let $x \in X$. Since

$$d(f_{n+1}(G_n(x)), G_{n+1}(x)) = d(f_{n+1}(G_n(x)), g_{n+1}(G_n(x))) < \delta,$$

for all $n \geq 0$, $\{G_n(x)\}_{n=0}^{\infty}$ is a δ pseudo orbit for $F = \{f_n\}_{n=0}^{\infty}$. Thus there exists a unique point $h(x) \in X$ whose F-orbit ε-traces $\{G_n(x)\}_{n=0}^{\infty}$. This defines a map $h\colon X \to X$ with

$$d(F_n(h(x)), G_n(x)) < \varepsilon,$$

for $n \geq 0$ and $x \in X$. Letting $n = 0$, we have

$$d(h(x), x) < \varepsilon,$$

for each $x \in X$.

Finally, we show that h is continuous. Let $\lambda > 0$. Then we can choose $N > 0$ such that,

$$0 \leq n \leq N, \quad d(F_n(x), F_n(y)) < e \Rightarrow d(x, y) < \lambda. \tag{3.3}$$

Suppose this is false. Let α be an open cover of X with diameter less than e. Then there exists an $\varepsilon > 0$ such that for each $j \geq 0$, there exist $x_j, y_j \in X$ with

$$d(x_j, y_j) > \varepsilon \;\; and \;\; A_{j,i} \in \alpha \; (0 \leq i \leq j)$$

with $F_i(x_j), F_i(y_j) \in A_{j,i}$, $0 \leq i \leq j$. Since X is a compact metric space, passing to a subsequence if necessary, without loss of generality, we can assume that there exist $x, y \in X$ such that $x_j \to x$ and $y_j \to y$. Note that $x \neq y$ as for each j, $d(x_j, y_j) > \varepsilon$.

Consider the sets $A_{j,0}$ for $j \geq 0$. Since X is compact therefore α has finite subcover and hence infinitely many $A_{j,0}$ coincide with some $A_0 \in \alpha$. Thus $x_j, y_j \in A_0$ for infinitely many j which implies $x, y \in \overline{A_0}$. Similarly for infinitely many $A_{j,n}$ they coincide with some $A_n \in \alpha$ and we have $F_n(x_j), F_n(y_j) \in A_n$ for infinitely many j. Thus F_n being continuous, $F_n(x), F_n(y) \in \overline{A_n}$. Since

$$diam(A_n) = diam(\overline{A_n}) < e,$$

we have

$$d(F_n(x), F_n(y)) < e,$$

for all $n \geq 0$, contradicting the fact that F is expansive.

Since $\{G_n \colon 0 \leq n \leq N\}$ is uniformly equicontinuous on X, we can choose $\eta > 0$ such that

$$d(x, y) < \eta \;\; implies \;\; d(G_n(x), G_n(y)) < \frac{e}{3},$$

for all n, $0 \leq n \leq N$. If $d(x, y) < \eta$ then for all n, $0 \leq n \leq N$, we have

$$\begin{aligned}
& d(F_n(h(x)), F_n(h(y))) \\
& \leq d(F_n(h(x)), G_n(x)) + d(G_n(x), G_n(y)) + d(G_n(y), F_n(h(y))) \\
& \leq \varepsilon + \frac{e}{3} + \varepsilon \\
& < e.
\end{aligned}$$

Thus by (3.3),

$$d(h(x), h(y)) < \lambda.$$

Therefore

$$d(x, y) < \eta \;\; implies \;\; d(h(x), h(y)) < \lambda$$

proving continuity of h.

Next we prove topological stability for an invertible nonautonomous discrete dynamical system :

Let (X, d) be a compact metric space, define standard bounded metric d_1 on X by

$$d_1(x, y) = \max\{d(x, y),\ 1\},\quad x, y \in X.$$

and $(\mathcal{H}(X), \eta)$ be the space of all homeomorphisms on X, where metric η is defined by

$$\eta(f, g) = \sup_{x \in X} d_1(f(x), g(x)),\quad f, g \in \mathcal{H}(X)$$

Let $\mathcal{G}(X)$ be the collection of all time varying homeomorphisms on X. We define a metric ρ on $\mathcal{G}(X)$ as follows:
For $F = \{f_n\}_{n=0}^{\infty}$ and $G = \{g_n\}_{n=0}^{\infty}$,

$$\rho(F, G) = \max\{\sup_{n \geq 0} \eta(f_n, g_n), \sup_{n \geq 0} \eta(f_n^{-1}, g_n^{-1})\}.$$

Definition 3.3.2 *A time varying homeomorphism F is said to be* **topologically stable** *in $\mathcal{G}(X)$, if for every $\varepsilon > 0$, there exists a δ, $0 < \delta < 1$ such that for a time varying homeomorphism G with $\rho(F, G) < \delta$ there is a continuous map h so that for all $x \in X$,*

$$d(h(x), x) < \varepsilon \ \text{ and } \ d(F_n(h(x)), G_n(x)) < \varepsilon,$$

for all $n \in \mathbb{Z}$.

Theorem 3.3.2 *Let (X, d) be a compact metric space and Let $F = \{f_n\}_{n=0}^{\infty}$ be a time varying homeomorphism on X which is expansive and has shadowing property then F is topologically stable in $\mathcal{G}(X)$.*

Proof Let $c > 0$ be an expansive constant for time varying homeomorphism $F = \{f_n\}_{n=0}^{\infty}$. Choose η such that $0 < \eta < \frac{c}{3}$.

Since F has shadowing property therefore for above η, we get δ, $0 < \delta < \min\{\frac{\epsilon}{3}, 1\}$ such that every δ-pseudo orbit of F can be η-traced by some F-orbit. Using expansiveness of F, one can prove that every δ-pseudo orbit is η-traced by unique $x \in X$.

Let $G = \{g_n\}_{n=0}^{\infty}$ be a time varying map on X such that $\rho(F, G) < \delta$. Let $x \in X$. Since

$$d(f_n(G_{n-1}(x)), G_n(x)) = d(f_n(G_{n-1}(x)), g_n(G_{n-1}(x))) < \delta, \quad for \ \ all \ \ n \geq 0$$

and

$$d(f_{-n}^{-1}(G_{n+1}(x)), G_n(x)) = d(f_{-n}^{-1}(G_n(x)), g_{-n}^{-1}(G_n(x))) < \delta, \quad for \ \ all \ \ n < 0$$

therefore $\{G_n(x)\}_{n=-\infty}^{\infty}$ is a δ pseudo orbit for $F = \{f_n\}_{n=0}^{\infty}$. Let $h(x) \in X$ be unique element of X whose F-orbit η-traces $G_n(x)$. So we get a map $h\colon X \to X$ with

$$d(F_n(h(x)), G_n(x)) < \eta,$$

for $n \in \mathbb{Z}$ and $x \in X$. Letting $n = 0$, we have

$$d(h(x), x) < \eta,$$

for each $x \in X$. Note that h is continuous by similar arguments given in Theorem 3.3.1.