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I wish her all the best.

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On a Unification of Generalized Mittag-Leffler Function and Family of Bessel Functions

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ABSTRACT

In the present work, a unification of certain functions of mathematical physics is proposed and its properties are studied. The proposed function unifies Lommel function, Struve function, the Bessel-Maitland function and its generalization, Dotsenko function, generalized Mittag-Leffler function etc. The properties include absolute and uniform convergence, differential recurrence relation, integral representations in the form of Euler-Beta transform, Mellin-Barnes transform, Laplace transform and Whittaker transform. The special cases namely the generalized hypergeometric function, generalized Laguerre polynomial, Fox H-function etc. are also obtained.

Keywords: Generalized Mittag-Leffler Function; Recurrence Relation; Wiman's Function

1. Introduction

In the present work, we propose an extension of a generalization of the Mittag-Leffler function due to A. K. Shukla and J. C. Prajapati [1], defined as

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^{n}}{\Gamma(\alpha n+\beta)n!},\tag{1.1}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\operatorname{Re}(\alpha,\operatorname{Re}(\beta,\gamma))>0$ and $q\in(0,1)\cup\mathbb{N}$. This is an entire function of order $(\operatorname{Re}\alpha-q+1)^{-1}$ if $\operatorname{Re}\alpha>q-1$ and absolutely convergent in $\{|z|<R,R<1\}$ if $\operatorname{Re}\alpha=q-1$. In fact (1.1) contains the $E_{\alpha}(z)$-Mittag-Leffler function [2], $E_{\alpha,\beta}(z)$-the generalized Mittag-Leffler function [3] and the function $E_{\alpha,\beta}^{\gamma}(z)$ due to Prabhakar [4].

Gorenflo *et al.* [5], Saigo and Kilbas [6] studied several interesting properties of these functions.

Another generalization of Mittag-leffler function due to T. O. Salim [7], given by

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}z^{n}}{\Gamma(\alpha n+\beta)(\delta)_{n}},\tag{1.1'}$$

where $z,\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and

$$\min\{\operatorname{Re}(\alpha,\beta,\delta)\}>0.$$

We state below the extended version in the form:

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}z^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}},\tag{1.2}$$

where $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\operatorname{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$, $\delta,\mu,p>0$. The function defined by (1.2) reduces to the one in (1.1) and (1.1') if $\rho=1$, $p=1$, $r=0$, $s=1$ and $\delta=1$, $\rho=\delta$, $p=1$, $r=0$, $s=1$ respectively.

It is noteworthy that the function in (1.2), besides containing the generalizations of the Mittag-Leffler function, also includes certain functions belonging to the family of Bessel function. To see this, take $s=0$, $r=0$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=\nu+1$, and replaced z by $\frac{-z^2}{4}$ in (1.2), then we find the well known Bessel function [8]:

$$J_{\nu}(z)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(1+\nu+n)}\left(\frac{z}{2}\right)^{\nu+2n}.$$

When $s=0$, $r=0$, $\alpha=\mu$, $\beta=\nu+1$, and z is replaced by $(-z)$ then we get the Bessel Maitland Function [8] given by $J_{\nu}^{\mu}(z)=\sum_{n=0}^{\infty}\frac{(-z)^{n}}{\Gamma(\nu+n\mu+1)n!}$. For $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=\mu$, $\beta=\nu+\lambda+1$, $\mu=1$, $\lambda=\lambda+1$, $\gamma=1$, $\delta=1$, and z is replaced by $\frac{-z^2}{4}$, we obtain the Generalized Bessel Maitland function [8]:

$$J^{\mu}_{\nu,\lambda}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu+n\mu+\lambda+1)\Gamma(n+\lambda+1)}\left(\frac{z}{2}\right)^{\nu+2\lambda+2n}.$$

The Dotsenko Function [8]:

$$ {}_2R_1(a,b;c,\omega;\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+n\frac{\omega}{\mu}\right)}{\Gamma\left(c+n\frac{\omega}{\mu}\right)}\frac{z^n}{n!}$$

occurs by substituting $s=1$, $r=-1$, $\rho=1$, $p=1$, $\alpha=c$, $\beta=\frac{\omega}{\mu}$, $\mu=\frac{\omega}{\mu}$, $\lambda=b$, $\gamma=a$, $\delta=1$ in (1.2).

The Lommel Function defined by [9]:

$$S_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 1/2(\mu-\nu+3), & 1/2(\mu+\nu+3),\end{matrix}\right)$$

is the special case $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=\frac{1}{2}(\mu-\nu+3)$, $\mu=1$, $\lambda=\frac{1}{2}(\mu+\nu+3)$, $\gamma=1$, $\delta=1$, and z is replaced by $\frac{-z^2}{4}$ of (1.2). On making substitutions $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=3/2$, $\mu=1$, $\lambda=3/2+\nu$, $\gamma=1$, $\delta=1$, and $z=\mp z^2/4$ in (1.2), provides us respectively, the Struve Function $H_\nu(z)$ [9] given by

$$H_\nu(z)=\frac{(z/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2+\nu)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 3/2 & 3/2+\nu\end{matrix}\right).$$

and the Modified Struve Function [9]:

$$L_\nu(z)=\frac{(z/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2+\nu)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 3/2 & 3/2+\nu\end{matrix}\right).$$

In what follows, we shall use the following definitions and formulas. Euler (Beta) transform [10]:

$$B\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)\mathrm{d}z. \quad (1.3)$$

Laplace transform [10]:

$$\mathcal{L}\{f(z)\}=\int_0^\infty \mathrm{e}^{-sz}f(z)\mathrm{d}z. \quad (1.4)$$

Mellin-Barnes transform [10]:

$$M\left[f(z);z\right]=\int_0^\infty z^{s-1}f(z)\mathrm{d}z=f^*(s),\ \mathrm{Re}(s)>0, \quad (1.5)$$

then

$$f(z)=M^{-1}\left[f^*(s);x\right]=\frac{1}{2\pi i}\int f^*(s)x^{-s}\mathrm{d}s. \quad (1.6)$$

Incomplete Gamma function [11]:

$$\gamma(\alpha,z)=\int_0^z \mathrm{e}^{-t}t^{\alpha-1}\mathrm{d}t,\ \mathrm{Re}(\alpha)>0. \quad (1.7)$$

The generalized hypergeometric function is denoted and defined by [11]

$${}_pF_q\left[\begin{matrix}a_1,a_2,\cdots,a_p;\\ b_1,b_2,\cdots,b_q;\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}, \quad (1.8)$$

where $b_1,b_2,\cdots,b_q$ are neither zero nor negative integers, and

$$(\lambda)_n=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1).$$

The series is convergent for 1) $|z|<\infty$ if $p\le q$, 2) $|z|<1$ if $p=q+1$.

