Chapter 5

Fractional q-integration and q-differentiation

5.1 Introduction

The principal object of this chapter is to introduce the operators involving the functions (3.1.3) and (3.1.4), given by

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q) = \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s\ (q;q)_n} z^n,$$

and

$$e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q) = \sum_{n=0}^{\infty} \frac{(q^{\alpha n+\beta};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s\ (q;q)_n} z^n,$$

respectively, in the space $L(a,b)$ of Lebesgue measurable real or complex functions. Certain properties of Riemann - Liouville fractional q-integral and q-differential operators associated with the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ and $e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ are studied.

5.2 Main results for $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$

5.2.1 Fractional q-operators

In this section the following results are proved.

Theorem 5.2.1. *Let $a \in [0,\infty)$ and $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N}$, $\eta > 0$ then for $x > a$*

$$\begin{aligned} & \left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x) \\ = \; & [x-|a]_{\beta+\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega\,[x-|aq^{\beta+\eta-1}]_{\alpha};s,r|q), \end{aligned} \tag{5.2.1}$$

and

$$\begin{aligned} & \left({}_qD^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x) \\ = \; & [x-|a]_{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_{\alpha};s,r|q). \end{aligned} \tag{5.2.2}$$

Proof. The proofs are straight forward.
To prove (5.2.1), one may begin with

$$\begin{aligned} l.h.s. &= \left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x) \\ &= \left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\; q^{pn(n-1)/2}\;[\Gamma_q(\gamma+\delta n)]^s\;\omega^n}{\Gamma_q(\beta+\alpha n)\;[\Gamma_q(\lambda+\mu n)]^r\;(q;q)_n}[t-|aq^{\beta-1}]_{\alpha n}\right. \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{pn}\; q^{pn(n-1)/2}\;[\Gamma_q(\gamma+\delta n)]^s\;\omega^n}{\Gamma_q(\beta+\alpha n)\;[\Gamma_q(\lambda+\mu n)]^r\;(q;q)_n}\;{}_qI^{\eta}_{a+}\big([t-|a]_{\alpha n+\beta-1}\big)(x) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{pn}\; q^{pn(n-1)/2}\;[\Gamma_q(\gamma+\delta n)]^s\;\omega^n}{\Gamma_q(\beta+\alpha n)\;[\Gamma_q(\lambda+\mu n)]^r\;(q;q)_n}\;\frac{\Gamma_q(\alpha n+\beta)}{\Gamma_q(\alpha n+\beta+\eta)}[x-|a]_{\alpha n+\beta+\eta-1} \\ &= [x-|a]_{\beta+\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega[x-|aq^{\beta+\eta-1}]_{\alpha};s,r|q) \end{aligned}$$

which is the right hand side.
Similarly for proving (5.2.2), one may consider

$$\begin{aligned} l.h.s. &= \left({}_qD^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x) \\ &= \left({}_qD^{\eta}_{a+}[t-|a]_{\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\; q^{pn(n-1)/2}\;[\Gamma_q(\gamma+\delta n)]^s\;\omega^n}{\Gamma_q(\beta+\alpha n)\;[\Gamma_q(\lambda+\mu n)]^r\;(q;q)_n}[t-|aq^{\beta-1}]_{\alpha n}\right. \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{pn}\; q^{pn(n-1)/2}\;[\Gamma_q(\gamma+\delta n)]^s\;\omega^n}{\Gamma_q(\beta+\alpha n)\;[\Gamma_q(\lambda+\mu n)]^r\;(q;q)_n}\;{}_qD^{\eta}_{a+}\big([t-|a]_{\alpha n+\beta-1}\big)(x) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} \frac{\Gamma_q(\alpha n+\beta)}{\Gamma_q(\alpha n+\beta-\eta)} [x-|a]_{\alpha n+\beta-\eta-1}$$
$$= [x-|a]_{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_\alpha; s, r|q)$$
$$= r.h.s.$$

□

Next is

Theorem 5.2.2. *Let* $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N},\ \eta > 0$ *then*

$$ {}_qI^{\eta}_{0+}\left[\nu\ E^{1,\delta}_{\alpha,1,\lambda,\mu}(\nu t^{\alpha}; s, r|q)\right](x) = \nu\ (x(1-q))^{\eta}\ E^{1,\delta}_{\alpha,\eta+1,\lambda,\mu}(\nu x^{\alpha}; s, r|q). \qquad (5.2.3)$$

Proof. Here

$$l.h.s.$$
$$= {}_qI^{\eta}_{0+}\left[\nu\ E^{1,\delta}_{\alpha,1,\lambda,\mu}(\nu t^{\alpha}; s, r|q)\right](x)$$
$$= {}_qI^{\eta}_{0+}\left[\nu \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ (\nu t^{\alpha})^n}{[(q^{1+\delta n};q)_\infty]^s}\right]$$
$$= \nu \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \nu^n}{[(q^{1+\delta n};q)_\infty]^s}\ {}_qI^{\eta}_{0+}(t^{\alpha n})(x)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn} q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \nu^{n+1}}{[(q^{1+\delta n};q)_\infty]^s}$$
$$\times \frac{1}{\Gamma_q(\eta)} \int_0^x t^{\alpha n}(x-|tq)_{\eta-1}\ \mathrm{d_q}t.$$

Now taking $t = xu$ and using (1.2.33), then

$$l.h.s. = \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \nu^{n+1}}{[(q^{1+\delta n};q)_\infty]^s}$$
$$\times \frac{x^{\alpha n+\eta}}{\Gamma_q(\eta)} \int_0^1 u^{\alpha n}\ (uq;q)_{\eta-1}\ \mathrm{d_q}u$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \nu^{n+1}}{[(q^{1+\delta n};q)_\infty]^s\ (q;q)_n}$$
$$\times \frac{x^{\eta+\alpha n}}{\Gamma_q(\eta)} \frac{(1-q)\ (q;q)_\infty\ (q^{\alpha n+\eta+1};q)_\infty}{(q^{\eta};q)_\infty\ (q^{\alpha n+1};q)_\infty}$$
$$= \nu\ (x(1-q))^{\eta} \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty\ \nu^n}{[(q^{1+\delta n};q)_\infty]^s\ (q;q)_n}$$

$$\times (q^{\alpha n+\eta+1};q)_\infty \, x^{\alpha n}$$
$$= \; r.h.s.$$

□

Theorem 5.2.3. *Let $a \in [0,\infty)$, $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N}$, $\eta,\nu > 0$ for $x > a$, then*

$$\Big({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_\alpha; s,r|q)\Big)(x)$$
$$= [x-|a]_{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_\alpha; s,r|q). \tag{5.2.4}$$

Proof. With the use of (1.6.6),

$$\begin{aligned}
&\Big({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_\alpha; s,r|q)\Big)(x)\\
=\;& \Big({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\,q^{pn(n-1)/2}\,[\Gamma_q(\gamma+\delta n)]^s\,\omega^n}{\Gamma_q(\beta+\alpha n)\,[\Gamma_q(\lambda+\mu n)]^r\,(q;q)_n}[t-|aq^{\beta-1}]_{\alpha n}\\
=\;& \sum_{n=0}^{\infty}\frac{(-1)^{pn}\,q^{pn(n-1)/2}\,[\Gamma_q(\gamma+\delta n)]^s\,\omega^n}{\Gamma_q(\beta+\alpha n)\,[\Gamma_q(\lambda+\mu n)]^r\,(q;q)_n}\,{}_qD^{\eta,\nu}_{a+}\big([t-|a]_{\alpha n+\beta-1}\big)(x)\\
=\;& \sum_{n=0}^{\infty}\frac{(-1)^{pn}\,q^{pn(n-1)/2}\,[\Gamma_q(\gamma+\delta n)]^s\,\omega^n}{\Gamma_q(\beta+\alpha n)\,[\Gamma_q(\lambda+\mu n)]^r\,(q;q)_n}\,\frac{\Gamma_q(\alpha n+\beta)}{\Gamma_q(\alpha n+\beta-\eta)}[x-|a]_{\alpha n+\beta-\eta-1}\\
=\;& [x-|a]_{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_\alpha; s,r|q).
\end{aligned}$$

□

A q-analogue of the operator $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}$ defined in (4.1.1) may be given as follows.

Definition 5.2.1. *Let $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N}$, $r \in \mathbb{N}\cup\{-1,0\}$, $s \in \mathbb{N}\cup\{0\}$, $\omega \in \mathbb{C}$ and $x > a$*

$$\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}f\right)(x) = \int_a^x (x-|tq)_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-tq^\beta)_\alpha; s,r|q)\, f(t)\,\mathrm{d_q}t, \tag{5.2.5}$$

where $\alpha^2 + r\mu^2 - s\delta^2 + 1 > 0$.