Wright generalized hypergeometric function [12]:

$${}_p\psi_q\left(\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{matrix}z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha i+A_i n)}{\prod_{j=1}^{q}\Gamma(\beta j+B_j n)}\frac{z^n}{n!} \quad (1.9)$$

Laguerre polynomial [12]:

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}{}_1F_1\left[\begin{matrix}-n;x\\ 1+\alpha;\end{matrix}\right]. \quad (1.10)$$

2. Main Results

In this section, we prove the following results for the function defined in (1.2).

Theorem 2.1. The series represented by the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$ converges absolutely for $|z|<n^{1/p(\mathrm{Re}(\mu)r+\Re(\alpha)p-\mathrm{Re}(\delta)s+p)}$.

Proof: Consider,

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}.$$

Take

$$u_n=\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

then

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \times \frac{\Gamma\left(\alpha\left(p(n+1)+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu(n+1)}\right]^r (\rho)_{p(n+1)}}{\left[(\gamma)_{\delta(n+1)}\right]^s z^{p(n+1)+\rho-1}}\right|$$

$$= \left|\frac{\left[\Gamma(\gamma+\delta n)\right]^s \left[\Gamma(\lambda)\right]^r \Gamma(\rho) z^{pn+\rho-1}}{\left[\Gamma(\gamma)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r \Gamma(\rho+pn)}\right.$$

$$\left.\times \frac{\left[\Gamma(\gamma)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha p\right)\left[\Gamma(\lambda+\mu n+\mu)\right]^r \Gamma(\rho+pn+p)}{\left[\Gamma(\gamma+\delta n+\delta)\right]^s \left[\Gamma(\lambda)\right]^r \Gamma(\rho) z^{p(n+1)+\rho-1}}\right|$$

$$= \left|\frac{\left[\Gamma(\gamma+\delta n)\right]^s \Gamma(\alpha n+\beta+\alpha p)\left[\Gamma(\lambda+\mu n+\mu)\right]^r \Gamma(\rho+pn+p)}{\left[\Gamma(\gamma+\delta n+\delta)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r \Gamma(\rho+pn) z^p}\right|$$

$$= \left|\frac{\left[(\lambda+\mu n)(\lambda+\mu n+1)(\lambda+\mu n+2)\cdots(\lambda+\mu n+\mu-1)\right]^r}{\left[(\gamma+\delta n)(\gamma+\delta n+1)(\gamma+\delta n+2)\cdots(\gamma+\delta n+\delta-1)\right]^s z^p}\right.$$

$$\times(\alpha pn+\alpha\rho-\alpha+\beta)(\alpha pn+\alpha\rho-\alpha+\beta+1)\cdots(\alpha pn+\alpha\rho-\alpha+\beta+\alpha-1)$$

$$\left.\times(\rho+pn)(\rho+pn+1)(\rho+pn+2)\cdots(\rho+pn+p-1)\right|$$

$$= \left|\frac{n^{\mu r}\left[\left(\frac{\lambda}{n}+\mu\right)\left(\frac{\lambda+1}{n}+\mu\right)\left(\frac{\lambda+2}{n}+\mu\right)\cdots\left(\frac{\lambda+\mu-1}{n}+\mu\right)\right]^r}{n^{\delta s}\left[\left(\frac{\gamma}{n}+\delta\right)\left(\frac{\gamma+1}{n}+\delta\right)\left(\frac{\gamma+2}{n}+\delta\right)\cdots\left(\frac{\gamma+\delta-1}{n}+\delta\right)\right]^s z^p}\right.$$

$$\times n^{\alpha p}\left(\alpha p+\frac{\alpha\rho-\alpha+\beta}{n}\right)\left(\alpha p+\frac{\alpha\rho-\alpha+1}{n}\right)\cdots\left(\alpha p+\frac{\alpha\rho-\alpha+\beta+\alpha p-1}{n}\right)$$

$$\left.\times n^p\left(\frac{\rho}{n}+p\right)\left(\frac{\rho+1}{n}+p\right)\left(\frac{\rho+2}{n}+p\right)\cdots\left(\frac{\rho+p-1}{n}+p\right)\right|.$$

Thus,

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|n^{\mu r+\alpha p-\delta s+p}/z^p\right|.$$

Hence,

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| > 1 \Rightarrow \lim_{n\to\infty}\left|n^{\mu r+\alpha p-\delta s+p}/z^p\right| > 1 \Rightarrow |z| < n^{1/p\left(\mathrm{Re}(\mu)r+\Re(\alpha)-\mathrm{Re}(\delta)s+p\right)}.$$

Theorem 2.2. For $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$; $\mathrm{Re}(\alpha)>0$ and $\delta,\mu,p>0$ the differential recurrence relation form:

$$\beta E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)+\alpha z\frac{\mathrm{d}}{\mathrm{d}z}E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r).$$

Proof.
Consider,

$$\beta E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)+\alpha z\frac{\mathrm{d}}{\mathrm{d}z}E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)$$

$$=\beta\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}+\alpha z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}$$

$$=\beta\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}+\alpha z\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(pn+\rho-1)(z)^{pn+\rho-2}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$$

As the series given in (1.2) converges uniformly in any compact subset of $\mathbb{C}$, the use of term by term differentiation under the sign of summation leads us to the following theorem.

Theorem 2.3. If $m\in\mathbb{N}$, $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$ and $\delta,\mu,p>0$ then

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha pm,\lambda+\mu m,\mu,\rho+pm-m,p}(z;s,r), \tag{2.3.1}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m}\left[z^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(\omega z^{\alpha};s,r\right)\right]=z^{\beta-m-1}E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu,\rho,p}\left(\omega z^{\alpha};s,r\right),\ \text{if }\mathrm{Re}(\beta-m)>0. \tag{2.3.2}$$

Proof. Consider

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,rho,p}(z;s,r)=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}z^{pn+\rho-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\left[(\rho)_{pn}\right]}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\left[(\rho)_{pn}\right]}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m}\left(z^{pn+\rho-1}\right)$$

$$=\sum_{n=m}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\Gamma(\rho)z^{pn+\rho-m-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\Gamma\left(pn+\rho-m\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta(n+m)}\right]^{s}\Gamma(\rho)z^{pn+pm+\rho-m-1}}{\left[\Gamma\left(\alpha\left(pn+pm+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu(n+m)}\right]^{r}\Gamma\left(pn+pm+\rho-m\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta m}\right]^{s}\left[(\gamma+\delta m)_{\delta n}\right]^{s}\Gamma(\rho)z^{pm-m}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha pm\right)\right]\left[(\lambda)_{\mu m}\right]^{r}\left[(\lambda+\mu m)_{\mu n}\right]^{r}}\frac{z^{pn+\rho-1}}{\Gamma\left(pn+pm+\rho-m\right)}\frac{\Gamma\left(\rho+pm-m\right)}{\Gamma\left(\rho+pm-m\right)}$$

$$=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}\times\sum_{n=0}^{\infty}\frac{\left[(\gamma+\delta m)_{\delta n}\right]^{s}z^{pn+\rho-1}}{\left[(\lambda+\mu m)_{\mu n}\right]^{r}\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha pm\right)\right](\rho+pm-m)_{pn}}$$

$$=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha pm,\lambda+\mu m,\mu,\rho+pm-m,p}(z;s,r).$$