And

Definition 5.2.2. *Let $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N}$, $r \in \mathbb{N}\cup\{-1,0\}$, $s \in \mathbb{N}\cup\{0\}$, $\omega \in \mathbb{C}$ and $x > a$*

$$\left({}_q\mathfrak{e}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}f\right)(x) = \int_a^x (x-|tq)_{\beta-1}\, e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-tq^\beta)_\alpha; s,r|q)\, f(t)\,\mathrm{d_q}t, \tag{5.2.6}$$

where $\left|(1-q)^{\alpha+r\mu-s\delta+1}\right| < 1$.

These q-operators are also turn out to be bounded. This is proved in

Theorem 5.2.4. *Let the function* ϕ *be in the space* $L(a,b) = \left\{f : {}_q\|f\|_1 = \int_a^b |f(t)|\ \mathrm{d_q}t < \infty\right\}$ *of Lebesgue measurable functions on a finite interval* $[a,b]$. *Then the integral operator* ${}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}$ *is bounded on* $L(a,b)$.

Proof. It is suffice to show that

$$
\begin{aligned}
&{}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}\phi\|_1 \\
=\ &\int_a^b \left|\int_a^x [x-tq]_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[x-tq^{\beta}]_{\alpha};s,r|q)\ \phi(t)\ \mathrm{d_q}t\right| \mathrm{d_q}x < \infty.
\end{aligned}
$$

By Fubini's theorem,

$$
\begin{aligned}
&{}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}\phi\|_1 \\
=\ &\int_a^b \left|\int_a^x [x-tq]_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[x-tq^{\beta}]_{\alpha};s,r)\ \phi(t)\ \mathrm{d_q}t\right| \mathrm{d_q}x \\
\leq\ &\int_a^b \left[\int_t^b [x-tq]_{\beta-1}\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\ \omega[x-tq^{\beta}]_{\alpha};s,r)\right| \mathrm{d_q}x\right]|\phi(t)|\ \mathrm{d_q}t \\
=\ &\left|\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{\alpha n+\beta};q)_{\infty}\ \omega^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\right| \\
&\times\int_a^b\int_t^b [x-tq]_{\alpha n+\beta-1}\ \mathrm{d_q}x\ |\phi(t)|\mathrm{d_q}t \\
=\ &\left|\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{\alpha n+\beta};q)_{\infty}\ \omega^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\right|\ \mathcal{I},
\end{aligned}
\tag{5.2.7}
$$

where

$$
\begin{aligned}
\mathcal{I}\ &=\ \int_a^b\int_t^b [x-tq]_{\alpha n+\beta-1}\ \mathrm{d_q}x\ |\phi(t)|\ d_qt \\
&\leq\ \int_a^b\left[\int_a^b [x-tq]_{\alpha n+\beta-1}\ \mathrm{d_q}x\right]\ |\phi(t)|\ d_qt
\end{aligned}
$$

$$\begin{aligned}
&= \int_a^b \left[[x-tq]_{\alpha n+\beta} \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \right]_a^b |\phi(t)| \, \mathrm{d_q}t \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \int_a^b \left([b-tq]_{\alpha n+\beta} - [a-tq]_{\alpha n+\beta} \right) |\phi(t)| \, \mathrm{d_q}t \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \left(\int_a^b [b-tq]_{\alpha n+\beta} |\phi(t)| \, \mathrm{d_q}t - \int_a^b [a-tq]_{\alpha n+\beta} |\phi(t)| \, \mathrm{d_q}t \right) \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) (\mathcal{I}_1 - \mathcal{I}_2), \; say. \qquad (5.2.8)
\end{aligned}$$

Here

$$\begin{aligned}
\mathcal{I}_1 &= \int_a^b [b-tq]_{\alpha n+\beta} |\phi(t)| \, d_q t \\
&= \left(\int_0^b [b-tq]_{\alpha n+\beta} |\phi(t)| \, \mathrm{d_q}t - \int_0^a [b-tq]_{\alpha n+\beta} |\phi(t)| \, \mathrm{d_q}t \right) \\
&= \int_0^b [b-tq]_{\alpha n+\beta} \, D_q \left(\int_0^t |\phi(u)| \, \mathrm{d_q}u \right) d_q t \\
&\quad - \int_0^a [b-tq]_{\alpha n+\beta} \, D_q \left(\int_0^t |\phi(u)| \, \mathrm{d_q}u \right) d_q t \\
&= \mathcal{I}_{11} - \mathcal{I}_{12}, \qquad (5.2.9)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_{11} &= \int_0^b [b-tq]_{\alpha n+\beta} \, D_q \left(\int_0^t |\phi(u)| \, \mathrm{d_q}u \right) \mathrm{d_q}t \\
&= \left[[b-tq]_{\alpha n+\beta} \left(\int_0^t |\phi(u)| \, \mathrm{d_q}u \right) \right]_0^b - \int_0^b \left(\int_0^t |\phi(qu)| \, \mathrm{d_q}u \right) \\
&\quad \times \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} [b-tq^2]_{\alpha n+\beta-1} \, \mathrm{d_q}t \\
&= [b-bq]_{\alpha n+\beta} \int_0^b |\phi(u)| \, \mathrm{d_q}u - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} \\
&\quad \times \int_0^b \left[\int_0^t |\phi(qu)| \, \mathrm{d_q}u \, [b-tq^2]_{\alpha n+\beta-1} \right] \mathrm{d_q}t \\
&\leq [b-bq]_{\alpha n+\beta} \int_0^b |\phi(u)| \, \mathrm{d_q}u - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q}
\end{aligned}$$

$$\times \int_0^b \left[\int_0^b |\phi(qu)| \; \mathrm{d_q}u \; [b - tq^2]_{\alpha n+\beta-1} \right] \mathrm{d_q}t.$$

Since $\phi \in L(a,b)$,

$$\int_0^b |\phi(u)| \; \mathrm{d_q}u = \mathcal{M}_1 (= a \; finite \; value)$$

hence

$$\begin{aligned}
\mathcal{I}_{11} &\leq [b - bq]_{\alpha n+\beta} \; \mathcal{M}_1 - \frac{(-q)(1 - q^{\alpha n+\beta})}{1-q} \int_0^b \mathcal{M}_1 \; [b - tq^2]_{\alpha n+\beta-1} \, \mathrm{d_q}t \\
&= \mathcal{M}_1 \Bigg([b - bq]_{\alpha n+\beta} - \frac{(-q)(1 - q^{\alpha n+\beta})}{1-q} \frac{1-q}{(-q)(1 - q^{\alpha n+\beta})} \\
&\quad \times [[b - tq]_{\alpha n+\beta}]_0^b \Bigg) \\
&= \mathcal{M}_1 \left([b - bq]_{\alpha n+\beta} - ([b - bq]_{\alpha n+\beta} - [b - 0q]_{\alpha n+\beta}) \right) \\
&= \mathcal{M}_1 \; b^{\alpha n+\beta}. \qquad (5.2.10)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{12} &= \int_0^a [b - tq]_{\alpha n+\beta} \; D_q \left(\int_0^t |\phi(u)| \; \mathrm{d_q}u \right) \mathrm{d_q}t \\
&= \left[[b - tq]_{\alpha n+\beta} \; \left(\int_0^t |\phi(u)| \; \mathrm{d_q}u \right) \right]_0^a - \int_0^a \left(\int_0^t |\phi(qu)| \; \mathrm{d_q}u \right) \\
&\quad \times \frac{(-q)(1 - q^{\alpha n+\beta})}{1-q} [b - tq^2]_{\alpha n+\beta-1} \, \mathrm{d_q}t \\
&= [b - aq]_{\alpha n+\beta} \int_0^a |\phi(u)| \; \mathrm{d_q}u - \frac{(-q)(1 - q^{\alpha n+\beta})}{1-q} \\
&\quad \times \int_0^a \left[\int_0^t |\phi(qu)| \; \mathrm{d_q}u \; [b - tq^2]_{\alpha n+\beta-1} \right] \mathrm{d_q}t \\
&\leq [b - aq]_{\alpha n+\beta} \int_0^a |\phi(u)| \; \mathrm{d_q}u - \frac{(-q)(1 - q^{\alpha n+\beta})}{1-q} \\
&\quad \times \int_0^b \left[\int_0^a |\phi(qu)| \; \mathrm{d_q}u \; [b - tq^2]_{\alpha n+\beta-1} \right] \mathrm{d_q}t.
\end{aligned}$$

Again since $\phi \in L(a,b)$,

$$\int\limits_0^a |\phi(u)|\ \mathrm{d_q}u = \mathcal{M}_2 (= a\ finite\ value)$$

on any finite interval $[0,a]$, and hence

$$\begin{aligned}
\mathcal{I}_{12} &\leq [b-bq]_{\alpha n+\beta}\ \mathcal{M}_2 - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} \int\limits_0^a \mathcal{M}_2\ [b-tq^2]_{\alpha n+\beta-1}\ \mathrm{d_q}t \\
&= \mathcal{M}_2 \left([b-aq]_{\alpha n+\beta} - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} \frac{1-q}{(-q)(1-q^{\alpha n+\beta})} \left[[b-tq]_{\alpha n+\beta}\right]_0^a \right) \\
&= \mathcal{M}_2 \left([b-aq]_{\alpha n+\beta} - ([b-aq]_{\alpha n+\beta} - [b-0q]_{\alpha n+\beta})\right) \\
&= \mathcal{M}_2\ b^{\alpha n+\beta}.
\end{aligned} \tag{5.2.11}$$

Using (5.2.10) and (5.2.11) in (5.2.9), one gets

$$\mathcal{I}_1 \leq \mathcal{M}_1\ b^{\alpha n+\beta} - \mathcal{M}_2\ b^{\alpha n+\beta}.$$

Similarly,

$$\mathcal{I}_2 \leq \mathcal{M}_1\ a^{\alpha n+\beta} - \mathcal{M}_2\ a^{\alpha n+\beta}.$$

Consequently (5.2.8) leads to

$$\begin{aligned}
\mathcal{I} &= \int_a^b \int\limits_t^b [x-tq]_{\alpha n+\beta-1}\ \mathrm{d_q}x\ |\phi(t)|\ \mathrm{d_q}t \\
&\leq \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right) \left(\left(\mathcal{M}_1\ b^{\alpha n+\beta} - \mathcal{M}_2\ b^{\alpha n+\beta}\right) - \left(\mathcal{M}_1\ a^{\alpha n+\beta} - \mathcal{M}_2\ a^{\alpha n+\beta}\right)\right) \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right) (\mathcal{M}_1 - \mathcal{M}_2)\ b^{\alpha n+\beta} - (\mathcal{M}_1 - \mathcal{M}_2)\ a^{\alpha n+\beta} \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right) (\mathcal{M}_1 - \mathcal{M}_2)\ \left(b^{\alpha n+\beta} - a^{\alpha n+\beta}\right).
\end{aligned}$$

Using this in (5.2.7), one finally finds

$$\begin{aligned}
&{}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}\phi\|_1 \\
&\leq \left| \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \right| \\
&\times \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1 - \mathcal{M}_2)\ \left(b^{\alpha n+\beta} - a^{\alpha n+\beta}\right)
\end{aligned}$$

$$\leq \sum_{n=0}^{\infty} \frac{q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ |\omega|^n}{[(q^{\gamma+\delta n};q)_\infty]^s}$$
$$\times \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1-\mathcal{M}_2)\ \left(b^{\alpha n+\beta}-a^{\alpha n+\beta}\right).$$

The proof follows in view of Theorem 3.2.1, Chapter-3 as the series on right hand side represents entire function. □

5.2.2 q-Gauss multiplication type formula

A q-analogue of Gauss multiplication theorem is given by [42, Eq.(2.27), p.483]

$$\prod_{k=1}^{m} \Gamma_{q^m}\left(z+\frac{k-1}{m}\right) = ([m]_q)^{\frac{1}{2}-mz} \left([2]_q\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}} \Gamma_q(mz), \qquad (5.2.12)$$

where in general, $m \in \mathbb{N},\ z \in \mathbb{C}$ and $[r]_q = \dfrac{1-q^r}{1-q}$.
The choice $z = n + \frac{\beta}{m}$ in (5.2.12) yields

$$\prod_{k=1}^{m} \Gamma_{q^m}\left(n+\frac{\beta}{m}+\frac{k-1}{m}\right) = ([m]_q)^{\frac{1}{2}-m\left(n+\frac{\beta}{m}\right)} \left([2]_q\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}$$
$$\times \Gamma_q\left(m\left(n+\frac{\beta}{m}\right)\right),$$

and replacing k by $k+1$, this becomes

$$\prod_{k=0}^{m-1} \Gamma_{q^m}\left(n+\frac{\beta}{m}+\frac{k}{m}\right) = ([m]_q)^{\frac{1}{2}-mn-\beta} \left([2]_q\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}} \Gamma_q\left(mn+\beta\right).$$

Alternatively,

$$\frac{1}{\Gamma_q\left(mn+\beta\right)} = \frac{([m]_q)^{\frac{1}{2}-\beta} \left([2]_q\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}{([m]_q)^{mn} \prod\limits_{k=0}^{m-1} \Gamma_{q^m}\left(n+\frac{\beta+k}{m}\right)}.$$

In view of this, the function (3.1.1) for $\alpha = m \in \mathbb{N}$ can be expressed as

$$E_{m,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q) = \sum_{n=0}^{\infty} \frac{(-1)^{cn}\ q^{cn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+mn)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n,$$

where $c = m^2 + r\mu^2 - s\delta^2 + 1 > 0$.
Now

$$\begin{aligned} E^{\gamma,\delta}_{m,\beta,\lambda,\mu}(z;s,r|q) &= \sum_{n=0}^{\infty} \frac{(-1)^{cn}\ q^{cn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{[\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} \\ &\quad \times \frac{([m]_q)^{\frac{1}{2}-\beta} \left([2]_q\ \Gamma^2_{q^2}\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}{([m]_q)^{mn} \prod\limits_{k=0}^{m-1} \Gamma_{q^m}\left(n+\frac{\beta+k}{m}\right)}\ z^n \\ &= \frac{\left([2]_q\ \Gamma^2_{q^2}\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}{([m]_q)^{\beta-\frac{1}{2}}} \left\{\prod_{k=0}^{m-1} \Gamma_{q^m}\left(\frac{\beta+k}{m}\right)\right\}^{-1} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^{cn}\ q^{cn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ z^n}{([m]_q)^{mn}\ \left(\frac{\beta+k}{m};q^m\right)_n\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}. \end{aligned} \tag{5.2.13}$$

This generalizes the result:

$$\begin{aligned} E^{\gamma}_{\alpha,\beta}(z|q) &= \frac{\left([2]_q\ \Gamma^2_{q^2}\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}{([m]_q)^{\beta-\frac{1}{2}}} \left\{\prod_{k=0}^{m-1} \Gamma_{q^m}\left(\frac{\beta+k}{m}\right)\right\}^{-1} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^{m^2 n}\ q^{m^2(n(n-1))/2}\ \Gamma_q(\gamma+n)\ z^n}{([m]_q)^{mn}\ \left(\frac{\beta+k}{m};q^m\right)_n\ (q;q)_n} \end{aligned}$$

due to Kilbas [31] with $r = 0, s = 1, \delta = 1$.

5.2.3 Fractional q-differential equations involving the q-analogue of Hilfer derivative operator

In this subsection, the fractional q-differential equations corresponding to the q-analogue of Hilfer derivative operator are obtained.
For that the following lemma is required.

Lemma 5.2.1. *In the notations of q-Laplace transform (1.5.1) and the operator defined by (5.2.5),*

$$\begin{aligned} \mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S) &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\ &\quad \times \frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}, \end{aligned}$$

in which the series is convergent for $r\mu^2 - s\delta^2 + 1 > 0$.