Now consider,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega z^{\alpha};s,r\right)\right]=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{\alpha(pn+\rho-1)}z^{\beta-1}\omega^{pn+\rho-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\omega^{pn+\rho-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left(z^{\alpha(pn+\rho-1)+\beta-1}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(\omega z^{\alpha}\right)^{(pn+\rho-1)}z^{\beta-m-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta-m\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}=z^{\beta-m-1}E_{\alpha,\beta-m,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega z^{\alpha};s,r\right)$$

Next, taking $f(z)=E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(zu^{\alpha};s,r\right)$, in the Euler (Beta) transform (1.3), one finds the following

Theorem 2.4. If $\alpha,\beta,\gamma,\lambda,\rho,\sigma,\eta,\nu\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho,\sigma,\eta,\nu)>0$ and $\delta,\mu,p>0$ then

$$\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(zu^{\alpha};s,r\right)\mathrm{d}u=E_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(z;s,r\right),\tag{2.4.1}$$

$$\frac{1}{\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\nu(s-t)^{\alpha};s,r\right]\mathrm{d}s=(x-t)^{\eta+\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\nu(x-t)^{\alpha};s,r\right],\tag{2.4.2}$$

$$\int_0^z t^{\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\omega t^{\alpha};s,r\right]\mathrm{d}t=z^{b}E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\omega t^{\alpha};s,r\right],\tag{2.4.3}$$

$$\frac{1}{\Gamma(\sigma)}\int_0^1 z^{\sigma-1}(1-z)^{\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(x(1-z)^{\alpha};s,r\right)\mathrm{d}z=E_{\alpha,\beta+\sigma,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(x;s,r\right).\tag{2.4.4}$$

Proof.
In (2.4.1),

$$L.H.S.=\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(zu^{\alpha}s,r\right)\mathrm{d}u=\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s u^{\alpha(pn+\rho-1)}z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}u$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{n}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\int_0^1 u^{\alpha(pn+\rho-1)+\beta-1}(1-u)^{\eta-1}\mathrm{d}u$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{n}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\frac{\Gamma(\eta)\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)+\eta\right]}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}=E_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(z;s,r\right)=R.H.S.$$

Now, denoting the *L.H.S.* of (2.4.2) by I, we have

$$I=\frac{1}{\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\beta-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\nu(s-t)^{\alpha};s,r\right]\mathrm{d}s$$

$$=\frac{1}{\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s(s-t)^{\alpha(pn+\rho-1)}\nu^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}s$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\nu^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\int_t^x (x-t)^{\eta-1}(s-t)^{\alpha(pn+\rho-1)+\beta-1}\mathrm{d}s.$$

Here, introducing u as a new variable of integration, by means of the relation

$$u = \frac{s-t}{x-t},$$

The further simplification gives,

$$I = \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn} \Gamma(\eta)} \int_0^1 (1-u)^{\eta-1} u^{\alpha(pn+\rho-1)+\beta-1} \mathrm{d}u$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn} \Gamma(\eta)} \frac{\Gamma(\eta)\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\right]}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}}$$

$$= (x-t)^{\eta+\beta-1} E_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\nu(x-t)^{\alpha};s,r\right].$$

as desired.

To prove (2.4.3) we begin with

$$\int_0^z t^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega t^{\alpha};s,r\right)\mathrm{d}t$$

$$= \int_0^z t^{\beta-1} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} t^{\alpha(pn+\rho-1)} \omega^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}} \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \omega^{n}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}} \int_0^z t^{\alpha(pn+\rho-1)+\beta-1} \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} z^{\alpha(pn+\rho-1)+\beta} (\omega)^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn} (\alpha(pn+\rho-1)+\beta)}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} z^{\alpha(pn+\rho-1)+\beta} \omega^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+1)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}} = z^{\beta} E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega z^{\alpha};s,r\right).$$

Hence the result.

Now, consider

$$\frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1} (1-z)^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(x(1-z)^{\alpha};s,r\right)\mathrm{d}z$$

$$= \frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1} (1-z)^{\beta-1} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} (1-z)^{\alpha(pn+\rho-1)} x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn}} \mathrm{d}z$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn} \Gamma(\sigma)} \int_0^1 (1-z)^{\alpha(pn+\rho-1)+\beta-1} z^{\sigma-1} \mathrm{d}z$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^{r} (\rho)_{pn} \Gamma(\sigma)} \frac{\Gamma(\sigma)\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)+\sigma\right]}$$

simplification of above series yields (2.4.4).

3. Mellin-Barnes Integral Representation of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$

Theorem 3.1. Let $\alpha\in\mathbb{R}_+$; $\beta,\gamma,\lambda,\rho,\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$ and $\delta,\mu>0$, $p\in\mathbb{N}$. Then the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$ is represented by the Mellin-Barnes integral as

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}{2\pi i[\Gamma(\gamma)]^s}\int_L\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\mathrm{d}S, \tag{3.1.1}$$

where $|\arg z|<\pi$; the contour of integration beginning at $-i\infty$ and ending at $+i\infty$, and indented to separate the poles of integrand at $S=-n$ for all $n\in\mathbb{N}_0$ (to the left) from those at $S=\frac{\gamma+n}{\delta}$ for all $n\in\mathbb{N}_0$ (to the right).

Proof.