Proof. Consider

$$\begin{aligned}
&\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\phi\right)(x)\\
= &\int_0^x (x-|tq)_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega\,(x-tq^\beta)_\alpha; s,r|q)\, 1(t)\, \mathrm{d_q}t\\
= &\int_0^x (x-|tq)_{\beta-1} \sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; (q^{n+1};q)_\infty\, (q^{\alpha n+\beta};q)_\infty\, \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\, [(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times (x-tq^\beta)_{\alpha n}\;\, 1(t)\, \mathrm{d_q}t\\
= &\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; [(q^{\lambda+\mu n};q)_\infty]^r\, (q^{\alpha n+\beta};q)_\infty\, \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \int_0^x (x-|tq)_{\alpha n+\beta-1}\, 1(t)\, \mathrm{d_q}t.
\end{aligned}$$

By applying q-Laplace transform on both the sides, one gets

$$\begin{aligned}
&\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S)\\
= &\mathcal{L}_q\Bigg(\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; [(q^{\lambda+\mu n};q)_\infty]^r\, (q^{\alpha n+\beta};q)_\infty\, \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times \int_0^x (x-|tq)_{\alpha n+\beta-1}\, 1(t)\, \mathrm{d_q}t\Bigg)\\
= &\sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; [(q^{\lambda+\mu n};q)_\infty]^r\, (q^{\alpha n+\beta};q)_\infty\, \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times \mathcal{L}_q\left(\int_0^x (x-|tq)_{\alpha n+\beta-1}\, 1(t)\, \mathrm{d_q}t\right).
\end{aligned} \tag{5.2.14}$$

From the definition of q-Laplace transforms (1.5.2), it follows that

$$\mathcal{L}_q\left(x^{\alpha n+\beta-1}\right) = \frac{1}{1-q}\int_0^\infty e_q(-Sx)\, x^{\alpha n+\beta-1}\, \mathrm{d_q}x. \tag{5.2.15}$$

Here letting $Sx = t$ and by making use of the q-integral formula:

$$\int_0^\infty t^{\alpha-1}\; e_q(-t)\, \mathrm{d_q}t = \frac{(1-q)\;(q;q)_\infty\, q^{-\alpha(\alpha-1)/2}}{(q^\alpha;q)_\infty\; S^{\alpha n+1}},$$

one gets

$$\mathcal{L}_q(x^{\alpha n+\beta-1}) = \frac{(q;q)_\infty \, q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta}\,(q^{\alpha n+\beta};q)_\infty}.$$

In (5.2.14) using the convolution theorem:

$$\mathcal{L}_q\left[\int_0^x f_1(t)\, f_2(x-tq)\,\mathrm{d_q}t\right] \;=\; F_{1_q}(S)\; F_{2_q}(S), \tag{5.2.16}$$

whenever $F_{1_q}(S), F_{2_q}(S)$ exist, with $F_{1_q}(S) = \mathcal{L}_q(f_1(x))(S)$ and $F_{2_q}(S) = \mathcal{L}_q(f_2(x))(S)$, one finds

$$\begin{aligned}\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S) &= \sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;[(q^{\lambda+\mu n};q)_\infty]^r\;(q^{\alpha n+\beta};q)_\infty\,\omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\ &\quad\times\frac{(q;q)_\infty\,q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta}\,(q^{\alpha n+\beta};q)_\infty}\frac{1}{S}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;[(q^{\lambda+\mu n};q)_\infty]^r\;\omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\ &\quad\times\frac{(q;q)_\infty\,q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}.\end{aligned}$$

□

Theorem 5.2.5. *If* $0<\eta<\ 1,\ 0\le\nu\le 1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left({}_qD^{\eta,\,\nu}_{0+}\;y\right)(x) \;=\; \xi\,\left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)\,(x)\;+\;f(x) \tag{5.2.17}$$

with the initial condition

$$\left({}_qI_0+^{(1-\nu)(1-\eta)}\;\;y\right)(0+) \;=\; C, \tag{5.2.18}$$

has solution

$$\begin{aligned}y(x) \;&=\; C\,\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\,x^{\beta+\eta}\\ &\times(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\;E^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\,(\omega\,(xq^{\eta+1})^{\alpha};\;s,\;r|q)\\ &+\frac{(1-q)1-\eta\,q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)\;(x-|tq)_{\eta-1}\,\mathrm{d_q}t,\end{aligned} \tag{5.2.19}$$

in the space $L(0,\infty)$ *wherein* C *is arbitrary constant.*

Proof. Apply q-Laplace transform on both the sides of (5.2.17) to get

$$\mathcal{L}_q\left({}_qD_{0+}^{\eta,\,\nu}\,y\right)(x)(S) \;=\; \xi\,\mathcal{L}_q\left({}_q\mathcal{E}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;0+}^{\gamma,\,\delta}\right)(x)(S) + \;\mathcal{L}_q\,f(x)(S).$$

In the light of Lemma 5.2.1 and formula (1.7.8), this gives

$$\begin{aligned} S^{\eta}\,Y(S) - C.S^{\nu(1-\eta)} \;=\; & \xi S^{-\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;[(q^{\lambda+\mu n};q)_\infty]^r\;(q^{n+1};q)_\infty\,\omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\ & \times\frac{(q;q)_\infty\,q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n}} + F(S). \end{aligned}$$

That is,

$$\begin{aligned} Y(S) \;=\; & C S^{\nu(1-\eta)-\eta} + \xi\,S^{-\beta-\eta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;[(q^{\lambda+\mu n};q)_\infty]^r\;(q^{n+1};q)_\infty\,\omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\ & \times\frac{(q;q)_\infty\,q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n}} + S^{-\eta}F(S). \end{aligned}$$

Here making an appeal to the inverse q-Laplace transform given by

$$\mathcal{L}_q^{-1}\big(\frac{1}{S^{\alpha n+\beta}}\big) = \frac{(q^{\alpha n+\beta};q)_\infty\;x^{\alpha n+\beta-1}}{q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\;(q;q)_\infty},$$

one gets

$$\begin{aligned} \mathcal{L}_q^{-1}(Y(S)) \;=\; & C\mathcal{L}_q^{-1}(S^{\nu(1-\eta)-\eta})(x) \\ & +\xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\;(q^{n+1};q)_\infty\,\omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\;[(q^{\gamma+\delta n};q)_\infty]^s\;(q;q)_n\;(q;q)_\infty} \\ & \times\mathcal{L}_q^{-1}\big(\frac{1}{S^{\alpha n+\beta+\eta+1}}\big) + \mathcal{L}_q^{-1}\big(\frac{1}{S^{\eta}}F(S)\big). \end{aligned}$$

Thus,

$$\begin{aligned} y(x) \;=\; & C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1} \\ & +\xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\;q^{pn(n-1)/2}\;q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\;(q^{n+1};q)_\infty\,\omega^n\;(q;q)_\infty}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\;[(q^{\gamma+\delta n};q)_\infty]^s} \\ & \times\frac{(q^{\alpha n+\beta+\eta+1};q)_\infty\;x^{\alpha n+\beta+\eta}\;q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}}{(q;q)_\infty} \\ & +\frac{(1-q)^{1-\eta}\;q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t \end{aligned}$$

$$
\begin{aligned}
&= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1} \\
&\quad +\xi x^{\beta+\eta}\sum_{n=0}^{\infty} q^{\eta(\eta+1)/2}q^{(1+\eta)(\alpha n+\beta)}(q^{n+1};q)_\infty \\
&\quad \times\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta+\eta+1};q)_\infty\ (\omega\, x^\alpha)^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad +\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t \\
&= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\, x^{\beta+\eta} \\
&\quad \times q^{\eta(\eta+1)/2+\beta(\eta+1)}\ E^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\,(\omega\,(xq^{\eta+1})^\alpha;s,r|q) \\
&\quad +\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t.
\end{aligned}
$$

□

Theorem 5.2.6. *The q-differential equation*

$$\left({}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\ \left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)\,+\,x^\beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(ax)^\alpha);s,r|q\big) \tag{5.2.20}$$

with the initial condition

$$\left({}_qI^{(1-\nu)(1-\eta)}_{0+}\ y\right)(0+)\ =\ C, \tag{5.2.21}$$

has solution in the space $L(0,\infty)$ which is given by

$$
\begin{aligned}
y(x) = C\ &\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+q^{\eta(\eta+1)/2}\ x^{\beta+\eta} \\
&\times(\xi\ (1-q)^{-\eta-1}\ q^{\beta(\eta+1)}+1)\ E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}(\omega(xq^{\eta+1})^\alpha;s,r|q)
\end{aligned}
\tag{5.2.22}
$$

in which C is arbitrary constant.

Proof. The choice

$$f(t)\ =\ t^\beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(tq^{\eta+1})^\alpha);s,r|q\big),$$

in Theorem 5.2.5, yields

$$y(x)\ =\ C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\, x^{\beta+\eta}$$

$$\times(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\ E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\ (\omega\ (xq^{\eta+1})^{\alpha};s,r|q)$$
$$+\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\ \mathcal{I}, \tag{5.2.23}$$

where

$$\mathcal{I}\ =\ \int_0^x t^{\beta}\ (x-|tq)_{\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(tq^{\eta+1})^{\alpha});s,r|q\big)\ \mathrm{d_q}t.$$

Now,

$$\begin{aligned}\mathcal{I}\ &=\ \int_0^x t^{\beta}\ (x-|tq)_{\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(tq^{\eta+1})^{\alpha});s,r|q\big)\ \mathrm{d_q}t\\
&=\ \int_0^x (x-|tq)_{\eta-1}\ t^{\beta}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}}{[(q^{\alpha n+\beta+1};q)_{\infty}]^{-1}\ [(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times(\omega\ (tq^{\eta+1})^{\alpha})^n\ \mathrm{d_q}t\\
&=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}\ (\omega\ (q^{\eta+1})^{\alpha})^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times\int_0^x (x-|tq)_{\eta-1}\ t^{\alpha n+\beta}\ \mathrm{d_q}t.\end{aligned}$$

On making substitution $t = xu$ and then using (1.2.33) and (1.2.34) in turn, this becomes

$$\begin{aligned}\mathcal{I}\ &=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}\ (\omega\ (q^{\eta+1})^{\alpha})^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times x^{\eta+\alpha n+\beta}\int_0^1 (1-|uq)_{\eta-1}\ u^{\alpha n+\beta}\ \mathrm{d_q}u\\
&=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}\ (\omega\ (q^{\eta+1})^{\alpha})^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times x^{\eta+\alpha n+\beta}\ \mathfrak{B}(\alpha n+\beta+1,\eta)\\
&=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}\ (\omega\ (q^{\eta+1})^{\alpha})^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times x^{\eta+\alpha n+\beta}\ \frac{(1-q)\ (q;q)_{\infty}\ (q^{\alpha n+\beta+\eta+1};q)_{\infty}}{(q^{\eta};q)_{\infty}\ (q^{\alpha n+\beta+1};q)_{\infty}}\\
&=\ \frac{(1-q)\ (q;q)_{\infty}\ x^{\beta+\eta}}{(q^{\eta};q)_{\infty}}\ E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}(\omega\ (xq^{\eta+1})^{\alpha};s,r|q).\end{aligned}$$

Using this in (5.2.23), it gives

$$\begin{aligned} y(x) &= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1} \\ &\quad +\xi\, x^{\beta+\eta}(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega\,(xq^{\eta+1})^{\alpha};s,r|q\right) \\ &\quad +\, q^{\eta(\eta+1)/2}\,x^{\beta+\eta}\,E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega\,(xq^{\eta+1})^{\alpha};\ s,\ r|q\right) \\ &= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\left(1+\frac{\xi\,q^{\beta(\eta+1)}}{(1-q)^{\eta+1}}\right) \\ &\quad \times q^{\eta(\eta+1)/2}\,x^{\beta+\eta}\,E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega\,(xq^{\eta+1})^{\alpha};s,r|q\right). \end{aligned}$$

□

Theorem 5.2.7. *If* $0<\eta<\ 1,\ 0\le\nu\le1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left(x\ {}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)(x) \tag{5.2.24}$$

with the initial condition

$$\left({}_qI_0+^{(1-\nu)(1-\eta)}\ y\right)(0+)\ =\ C, \tag{5.2.25}$$

has solution in the space $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) &= (q^{\eta+1}-q)\sum_{j=0}^{\infty}q^j\,y(xq^j) \\ &\quad +C\frac{q^{\eta}(1-q^{-\nu(1-\eta)})}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)-1}\ (q^{\eta-\nu(1-\eta)+1};q)_\infty}{(q;q)_\infty\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right) \\ &\quad +\frac{x^{\beta+\eta-1}}{q^{-\eta(\eta-1)/2-\beta(\eta+1)}}E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}(\omega\,(xq^{\eta+1})^{\alpha};s,r|q), \end{aligned} \tag{5.2.26}$$

wherein C *is arbitrary constant.*

Proof. Applying q-Laplace transform on both the sides of (5.2.24), one gets

$$\mathcal{L}_q\left(x\ {}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)(S)\ =\ \xi\,\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S).$$

In this, the left hand side simplification is given by

$$\begin{aligned} \mathcal{L}_q\left(x\ {}_qD^{\eta,\nu}_{0+}\ y\right)(x)(S) &= \xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\ &\quad \times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}. \end{aligned} \tag{5.2.27}$$

Now from the formulas (1.5.5), (5.2.25), (1.2.41), and (1.2.39) in turn, it follows that

$$
\begin{aligned}
& \mathcal{L}_q \left[x \left({}_qD_{0+}^{\eta;\,\nu}\, y\right)(x)\right] \\
= & -\frac{1}{q}\, \Delta_q \left(\mathcal{L}_q \left({}_qD_{0+}^{\eta;\,\nu}\, y\right)(x)(S)\right) \\
= & -\frac{1}{q}\, \Delta_q \left(S^\eta\, \mathcal{L}_q[y(x)](S) - S^{\nu(1-\eta)}({}_qI_0^{(1-\nu)(1-\eta)} f)(0+)\right) \\
= & -\frac{1}{q}\, \Delta_q \left(S^\eta\, Y_q(S) - C S^{\nu(1-\eta)}\right) \\
= & -\frac{1}{q}\, \left(\Delta_q \left(S^\eta\, Y_q(S)\right) - C\Delta_q \left(S^{\nu(1-\eta)}\right)\right) \\
= & -\frac{1}{q}\, \left((Sq^{-1})^\eta\, \Delta_q(Y_q(S)) + Y_q(S)\, \Delta_q(S^\eta)\right) - C\Delta_q \left(S^{\nu(1-\eta)}\right) \\
= & -\frac{1}{q}\left((Sq^{-1})^\eta \Delta_q(Y_q(S)) + Y_q(S)\, \frac{S^\eta - (Sq^{-1})^\eta}{S - Sq^{-1}} - C\frac{S^{\nu(1-\eta)} - S^{\nu(1-\eta)}q^{-\nu(1-\eta)}}{S - Sq^{-1}}\right) \\
= & -\frac{1}{q}\, \left(S^\eta q^{-\eta}\, \Delta_q(Y_q(S)) + Y_q(S)\, \frac{S^\eta - S^\eta q^{-\eta}}{S - Sq^{-1}} - C\frac{S^{\nu(1-\eta)} - S^{\nu(1-\eta)}q^{-\nu(1-\eta)}}{S - Sq^{-1}}\right) \\
= & -\frac{1}{q}\, \left(q^{-\eta}\, \Delta_q(Y_q(S)) + \frac{1-q^{-\eta}}{1-q^{-1}} Y_q(S)\frac{1}{S} - C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}} S^{\nu(1-\eta)-\eta-1}\right).
\end{aligned}
$$

Using this in (5.2.27), one gets

$$
\begin{aligned}
& -\frac{1}{q}\, \left(q^{-\eta}\, \Delta_q(Y_q(S)) + \frac{1-q^{-\eta}}{1-q^{-1}} Y_q(S)\frac{1}{S} - C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}} S^{\nu(1-\eta)-\eta-1}\right) \\
= & \sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; [(q^{\lambda+\mu n}; q)_\infty]^r\; \omega^n}{[(q^{\gamma+\delta n}; q)_\infty]^s} \frac{(q;q)_\infty\, q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+\eta+1}}.
\end{aligned}
$$

Here applying the inverse q-Laplace transforms, one finds

$$
\begin{aligned}
& \frac{-q^{-\eta}}{q} \mathcal{L}_q^{-1} \left(\Delta_q(Y_q(S))\right) + \frac{1-q^{-\eta}}{1-q^{-1}} \mathcal{L}_q^{-1} \left(Y_q(S)\frac{1}{S}\right) \\
& -C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}} \mathcal{L}_q^{-1} \left(\frac{1}{S^{\eta-\nu(1-\eta)+1}}\right) \\
= & \sum_{n=0}^{\infty} \frac{(-1)^{pn}\; q^{pn(n-1)/2}\; [(q^{\lambda+\mu n}; q)_\infty]^r\; \omega^n}{[(q^{\gamma+\delta n}; q)_\infty]^s} \frac{(q;q)_\infty}{q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}}\; \mathcal{L}_q^{-1} \left(\frac{1}{S^{\alpha n+\beta+\eta+1}}\right).
\end{aligned}
$$

Once again using (1.5.5) and convolution theorem (5.2.16), it gives

$$
\frac{-q^{-\eta}}{q} \left(-q\; x\; y(x)\right) + \frac{1-q^{-\eta}}{1-q^{-1}} \left(\int_0^x y(t)\, \mathrm{d}_q t\right)
$$

$$
\begin{aligned}
& -C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)}\ (q^{\eta-\nu(1-\eta)+1};q)_\infty}{(q;q)_\infty\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right)\\
= & \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\frac{(q;q)_\infty}{q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}}\\
& \times\frac{x^{\alpha n+\beta+\eta}\ (q^{\alpha n+\beta+\eta+1};q)_\infty}{q^{-(\alpha n+\beta+\eta+1)(\alpha n+\beta+\eta)/2}\ (q;q)_\infty}.
\end{aligned}
$$