We shall evaluate the integral on the *R.H.S.* of (3.1.1) as the sum of the residues at the poles $S=0,-1,-2,\cdots$ In fact, in view of the definition of residue, we have

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_L\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\mathrm{d}S\\
&=\sum_{n=0}^{\infty}S\mathrm{Res}n_{S=-n}\left[\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\right]\\
&=\sum_{n=0}^{\infty}\lim_{S\to-n}\frac{\pi(S+n)}{\sin\pi pS}\frac{[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^{pn}}{p}\frac{[\Gamma(\gamma-\delta S)]^s}{\Gamma(\beta+\alpha(pn+\rho-1))[\Gamma(\lambda+\mu n)]^r\Gamma(\rho+pn)}(-z)^{pn}.\end{aligned}$$

This gives,

$$\begin{aligned}I&=\frac{[\Gamma(\gamma)]^s}{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r(\rho)_{pn}}\\
&=\frac{[\Gamma(\gamma)]^s}{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r).\end{aligned}$$

4. Integral Transforms of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$

In this section, we discussed some useful integral transforms like Euler transforms, Laplace transforms, Mellin transforms, Whittaker transforms,

For the convenience, we introduce the Notation:

$$(\beta,\alpha)^r=\frac{[\Gamma(\beta+\alpha n)]^r}{[\Gamma(\beta)]^r}$$

Theorem 4.1. (*Euler*(*Beta*) *transforms*)

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(xz^{\sigma};s,r)\mathrm{d}z$$
$$=\frac{x^{\rho-1}[\Gamma(\lambda)]^r\Gamma(b)\Gamma(\rho)}{[\Gamma(\gamma)]^s}\cdot{}_{s+2}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\sigma\rho-\sigma+a,\sigma p), & (1,1); & x\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p),\ (a+b,\sigma); & \end{matrix}\right],$$

where $\alpha,\beta,\gamma,\lambda,\sigma,a,b\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,a,b)>0$ and $\delta,\mu p>0$.

Proof.

$$\int_0^1 z^{a-1}(1-z)^{b-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z = \int_0^1 z^{a-1}(1-z)^{b-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}z^{\sigma(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^1 z^{\sigma(pn+\rho-1)+a-1}(1-z)^{(b-1)}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma\left[(\gamma+\delta n)\right]^s\left[\Gamma(\lambda)\right]^r\Gamma(b)\Gamma\left(\sigma(pn+\rho-1)+a\right)x^{pn+\rho-1}}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r\Gamma\left(\sigma(pn+\rho-1)+a+b\right)(\rho)_{pn}}$$

$$=\frac{x^{\rho-1}\left[\Gamma(\lambda)\right]^r\Gamma(b)\Gamma(\rho)}{\left[\Gamma(\lambda)\right]^s}\times {}_{s+2}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\sigma\rho-\sigma+a,\sigma p), & (1,1); & x\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p),(a+b,\sigma); & \end{matrix}\right].$$

Theorem 4.2. (*Laplace transforms*)

$$\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z$$

$$=\frac{s^{-\sigma(\rho-1)}x^{(\rho-1)-a}\left[\Gamma(\lambda)\right]^r\Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s}\,{}_{s+2}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, & (a,\sigma), & (1,1); & \frac{x}{s^{\sigma p}}\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p); & \end{matrix}\right],$$

where $\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b)>0$ and $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b)>0$.

Proof. We begin with

$$I=\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z=\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}z^{\sigma(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s x^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r(\rho)_{pn}}\int_0^{\infty}\mathrm{e}^{-sz}z^{\sigma(pn+\rho-1)+a-1}\mathrm{d}z.$$

On making substitution $t=sz$, we get

$$I=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^{\infty}\mathrm{e}^{-t}(t/s)^{\sigma(pn+\rho-1)+a-1}(1/s)\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{(pn+\rho-1)}s^{-\sigma(pn+\rho-1)-a}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\Gamma\left(\sigma(pn+\rho-1)+a\right)$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma\left[(\gamma+\delta n)\right]^s\left[\Gamma(\lambda)\right]^r x^n s^{-\sigma n-a}\Gamma\left(\sigma(pn+\rho-1)+a\right)\Gamma(\rho)\Gamma(n+1)}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r\Gamma(\rho+pn)n!}$$

$$=\frac{s^{-\sigma(\rho-1)}x^{\rho-1}\left[\Gamma(\lambda)\right]^r}{\left[\Gamma(\gamma)\right]^s}\,{}_{s+2}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, & (a,\sigma), & (1,1); & \frac{x}{s^{\sigma p}}\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p), & \end{matrix}\right].$$

In proving the following theorem we use the integral formula involving the Whittaker function:

$$\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t/2}W_{\lambda,\mu}(t)\mathrm{d}t=\frac{\Gamma(1/2+\mu+\nu)\Gamma(1/2-\mu+\nu)}{\Gamma(1/2-\lambda+\nu)},\ \mathrm{Re}(\nu\pm\mu)>-1/2.$$

Theorem 4.3. (*Whittaker transforms*)

$$\int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega t^\sigma; s, r)\mathrm{d}t$$

$$= \frac{[\Gamma(\lambda)]^r \omega^{\rho-1}\Gamma(\rho)}{[\Gamma(\gamma)]^s q^{\sigma(\rho-1)-\psi}} \times {}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma), & (1;1); & \dfrac{\omega^p}{q^{\sigma p}} \\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma), & (\rho,p); & \end{matrix}\right],$$

where $\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b \in \mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b) > 0$ and $\delta,\mu,p > 0$.

Proof. Let

$$I = \int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega t^\sigma; s, r)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1} t^{\sigma(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}} \mathrm{d}t$$

$$= \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}} \int_0^\infty \mathrm{e}^{-qt/2} t^{\sigma(pn+\rho-1)+\psi-1} W_{\eta,\nu}(qt)\mathrm{d}t$$

then using the substitution $\xi = qt$, we get

$$I = \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1} q^{-\sigma(pn+\rho-1)-\psi}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}} \int_0^\infty \mathrm{e}^{-\xi/2} \xi^{\sigma(pn+\rho-1)+\psi-1} W_{\eta,\nu}(\xi)\mathrm{d}\xi$$

$$= \sum_{n=0}^\infty \frac{\Gamma[(\gamma+\delta n)]^s [\Gamma(\lambda)]^r \omega^{pn+\rho-1} q^{-\sigma(pn+\rho-1)-\psi}}{[\Gamma(\gamma)]^s \Gamma(\alpha(pn+\rho-1)+\beta)[\Gamma(\lambda+\mu n)]^r (\rho)_{pn}} \times \frac{\Gamma(\sigma(pn+\rho-1)+\psi+\nu+1/2)\Gamma(\sigma n+\psi-\nu+1/2)}{\Gamma(\sigma(pn+\rho-1)+\psi-\eta+1)}$$

$$= \frac{[\Gamma(\lambda)]^r \omega^{\rho-1}\Gamma(\rho)}{[\Gamma(\gamma)]^s q^{\sigma(\rho-1)-\psi}} \times {}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma), & (1;1); & \dfrac{\omega^p}{q^{\sigma p}} \\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma), & (\rho,p); & \end{matrix}\right],$$

Theorem 4.4. (*Mellin transforms*)

$$\int_0^\infty t^{pS-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(-\omega t; s, r)\mathrm{d}t = \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s}{\omega^{pS}\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)} \tag{4.4.1}$$

where $\alpha,\beta,\gamma,\lambda,\rho,S \in \mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho,S) > 0$ $\delta,\mu,p > 0$.

Proof. Putting $z = -\omega t$ in (3.1.1), we get

$$\begin{aligned} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z; s, r) &= \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p z^{\rho-1}}{2\pi i[\Gamma(\gamma)]^s} \int_L \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s (-\omega t)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)} \mathrm{d}S \\ &= \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p z^{\rho-1}}{2\pi i[\Gamma(\lambda)]^s} \int_L f^*(S) t^{-pS}\mathrm{d}S, \end{aligned} \tag{4.4.2}$$

in which

$$f^*(S) = \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s}{\omega^{pS}\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)}$$

using (1.5) and (1.6) in (4.4.2), immediately leads us to (4.4.1).

5. Generalized Hypergeometric Function Representation of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$

Taking $\alpha = a$, $\delta = l$, $\mu = m$ in (1.2), we get

$$E^{\gamma,l}_{k,\beta,\lambda,m,\rho,p}(z;s,r) = \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\ln}\right]^s z^{pn+\rho-1}}{\Gamma(a(pn+\rho-1)+\beta)\left[(\lambda)_{mn}\right]^r (\rho)_{pn}} = \frac{1}{\Gamma(\beta+a\rho-a)} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\ln}\right]^s z^{pn+\rho-1}(1)_n}{(\beta+a\rho-a)_{apn}\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}\, n!}$$

$$= \frac{z^{\rho-1}}{\Gamma(\beta+a\rho-a)} \sum_{n=0}^{\infty} \frac{\left[\prod_{i=1}^{l}\left(\frac{\gamma+i-1}{l}\right)_n\right]^s \left[\prod_{k=1}^{m}\left(\frac{\lambda+k-1}{m}\right)_n\right]^{-r} (1)_n}{\prod_{j=1}^{a}\left(\frac{\beta+a\rho-a+j-1}{ap}\right)_n \prod_{h=1}^{p}\left(\frac{\rho+h-1}{p}\right)_n} \times \frac{\frac{l^{lsn} z^{pn}}{m^{mrn} p^{pn} (ap)^{apn}}}{n!}$$

$$= \frac{z^{\rho-1}}{\Gamma(\beta+a\rho-a)} \times {}_{l^s}F_{ap+m^r+p}\left[\begin{matrix} \Delta(l,\gamma)^s, & 1; & \dfrac{l^{ls} z^p}{m^{mr} p^p (ap)^{ap}} \\ \Delta(ap,\beta+a\rho-a), & \Delta(m,\lambda)^r, \Delta(p,\rho); & \end{matrix}\right]$$

where $\Delta(n;\alpha)$ is a n-tuple $\frac{\alpha}{n}, \frac{\alpha+1}{n}, \cdots, \frac{\alpha+n-1}{n}$.

6. Relationship with Some Known Special Functions (Generalized Laguerre Polynomial, Fox H-Function, Wright Hypergeometric Function)

6.1. Relationship with Generalized Laguerre Polynomials

Putting $\alpha = k$, $\beta = \nu + 1$, $\gamma = -m$, $r = 0$, $s = 1$, $\rho = 1$, $p = 1$ and replacing δ by $q \in \mathbb{N}$ and z by z^k in (1.2), we get

$$E^{-m,q}_{k,\nu+1,\lambda,\mu,1,1}(z;1,0) = \sum_{n=0}^{\left[\frac{m}{q}\right]} \frac{\left[(-m)_{qn}\right]}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!} = \sum_{n=0}^{\left[\frac{m}{q}\right]} \frac{(-1)^{qn} m!}{(m-qn)!} \frac{1}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!}$$
$$= \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} \sum_{n=0}^{\left[\frac{m}{q}\right]} \frac{(-1)^{qn}}{(m-qn)!} \frac{\Gamma(km+\nu+1)}{\Gamma(kn+\nu+1)} \frac{z^{kn}}{n!} = \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} Z^{(\nu)}_{\left[\frac{m}{q}\right]}(z,k) \quad (6.1.1)$$

where $Z^{(\nu)}_{\left[\frac{m}{q}\right]}(z,k)$ is polynomial of degree $\left[\frac{m}{q}\right]$ in z^k.

In particular, $Z^{(\nu)}_m(z,1) = L^{(\nu)}_m(z)$, so that

$$E^{-m,1}_{k,\nu+1,\lambda,\mu,1,1}(z;1,0) = \frac{\Gamma(m+1)}{\Gamma(km+\nu+1)} L^{(\nu)}_m(z) \quad (6.1.2)$$

6.2. Relationship with Fox H-Function

From (3.1.1), we have

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r) = \frac{\left[\Gamma(\lambda)\right]^r \Gamma(\rho)}{2\pi i\left[\Gamma(\gamma)\right]^s} \times \int_L \frac{\Gamma(S)\left[\Gamma(\gamma-\delta S)\right]^s}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)\left[\Gamma(\lambda-\mu S)\right]^r \Gamma(\rho-pS)} (-z)^{-S}\, dS$$
$$= \frac{\left[\Gamma(\lambda)\right]^r \Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s} \times H^{1,s}_{s,r+3}\left(-z \left| \begin{matrix} \left[(1-\gamma,\delta)\right]^s; & & & \\ (0,1), & (1-\beta-\alpha\rho+\alpha,\alpha p), & \left[(1-\lambda,\mu)\right]^r, & (1-\rho,p); \end{matrix} \right.\right). \quad (6.2.1)$$

6.3. Relationship with Wright Function

If $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$, $\delta,\mu,p>0$ (1.2) can be written as

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)}{[\Gamma(\gamma)]^s}\sum_{n=0}^{\infty}\frac{[\Gamma(\gamma+\delta n)]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda+\mu n)]^r\Gamma(\rho+pn)}, \tag{6.3.1}$$

from (1.9) for (6.3.1), we get

$$=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)z^{\rho-1}}{[\Gamma(\gamma)]^s}\,{}_{s+1}\Psi_{r+2}\left[\begin{matrix}[(\gamma,\delta)]^s, & (1,1); & z^p\\ (\beta+\alpha\rho-\alpha,\alpha p), & [(\lambda,\mu)]^r, \ (\rho,p); & \end{matrix}\right]$$

7. Summary

In Section 1, an extended version of Mittag-Leffler function of 10 indices established as an Equation (1.2) including with some necessary information of Bessel function, some well-known Integral transforms and generalized hypergeometric functions with their family. Results obtained in Sections 2 to 6 are interesting generalizations of (Shukla and Prajapati [1]) and stimulate the scope of further research in the field of generalization Mittag-Leffler function.