Finally, the q-integral formula (1.2.42):

$$
\int_0^x f(t)\ \mathrm{d_q}t\ =\ x(1-q)\sum_{k=0}^{\infty}q^k\ f(xq^k),
$$

simplifies this to

$$
\begin{aligned}
& x\ y(x)+(q-q^{\eta+1})\ x\ \sum_{j=0}^{\infty}q^j\ y(xq^j)\\
& -C\frac{q^\eta(1-q^{-\nu(1-\eta)})}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)}\ (q^{\eta-\nu(1-\eta)+1};q)_\infty}{(q;q)_\infty\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right)\\
= & \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ x^{\alpha n+\beta+\eta}\ (q^{\alpha n+\beta+\eta+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s\ q^{-\alpha n\eta+1}\ q^{-\eta(\eta-1)/2-\beta(\eta+1)}}\\
= & \ q^{\eta(\eta-1)/2-\beta(\eta+1)}x^{\beta+\eta}E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega\ (xq^{\eta+1})^\alpha;s,r|q\right).
\end{aligned}
$$

This is (5.2.26). □

5.2.4 Properties of $E_t(c,\nu,\gamma,\delta,\lambda,\mu|q)$ and $E_t(c,-\eta,\gamma,\delta,\lambda,\mu|q)$

In this section, certain properties of the function

$$
f(t)\ =\ \sum_{n=0}^{\infty}\frac{(-1)^{\ell n}\ q^{\ell n(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ [(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta n};q)_\infty]^s}\ (ct)^n
$$

are studied with reference to the function (3.1.1). Here the function is entire function with the condition $\ell=r\mu^2-s\delta^2+2,\ \Re(\ell)>0,\ |q|<1$ and c is arbitrary constant. At the first place the q-Riemann-Liouville fractional integral operator (1.7.1) given by:

$$
{}_qI^\mu_{a+}f(x)\ =\ \frac{1}{\Gamma_q(\mu)}\int_a^x(x-|yq)_{\mu-1}\ f(y)\ \mathrm{d}_qy
$$

is applied on $f(t)$ to get

$$
{}_qI^\nu_{0+}\ f(t)\ =\ \frac{1}{\Gamma_q(\nu)}\int_0^t(t-\xi\ q)_{\nu-1}
$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, (c\xi)^n\, [(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta\, n};q)_\infty]^s}\, \mathrm{d}_q\xi$$

$$= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, c^n\, [(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta\, n};q)_\infty]^s}$$

$$\times \int_0^t \xi^n (t-\xi\, q)_{\nu-1}\, \mathrm{d}_q\xi.$$

The substitution $\xi = tu$, simplifies this as follows.

$$\begin{aligned}
{}_qI^{\nu}_{0+}\, f(t) &= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, c^n\, [(q^{n+1};q)_\infty]^2\, t^{n+\nu}}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
&\quad \times \int_0^1 u^n\, (1-uq)_{\nu-1}\, \mathrm{d}_q u \\
&= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, c^n\, [(q^{n+1};q)_\infty]^2\, t^{n+\nu}}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
&\quad \times \int_0^1 u^{n+1-1} \frac{(uq;q)_\infty}{(uq^\nu;q)_\infty} \mathrm{d}_q u \\
&= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, c^n\, [(q^{n+1};q)_\infty]^2\, t^{n+\nu}}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
&\quad \times \mathfrak{B}_q(n+1,\nu) \\
&= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, c^n\, [(q^{n+1};q)_\infty]^2\, t^{n+\nu}}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
&\quad \times \frac{(1-q)\,(q;q)_\infty\,(q^{n+\nu+1};q)_\infty}{(q^{n+1};q)_\infty\,(q^\nu;q)_\infty} \\
&= t^\nu (1-q)^\nu \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, (q^{\nu+n+1};q)_\infty\, (q^{n+1};q)_\infty\, (ct)^n}{[(q^{\lambda+\mu\, n};q)_\infty]^{-r}\, [(q^{\gamma+\delta\, n};q)_\infty]^s}.
\end{aligned}$$

Finally, in view of (3.1.1), this gives

$${}_qI^{\nu}_{0+}\, f(t) = t^\nu\, (1-q)^\nu\, E^{\gamma,\delta}_{1,1+\nu,\lambda,\mu}(ct;s,r|q). \tag{5.2.28}$$

Here the function on the right hand side may be denoted by $E_t(c,\nu,\gamma,\delta,\lambda,\mu|q)$ to get

$${}_qI^{\nu}_{0+}\, f(t) \;=\; E_t(c,\nu,\gamma,\delta,\lambda,\mu|q). \tag{5.2.29}$$

Next, using the fractional q-differential operator (1.7.5) of order η:

$$\left({}_qD^{\alpha}_{a+} f\right)(x) = \left(\frac{\mathrm{d}_q}{\mathrm{d}_q x}\right)^n \left({}_qI^{n-\alpha}_{a+} f\right)(x)$$

on $f(t)$, and then using (5.2.28), one gets

$$
\begin{aligned}
{}_qD^{\eta}_{0+}\, f(t) &= D^k_q \left\{ {}_qI^{k-\eta}_{0+} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, [(q^{\lambda+\mu\, n};q)_\infty]^r\, (ct)^n\, [(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \right\} \\
&= D^k_q \left\{ t^{k-\eta}\, (1-q)^{k-\eta}\, E^{\gamma,\delta}_{1,1+k-\eta,\lambda,\mu}(ct;s,r|q) \right\} \\
&= t^{-\eta}\, (1-q)^{-\eta}\, E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu}(ct;s,r|q).
\end{aligned} \tag{5.2.30}
$$

With the notation

$$
E_t(c,-\eta,\gamma,\delta,\lambda,\mu|q) = t^{-\eta}\,(1-q)^{-\eta}\, E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu}(ct;s,r|q) \tag{5.2.31}
$$

it gives

$$
{}_qD^{\eta}_{0+}\, f(t) = E_t(c,-\eta,\gamma,\delta,\lambda,\mu|q). \tag{5.2.32}
$$

In the following theorems, the act of fractional q-integral and fraction q-differential operators are examined.

Theorem 5.2.8. *For $\gamma,\lambda,\delta,\mu \in \mathbb{N}, \nu > 0$, $r \in \mathbb{N} \cup \{-1,0\}$, $s \in \mathbb{N} \cup \{0\}$, c is arbitrary constant and fractional q-integral and q-differential operators are of order σ then*

$$
{}_qI^{\sigma}_{0+}\, E_t(c,\nu,\gamma,\delta,\lambda,\mu|q) = E_t(c,\sigma+\nu,\gamma,\delta,\lambda,\mu|q). \tag{5.2.33}
$$

$$
{}_qD^{\sigma}_{0+} E_t(c,\nu,\gamma,\delta,\lambda,\mu) = E_t(c,\nu-\sigma,\gamma,\delta,\lambda,\mu). \tag{5.2.34}
$$

Proof. From (1.7.1),

$$
{}_qI^{\sigma}_{0+}\, E_t(c,\nu,\gamma,\delta,\lambda,\mu|q) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-|\xi q)_{\sigma-1}\, E_\xi(c,\nu,\gamma,\delta,\lambda,\mu|q)\, \mathrm{d_q}\xi.
$$

Using (5.2.29), the above equation becomes

$$
\begin{aligned}
{}_qI^{\sigma}_{0+}\, E_t(c,\nu,\gamma,\delta,\lambda,\mu|q) &= \frac{1}{\Gamma_q(\sigma)} \int_0^t (t-|\xi q)_{\sigma-1}\, \xi^{\nu}\, (1-q)^{\nu} \\
&\times \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\, q^{\ell n(n-1)/2}\, (q^{\nu+n+1};q)_\infty\, [(q^{\lambda+\mu\, n};q)_\infty]^r}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
&\times (q^{n+1};q)_\infty\, (c\xi)^n \mathrm{d_q}\xi.
\end{aligned}
$$

In this the substitution $\xi = xt$, gives

$$
\begin{aligned}
& {}_qI^{\sigma}_{0+}\ E_t(c,\nu,\gamma,\delta,\lambda,\mu|q) \\
= & \frac{(1-q)^{\nu}}{\Gamma_q(\sigma)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{\nu+n+1};q)_\infty [(q^{\lambda+\mu\, n};q)_\infty]^r (q^{n+1};q)_\infty\ t^{\sigma+\nu+n}\ c^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
& \times \int_0^1 (1-|xq)_{\sigma-1}\ x^{\nu+n}\ \mathrm{d_q}x \\
= & \frac{(1-q)^{\nu}}{\Gamma_q(\sigma)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{\nu+n+1};q)_\infty [(q^{\lambda+\mu\, n};q)_\infty]^r (q^{n+1};q)_\infty\ t^{\sigma+\nu+n}\ c^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
& \times \int_0^1 x^{\nu+n+1-1}\ \frac{(xq;q)_\infty}{(xq^{\sigma};q)_\infty}\ \mathrm{d_q}x \\
= & \frac{(1-q)^{\nu}}{\Gamma_q(\sigma)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{\nu+n+1};q)_\infty [(q^{\lambda+\mu\, n};q)_\infty]^r (q^{n+1};q)_\infty\ t^{\sigma+\nu+n}\ c^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
& \times \mathfrak{B}_q(\sigma,\nu+n+1) \\
= & \frac{(1-q)^{\nu}}{\Gamma_q(\sigma)} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{\nu+n+1};q)_\infty [(q^{\lambda+\mu\, n};q)_\infty]^r (q^{n+1};q)_\infty\ t^{\sigma+\nu+n}\ c^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s} \\
& \times \frac{(1-q)\ (q;q)_\infty\ (q^{\sigma+\nu+n+1};q)_\infty}{(q^{\sigma};q)_\infty\ (q^{\nu+n+1};q)_\infty} \\
= & (1-q)^{\nu+\sigma}\ t^{\nu+\sigma} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{\sigma+\nu+n+1};q)_\infty (q^{n+1};q)_\infty\ (ct)^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s\ [(q^{\lambda+\mu\, n};q)_\infty]^{-r}}.
\end{aligned}
$$

Once again use of (5.2.29) gives (5.2.33). □

Now, from (1.7.5) and using (5.2.33),

$$
\begin{aligned}
& {}_qD^{\sigma}_{0+}E_t(c,\nu,\gamma,\delta,\lambda,\mu) \\
= & D^k_q\left[{}_qI^{k-\sigma}_{0+}\ E_t(c,\nu,\gamma,\delta,\lambda,\mu|q)\right] \\
= & D^k_q\left[E_t(c,k-\sigma+\nu,\gamma,\delta,\lambda,\mu|q)\right] \\
= & D^k_q\ \left\{t^{k-\sigma+\nu}\ E^{\gamma,\delta}_{1,k-\sigma+\nu+1,\lambda,\mu}(ct;s,r|q)\right\} \\
= & D^k_q\ \left\{t^{k-\sigma+\nu} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n-1)/2}\ (q^{n+k-\sigma+\nu+1};q)_\infty\ (q^{n+1};q)_\infty\ (ct)^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s\ [(q^{\lambda+\mu\, n};q)_\infty]^{-r}}\right\} \\
= & t^{-\sigma+\nu} \sum_{n=0}^{\infty} \frac{(-1)^{\ell n}\ q^{\ell n(n-1)/2}\ (q^{n-\sigma+\nu+1};q)_\infty\ [(q^{\lambda+\mu\, n};q)_\infty]^r\ (q^{n+1};q)_\infty\ (ct)^n}{[(q^{\gamma+\delta\, n};q)_\infty]^s}.
\end{aligned}
$$

Using (5.2.31), (5.2.34) is obtained.

In the light of Theorem 5.2.8, the following theorem may be proved.

Theorem 5.2.9. *$\gamma, \lambda, \delta, \mu \in \mathbb{N}, \eta > 0, r \in \mathbb{N}\cup\{-1, 0\},\ s \in \mathbb{N}\cup\{0\},$ c is arbitrary constant and fractional integral and differential operators are of order σ then*

$$ {}_qI^{\sigma}_{0+}\ E_t(c, -\eta, \gamma, \delta, \lambda, \mu|q) \;=\; E_t(c, \sigma - \eta, \gamma, \delta, \lambda, \mu|q) \qquad (5.2.35)$$

$$ {}_qD^{\sigma}_{0+}\ E_t(c, -\eta, \gamma, \delta, \lambda, \mu|q) \;=\; E_t(c, -\sigma - \eta, \gamma, \delta, \lambda, \mu|q). \qquad (5.2.36)$$

5.3 Main results for $e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$

Since the function $e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ does not involve the factor $q^{pn(n-1)/2}$, the results involving this function will be free from this factor; the techniques of derivations of the corresponding results remain same. Hence for the sake of bravity, they are just stated below.

5.3.1 Fractional q-operators

Theorem 5.3.1. *Let $a \in [0, \infty)$ and $\alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{N},\ \eta > 0$ then for $x > a$*

$$\begin{aligned}&\left({}_qI^{\eta}_{a+}[t - |a]_{\beta-1}\, e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t - |aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x)\\ &= [x - |a]_{\beta+\eta-1}\, e^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega[x - |aq^{\beta+\eta-1}]_{\alpha}; s, r|q),\end{aligned} \qquad (5.3.1)$$

and

$$\begin{aligned}&\left({}_qD^{\eta}_{a+}[t - |a]_{\beta-1}\, e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t - |aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x)\\ &= [x - |a]_{\beta-\eta-1}\, e^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x - |aq^{\beta-\eta-1}]_{\alpha}; s, r|q).\end{aligned} \qquad (5.3.2)$$

Theorem 5.3.2. *Let $\alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{N},\ \eta > 0$ then*

$$ {}_qI^{\eta}_{0+}\ \left[\nu\, e^{1,\delta}_{\alpha,1,\lambda,\mu}(\nu t^{\alpha}; s, r|q)\right](x) = \nu\ (x(1-q))^{\eta}\ e^{1,\delta}_{\alpha,\eta+1,\lambda,\mu}(\nu x^{\alpha}; s, r|q). \qquad (5.3.3)$$

Theorem 5.3.3. *Let $a \in [0, \infty),\ \alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{N},\ \eta, \nu > 0$ for $x > a$, then*

$$\left({}_qD^{\eta,\nu}_{a+}[t - |a]_{\beta-1}\, e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t - |aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x)$$

$$= \; [x-|a]_{\beta-\eta-1}\, e^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_{\alpha};s,r|q). \tag{5.3.4}$$

Theorem 5.3.4. *Let the function* ϕ *be in the space* $L(a,b) = \left\{f : {}_q\|f\|_1 = \int_a^b |f(t)|\, d_qt < \infty\right\}$ *of Lebesgue measurable functions on a finite interval* $[a,b]$. *Then the integral operator* ${}_q\mathfrak{e}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;a+}$ *is bounded on* $L(a,b)$.

5.3.2 q-Gauss multiplication type formula

Another q-form of Gauss multiplication formula may be given by

$$e^{\gamma,\delta}_{m,\,\beta,\,\lambda,\,\mu}(z;s,r|q) \;=\; \frac{\left([2]_q\,\Gamma^2_{q^2}\left(\frac{1}{2}\right)\right)^{\frac{m-1}{2}}}{([m]_q)^{\beta-\frac{1}{2}}} \prod_{k=0}^{m-1} \frac{1}{\Gamma_{q^m}\left(\frac{\beta+k}{m}\right)} \times \sum_{n=0}^{\infty} \frac{[\Gamma_q(\gamma+\delta n)]^s\, z^n}{([m]_q)^{mn}\, \left(\frac{\beta+k}{m};q^m\right)_n\, [\Gamma_q(\lambda+\mu n)]^r\, (q;q)_n}. \tag{5.3.5}$$

5.3.3 Fractional q-differential equations involving the q-analogue of Hilfer derivative operator

In the following, the fractional q-differential equations corresponding to the q-analogue of Hilfer derivative operator are stated which use

Lemma 5.3.1.

$$\mathcal{L}_q\left({}_q\mathfrak{e}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;0+}\right)(x)(S) \;=\; \sum_{n=0}^{\infty} \frac{[(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \times \frac{(q;q)_\infty\, q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}$$

in which the series is convergent for $\Re(\alpha^2) < 0$.