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RECURRENCE RELATION OF A UNIFIED GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. The present work incorporates a recurrence relation and an integral representation of a further generalization of a generalized Mittag-Leffler fuction due to A.K. Shukla and J.C. Prajapati[Surveys in Mathematics and its Applications, Volume 4(2009), 133-138]. At the end, several special cases have also been discussed.

1. Introduction, definitions and Preliminaries

The Mittag-Leffler function has been studied by many researchers either in context with obtaining new properties or by introducing a new generalization and then deriving its properties ([5], [7], [4]). Recently, we have also studied various properties of our newly introduced unification of Generalized Mittag-Leffler function in the form [2]

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (cz)^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\ [(\lambda)_{\mu n}]^r\ (\rho)_{pn}}, \tag{1.1}$$

wherein $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho\ \in \mathbb{C},\ Re(\alpha,\beta,\gamma,\lambda,\rho) > 0;\ \delta,\ \mu,\ p,\ c > 0$. If $p = 1,\ \rho = 1,\ r = 0,\ s = 1,\ \delta = q,\ s = 1,\ c = 1$, then this yields the generalization due to Shukla and Prajapati [5]. In the next section, we prove the main results.

2. Recurrence Relation

We begin by stating the main theorem.

Theorem 1. For $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho\ \in \mathbb{C},\ Re((\alpha+a),(\beta+b),\gamma,\lambda,\rho) > 0,$ $\delta,\ \mu,\ p,\ c > 0,$ we get

$$\begin{aligned} &E^{\gamma,\delta}_{\alpha+a,\,\beta+b+1,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) - E^{\gamma,\delta}_{\alpha+a,\,\beta+b+2,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) \\ &\quad = (\alpha+a)^2\, z^2 \ddot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) \\ &\qquad + z\,(\alpha+a)\big[\alpha+a+2(\beta+b+1)\big]\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) \\ &\qquad + (\beta+b)\ (\beta+b+2)\ E^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r), \end{aligned} \tag{2.1}$$

where, $\dot{E}^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) = \dfrac{d}{dz}E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r),$

$\ddot{E}^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) = \dfrac{d^2}{dz^2}E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r).$

It is easy to obtain the following corollary by letting $\alpha + a = k$ and $\beta + b = m$.

Corollary: We have, for $k,\ m\ \in \mathbb{N}$,

$$E^{\gamma,\delta}_{k,\ m+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = E^{\gamma,\delta}_{k,\ m+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) + m(m+2)E^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\ +\ k^2\ z^2\ddot{E}^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) +k\,[k+2(m+1)]\ z\dot{E}^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r). \tag{2.2}$$

Proof of Theorem 1. By substituting $\alpha\ =\ \alpha+a\ ,\beta\ =\ \beta+b+1$ in (1.1) and applying the fundamental relation of the Gamma function $\Gamma(z+1)\ =\ z\Gamma(z)$, we have

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b))^{-1}(cz)^{(p\,n+\rho-1)}}{((\alpha+a)(p\,n+\rho-1)+\beta+b)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \tag{2.3}$$

and

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{((\alpha+a)(p\,n+\rho-1)+\beta+b)[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \cdot\frac{(\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+2))^{-1}}{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}. \tag{2.4}$$

Equation (2.4) can be written as follows:

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty}\left[\frac{1}{((\alpha+a)(p\,n+\rho-1)+\beta+b)}\ -\ \frac{1}{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}\right] \cdot\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}$$

$$= E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\ -\ \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b)} \cdot\frac{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}. \tag{2.5}$$

For the sake of convenience, we denote the last summation in (2.5) by S, then

$$S\ =\ E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) - E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r). \tag{2.6}$$

Applying the following(evident):

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{(u+1)}$$

and then taking $u = ((\alpha+a)(p\,n+\rho-1)+\beta+b+1)$ to (2.6), we obtain

$$S = \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} + \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \cdot\frac{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)}$$

$$= (\alpha+a)\ \sum_{n=1}^{\infty}\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn-1}}$$

$$+\ (\beta+b)\ \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}$$

$$+ (\alpha+a)^2 \sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,(cz)^{(p\,n+\rho-1)}\,((\alpha+a)(p\,n+\rho-1)+\beta+b)}{[(\gamma)_{\delta\,n}]^{-s}\,\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$+ x \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(cz)^{(p\,n+\rho-1)}\,((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$+ y \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}}, \tag{2.7}$$

where, $x = (\alpha+a)\,(2\beta+2b+1)$ and $y = (\beta+b)(\beta+b+1)$.
We now express each summation in the right hand side of (2.7) as follows:

$$\frac{d^2}{dz^2}\left[z^2\,E^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\right]$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(p\,n+\rho+1)\,(p\,n+\rho)\,(cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}}. \tag{2.8}$$

From (2.8) we find that

$$\sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) + 4z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)$$

$$- 3 \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}\,(p\,n+\rho-1)\,z^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}. \tag{2.9}$$

Now,

$$\frac{d}{dz}\left[z\,E^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\right]$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(p\,n+\rho)\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}},$$

and

$$\sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r). \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) + z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) \tag{2.11}$$

Applying (2.10) and (2.11) to (2.7), we find that

$$S = (\alpha+t)^2\,z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) + z\left[(\alpha+a)^2+(\alpha+a)+x\right]$$

$$.\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r) + (\beta+b+y)\,E^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r).$$

Now setting this last identity for S in (2.6), completes the proof of Theorem 1.

3. Integral Representation:

Theorem 2. For $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho \in \mathbb{C},\ Re((\alpha+a),(\beta+b),\gamma,\lambda,\rho) > 0$
$\delta,\ \mu,\ p,\ c > 0,$ we get

$$\int_0^1 (cu)^{\beta+b}\,E^{\gamma,\delta}_{\alpha+a,\,\beta+b,\,\lambda,\,\mu,\,\rho,\,p}((cu)^{\alpha+a};s,r)du$$

$$= c^{\beta+b}\left(E^{\gamma,\delta}_{\alpha+a,\,\beta+b+1,\,\lambda,\,\mu,\,\rho,\,p}(c^{\alpha+a};s,r) - E^{\gamma,\delta}_{\alpha+a,\,\beta+b+2,\,\lambda,\,\mu,\,\rho,\,p}(c^{\alpha+a};s,r)\right). \tag{3.1}$$

Setting $\alpha+a = k \in \mathbb{N}$ and $\beta+b = m \in \mathbb{N}$ in (3.1) yields

Corollary:

$$\int_0^1 (cu)^m\, E^{\gamma,\delta}_{k,\, m,\, \lambda,\, \mu,\, \rho,\, p}((cu)^k; s, r)du$$
$$= c^m \Big(E^{\gamma,\delta}_{k,\, m+1,\, \lambda,\, \mu,\, \rho,\, p}(c^k; s, r) \;-\; E^{\gamma,\delta}_{k,\, m+2,\, \lambda,\, \mu,\, \rho,\, p}(c^k; s, r) \Big). \qquad (3.2)$$

Proof of the Theorem 2. Putting z=1 in (2.6) gives

$$\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (c)^{(p\, n+\rho-1)}}{\Gamma((\alpha+a)(p\, n+\rho-1)+\beta+b)} \cdot \frac{((\alpha+a)(p\, n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r\, (\rho)_{pn}}$$
$$= E^{\gamma,\delta}_{\alpha+a,\, \beta+b+1,\, \lambda,\, \mu,\, \rho,\, p}(c; s, r) - E^{\gamma,\delta}_{\alpha+a,\, \beta+b+2,\, \lambda,\, \mu,\, \rho,\, p}(c; s, r). \qquad (3.3)$$

It is easy to find that

$$\int_0^z (cu)^{\beta+b}\, E^{\gamma,\delta}_{\alpha+a,\, \beta+b,\, \lambda,\, \mu,\, \rho,\, p}((cu)^{\alpha+a}; s, r)du$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, c^{(\alpha+a)(p\, n+\rho-1)+\beta+b}\, z^{(\alpha+a)(p\, n+\rho-1)+\beta+b+1}}{\Gamma((\alpha+a)(p\, n+\rho-1)+\beta+s)} \cdot \frac{((\alpha+a)(p\, n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r\, (\rho)_{pn}}. \qquad (3.4)$$

On comparing (3.3) with the identity obtaining by setting z=1 in (3.4) completes the proof of Theorem 2.

4. Special Cases:

1. Setting $r = 0, \rho = p = c = s = 1, \delta = q$ in (2.1), we get recurrence relation of $E^{\gamma,q}_{\alpha,\, \beta}(z)$ [6]:

$$E^{\gamma,\, q}_{\alpha+a,\, \beta+b+1}(z) \;-\; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+2}(z) = (\alpha+a)^2\, z^2 \ddot{E}^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z)$$
$$+z\,(\alpha+a)\big[\alpha+a+2(\beta+b+1)\big]\dot{E}^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z)$$
$$+\,(\beta+b)\,(\beta+b+2)\, E^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z), \qquad (4.1)$$

where, $\dot{E}^{\gamma,\, q}_{\alpha,\, \beta}(z) \;=\; \dfrac{d}{dz} E^{\gamma,\, q}_{\alpha,\, \beta}(z)$ and $\ddot{E}^{\gamma,\, q}_{\alpha,\, \beta}(z) \;=\; \dfrac{d^2}{dz^2} E^{\gamma,\, q}_{\alpha,\, \beta}(z)$.

2. Putting $r = a = 0, \gamma = \delta = \rho = p = s = 1; \beta + b = m \in \mathbb{N}$ in (2.1) reduces to a known recurrence relation of $E_{\alpha,\, \beta}(z)$ [1]:

$$E_{\alpha,\, m+1}(z) \;=\; \alpha^2\, z^2 \ddot{E}_{\alpha,\, m+3}(z) \;+\; \alpha(\alpha+2m+2)\, z\, \dot{E}_{\alpha,\, m+3}(z)$$
$$+\,m(m+2)\, E_{\alpha,\, m+3}(z) \;+\; E_{\alpha,\, m+2}(z), \qquad (4.2)$$

where, $\dot{E}_{\alpha,\, \beta}(z) \;=\; \dfrac{d}{dz} E_{\alpha,\, \beta}(z)$ and $\ddot{E}_{\alpha,\, \beta}(z) \;=\; \dfrac{d^2}{dz^2} E_{\alpha,\, \beta}(z)$.

3. Substituting $r = 0, \rho = p = c = s = 1, \delta = q$ in (3.1), we get integral representation of $E^{\gamma,q}_{\alpha,\, \beta}(z)$ [6]:

$$\int_0^1 u^{\beta+b}\, E^{\gamma,\, q}_{\alpha+a,\, \beta+b}(u^{\alpha+a})du \;=\; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+1}(1) \;-\; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+2}(1) \qquad (4.3)$$

4. Substituting $(r = 0, \rho = p = c = \delta = \gamma = s = k = m = 1)$ and $(r = 0,\ \rho = p = c = \delta = s = k = m = 1\,, \gamma \;=\; 2)$ in (3.2) respectively, yields

$$\int_0^1 u\, e^u\, du \;=\; E_{1,\, 2}(1) \;-\; E_{1,\, 3}(1)$$

and

$$\int_0^1 u\, E^{2,1}_{1,\, 1}(1) \;=\; E^{2,1}_{1,\, 2}(1) \;-\; E^{2,1}_{1,\, 3}(1).$$

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FRACTIONAL CALCULUS OF A UNIFIED MITTAG-LEFFLER FUNCTION

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UDC 517.9

The main aim of the paper is to introduce an operator in the space of Lebesgue measurable real or complex functions $L(a,b)$. Some properties of the Riemann–Liouville fractional integrals and differential operators associated with the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(cz;s,r)$ are studied and the integral representations are obtained. Some properties of a special case of this function are also studied by the means of fractional calculus.

1. Introduction, Definitions, and Preliminaries

The Mittag-Leffler function was studied by numerous researchers with an aim either to establish its new properties or to introduce its new generalization and then study its properties [9, 11, 13]. In [7], we have also studied various properties of our newly introduced generalization of the Mittag-Leffler function in the form

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(cz;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s\,(cz)^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\,[(\lambda)_{\mu n}]^r\,(\rho)_{pn}},\tag{1.1}$$

where $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C},\ \mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0,\ \delta,\ \mu,p,c>0,$ and

$$(\gamma)_{q\,n}=\frac{\Gamma(\gamma+q\,n)}{\Gamma(\gamma)}$$

is the generalized Pochhammer symbol [8]. In particular, for $q\in\mathbb{N}.$ it takes the form

$$(\gamma)_{q\,n}=q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n.$$

If $p=1,\ \rho=1,\ r=0,\ s=1,\ \delta=q,\ s=1,\ c=1,$ then relation (1.1) yields the generalization due to Shukla and Prajapati [11]. Here, we also introduce an operator denoted and defined by as follows:

$$\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}f\right)(x)=\int_a^x(x-t)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^{\alpha};s,r\big)f(t)\,dt,\tag{1.2}$$

where $\alpha,\beta,\gamma,\lambda,\rho,\omega\in\mathbb{C},\ \mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0,\ \delta,\mu,p>0,$ and $x>a.$