Theorem 5.3.5. *If* $0<\eta<\ 1,\ 0\le\nu\le 1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left({}_qD^{\eta,\,\nu}_{0+}\ y\right)(x) \;=\; \xi\left({}_q\mathfrak{e}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)(x) \;+\; f(x) \tag{5.3.6}$$

with the initial condition

$$\left({}_qI^{(1-\nu)(1-\eta)}_{0+}\ y\right)(0+) \;=\; C, \tag{5.3.7}$$

has solution

$$\begin{aligned} y(x) &= C\,\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\,x^{\beta+\eta} \\ &\times(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\,e^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\,(\omega\,(xq^{\eta+1})^{\alpha};\ s,\ r|q) \\ &+\frac{(1-q)1-\eta\,q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int\limits_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t, \end{aligned} \tag{5.3.8}$$

in the space $L(0,\infty)$ *wherein* C *is arbitrary constant.*

Theorem 5.3.6. *The q-differential equation*

$$\left({}_qD^{\eta,\,\nu}_{0+}\,y\right)(x)\ =\ \xi\,\left({}_q\mathfrak{e}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu\,\omega;0+}\right)(x)\ +\ x^{\beta}\,e^{\gamma,\delta}_{\alpha,\,\beta+1,\,\lambda,\,\mu}\big((\omega(ax)^{\alpha});s,r|q\big) \tag{5.3.9}$$

with the initial condition

$$\left({}_qI_0+^{(1-\nu)(1-\eta)}\ y\right)(0+)\ =\ C, \tag{5.3.10}$$

has solution in the space $L(0,\infty)$ *which is given by*

$$y(x)=C\,\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+q^{\eta(\eta+1)/2}\,x^{\beta+\eta}$$

$$\times\,(\xi\,(1-q)^{-\eta-1}\,q^{\beta(\eta+1)}\ +\ 1)\,e^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\,(\omega\,(xq^{\eta+1})^{\alpha};\ s,\ r|q) \tag{5.3.11}$$

in which C *is arbitrary constant.*

Theorem 5.3.7. *If* $0<\eta<\ 1,\ 0\le\nu\le1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left(x\ {}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\,\left({}_q\mathfrak{e}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)(x) \tag{5.3.12}$$

with the initial condition

$$\left({}_qI_0+^{(1-\nu)(1-\eta)}\ y\right)(0+)\ =\ C, \tag{5.3.13}$$

has solution in the space $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) &= (q^{\eta+1}-q)\,\sum_{j=0}^{\infty}q^j\,y(xq^j) \\ &+C\frac{q^{\eta}(1-q^{-\nu(1-\eta)})}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)-1}\ (q^{\eta-\nu(1-\eta)+1};q)_{\infty}}{(q;q)_{\infty}\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right) \end{aligned}$$

$$+\frac{x^{\beta+\eta-1}}{q^{-\eta(\eta-1)/2-\beta(\eta+1)}}e^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega\left(xq^{\eta+1}\right)^{\alpha};s,r|q\right), \tag{5.3.14}$$

wherein C is arbitrary constant.

5.3.4 Properties of $e_t(c,\nu,\gamma,\delta,\lambda,\mu|q)$ and $e_t(c,-\eta,\gamma,\delta,\lambda,\mu|q)$

Here certain properties of the function

$$f(t) = \sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu n};q)_\infty]^r\,[(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta n};q)_\infty]^s}(ct)^n$$

are stated which are suggested by the function (3.1.2). Here the infinite series converges absolutely when $|t| < \left|(1-q)^{(s\delta-r\mu-2)}\right|$ and $|q|<1$ where c is arbitrary constant.

Now using q-Riemann-Liouville fractional integral operator (1.7.1) on $f(t)$ one gets

$$\begin{aligned}
{}_qI^{\nu}_{0+}\,f(t) &= \frac{1}{\Gamma_q(\nu)}\int_0^t (t-\xi\,q)_{\nu-1}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,(c\xi)^n\,[(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\,\mathrm{d}_q\xi\\
&= \frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,c^n\,[(q^{n+1};q)_\infty]^2}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\int_0^t \xi^n(t-\xi\,q)_{\nu-1}\,\mathrm{d}_q\xi.
\end{aligned}$$

Taking $\xi = tu$, this gives

$$\begin{aligned}
{}_qI^{\nu}_{0+}\,f(t) &= \frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,c^n\,[(q^{n+1};q)_\infty]^2\,t^{n+\nu}}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\int_0^1 u^n\,(1-uq)_{\nu-1}\,\mathrm{d}_qu\\
&= \frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,c^n\,[(q^{n+1};q)_\infty]^2\,t^{n+\nu}}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\int_0^1 u^{n+1-1}\frac{(uq;q)_\infty}{(uq^\nu;q)_\infty}\,\mathrm{d}_qu\\
&= \frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,c^n\,[(q^{n+1};q)_\infty]^2\,t^{n+\nu}}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\,\mathfrak{B}_q(n+1,\nu)\\
&= \frac{1}{\Gamma_q(\nu)}\sum_{n=0}^{\infty}\frac{[(q^{\lambda+\mu\,n};q)_\infty]^r\,c^n\,[(q^{n+1};q)_\infty]^2\,t^{n+\nu}}{[(q^{\gamma+\delta\,n};q)_\infty]^s}\\
&\quad\times\frac{(1-q)\,(q;q)_\infty\,(q^{n+\nu+1};q)_\infty}{(q^{n+1};q)_\infty\,(q^\nu;q)_\infty}\\
&= t^\nu(1-q)^\nu\sum_{n=0}^{\infty}\frac{(q^{\nu+n+1};q)_\infty\,[(q^{\lambda+\mu\,n};q)_\infty]^r\,(q^{n+1};q)_\infty\,(ct)^n}{[(q^{\gamma+\delta\,n};q)_\infty]^s}.
\end{aligned}$$

In view of (3.1.2), this leads to

$$\begin{aligned} {}_qI_{0+}^{\nu}\, f(t) &= t^{\nu}\,(1-q)^{\nu}\, e_{1,1+\nu,\lambda,\mu}^{\gamma,\delta}(ct;s,r|q) \\ &= e_t(c,\nu,\gamma,\delta,\lambda,\mu|q), \quad \text{say.} \end{aligned} \tag{5.3.15}$$

Next, using the fractional q-differential operator (1.7.5) of order η on $f(t)$, and then using (5.3.15), one gets

$$\begin{aligned} {}_qD_{0+}^{\eta}\, f(t) &= D_q^k \left\{ I_q^{k-\eta} \sum_{n=0}^{\infty} \frac{[(q^{\lambda+\mu\, n};q)_{\infty}]^r\,(ct)^n\,[(q^{n+1};q)_{\infty}]^2}{[(q^{\gamma+\delta\, n};q)_{\infty}]^s} \right\} \\ &= D_q^k \left\{ t^{k-\eta}\,(1-q)^{k-\eta}\, e_{1,1+k-\eta,\lambda,\mu}^{\gamma,\delta}(ct;s,r|q) \right\} \\ &= t^{-\eta}\,(1-q)^{-\eta}\, e_{1,1-\eta,\lambda,\mu}^{\gamma,\delta}(ct;s,r|q) \\ &= e_t(c,-\eta,\gamma,\delta,\lambda,\mu|q), \quad \text{say.} \end{aligned} \tag{5.3.16}$$

In parallel to Theorem-5.2.8 and Theorem-5.2.9, the effect of fractional q-integral and fraction q-differential operators are stated in the following theorems.

Theorem 5.3.8. *For $\gamma,\lambda,\delta,\mu \in \mathbb{N}, \nu > 0,\ r \in \mathbb{N}\cup\{-1,0\},\ s \in \mathbb{N}\cup\{0\},\ c$ is arbitrary constant and fractional q-integral and$q$-differential operators are of order σ then*

$${}_qI_{0+}^{\sigma}\, e_t(c,\nu,\gamma,\delta,\lambda,\mu|q) = e_t(c,\sigma+\nu,\gamma,\delta,\lambda,\mu|q). \tag{5.3.17}$$

$${}_qD_{0+}^{\sigma} e_t(c,\nu,\gamma,\delta,\lambda,\mu) = e_t(c,\nu-\sigma,\gamma,\delta,\lambda,\mu). \tag{5.3.18}$$

Theorem 5.3.9. *$\gamma,\lambda,\delta,\mu \in \mathbb{N}, \eta > 0,\ r \in \mathbb{N}\cup\{-1,0\},\ s \in \mathbb{N}\cup\{0\},\ c$ is arbitrary constant and fractional integral and differential operators are of order σ then*

$${}_qI_{0+}^{\sigma}\, e_t(c,-\eta,\gamma,\delta,\lambda,\mu|q) = e_t(c,\sigma-\eta,\gamma,\delta,\lambda,\mu|q) \tag{5.3.19}$$

$${}_qD_{0+}^{\sigma}\, e_t(c,-\eta,\gamma,\delta,\lambda,\mu|q) = e_t(c,-\sigma-\eta,\gamma,\delta,\lambda,\mu|q). \tag{5.3.20}$$