We enlist the following definitions and well-known formulas to study the properties of the Riemann–Liouville (R–L) fractional integrals and differential operators associated with our generalization (1.1), as well as th operator (1.2):

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The space $L(a,b)$ of (real or complex-valued) Lebesgue measurable functions [4, 10] is given by

$$L(a,b) = \left\{ f : \|f\|_1 = \int_a^b |f(t)|dt < \infty \right\}. \tag{1.3}$$

For $f(x) \in L(a,b),\ \mu \in \mathbb{C},$ and $\mathrm{Re}(\mu) > 0,$ the R–L fractional integrals of order μ [10] are defined as follows:

The left-sided R–L fractional integral operator of order μ is defined as

$$_aI_x^{\mu} f(x) = I_{a+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x > a, \tag{1.4}$$

where the right-sided R–L fractional integral operator of order μ is defined as

$$_xI_b^{\mu} f(x) := I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x < b. \tag{1.5}$$

Further, if $\mu, \beta \in \mathbb{C},\ \mathrm{Re}(\mu, \beta) > 0,$ then [6, 10]

$$I_{a+}^{\mu}[(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\mu+\beta)}(x-a)^{\mu+\beta-1}. \tag{1.6}$$

For $\mu \in \mathbb{C},\ \mathrm{Re}(\mu) > 0;\ n = [\mathrm{Re}(\mu)] + 1,$ the R–L fractional derivative is

$$(D_{a+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha} f)(x). \tag{1.7}$$

Then, for $\alpha, \beta, \gamma, \lambda, \rho, \in \mathbb{C},\ \mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho, (\beta - m)) > 0,$ and $\delta, \mu, p, m \in \mathbb{N},$ we can show that [7]

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\, E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\big(\omega\ (cz)^{\alpha}; s, r\big)\right] = z^{\beta-m-1}\, E_{\alpha,\beta-m,\lambda,\mu,\rho,p}^{\gamma,\delta}\big(\omega\ (cz)^{\alpha}; s, r\big). \tag{1.8}$$

The fractional integral operator investigated by Erdélyi–Kober is defined and represented as

$$I_x^{\eta,\nu}\{f(x)\} = \frac{x^{-\eta-\nu+1}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt, \qquad \mathrm{Re}(\nu) > 0, \quad \eta > 0. \tag{1.9}$$

This is a generalization of the R–L fractional integral operator (1.5).

Hilfer [2, 3] generalized the R–L fractional derivative operator D_{a+}^{μ} in (1.6) by introducing a right-sided fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ with respect to x as follows:

$$(D_{a+}^{\mu,\nu} f)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx} (I_{a+}^{(1-\nu)(1-\mu)} f)\right)(x). \tag{1.10}$$

The difference between the fractional derivatives of various types becomes apparent from the following formula involving the Laplace transformation [2, 3]:

$$\mathcal{L}[D_0^{\mu,\nu} f(x)](s) = s^{\mu}\,\mathcal{L}[f(x)](s) - s^{\nu(1-mu)}\big(I_0^{(1-\nu)(1-\mu)} f\big)(0+), \tag{1.11}$$

where $0 < \mu < 1$, and the initial-value term: $\big(I_0^{(1-\nu)(1-\mu)} f\big)(0+)$ involves the R–L fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0+$. Here, as usual

$$\mathcal{L}[f(x)](s) = \int_0^{\infty} e^{-sx}\, f(x)\, dx, \tag{1.12}$$

provided that the defining integral exists.

Prajapati, Dave, and Nathwani [7] showed that the Mellin–Barnes integral for the function defined by (1.1) is given by

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z; s, r) = \frac{[\Gamma(\lambda)]^r\,\Gamma(\rho)\,p\,z^{\rho-1}}{2\pi i\,[\Gamma(\gamma)]^s} \int_L \frac{\Gamma(-p\xi)\,\Gamma(1+p\xi)\,[\Gamma(\gamma+\delta\xi)]^s\,(-z)^{p\xi}}{\Gamma(\beta+\alpha\rho-\alpha+\alpha p\xi)\,[\Gamma(\lambda+\mu\xi)]^r\,\Gamma(\rho+p\xi)} d\xi. \tag{1.13}$$

Wright's generalized hypergeometric function [1] is defined as

$${}_p\psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); & z\\ (b_1, B_1), \ldots, (b_q, B_q); & \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + rA_j)}{\prod_{j=1}^{q} \Gamma(b_j + rB_j)} \frac{z^r}{r!} \tag{1.14}$$

$$= H_{p,q+1}^{1,p}\left[-z \,\middle|\, \begin{matrix}(1-a_1, A_1), \ldots, (1-a_p, A_p)\\ (0,1), (1-b_1, B_1), \ldots, (1-b_q, B_q)\end{matrix}\right], \tag{1.15}$$

where

$$H_{p,q}^{m,n}\left[-z \,\middle|\, \begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\right]$$

denotes the Fox H-function and $a_i, b_j \in \mathbb{C},\ A_i, B_j \in \mathbb{R},\ i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q,\ 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0$.

2. Main Results

In this section, we prove the following results.

Theorem 2.1. *Let* $a \in \mathbb{R}_+ = [0, \infty),\ \alpha, \beta, \gamma, \lambda, \rho, \eta \in \mathbb{C},\ \operatorname{Re}(\alpha, \beta, \gamma, \lambda, \rho, \eta) > 0;\ \delta,\ \mu, p > 0$ *for* $x > a$. *Then*

$$\Big(I_{a+}^{\eta}\,(t-a)^{\beta-1}\,E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega(c(t-a))^{\alpha}; s, r\Big)(x)$$

$$= (x-a)^{(\eta+\beta-1)} E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}(\omega(c(x-a))^{\alpha}; s, r) \tag{2.1}$$

and

$$\Big(D^{\eta}_{a+}\ (t-a)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r)\Big)(x)$$
$$= (x-a)^{(\beta-\eta-1)} E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu,\rho,p}(\omega(c(x-a))^{\alpha}; s, r). \tag{2.2}$$

Proof. We now apply (1.1) to the left-hand side of (1.13) and then use (1.6). This yields

$$\Big(I^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r\Big)(x)$$
$$= (x-a)^{\beta+\eta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^{s} (\omega\ (c\ (x-a))^{\alpha})^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\ [(\lambda)_{\mu n}]^{r}\ (\rho)_{pn}}.$$

Thus, by using (1.1) once again, we find (2.1).

We now apply (1.7) to the left-hand side of (2.2) and then use (2.1). As a result, we get

$$\Big(D^{\eta}_{a+}\ (t-a)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(t-a))^{\alpha}; s, r)\Big)(x)$$
$$= \left(\frac{d}{dx}\right)^{n} \big[(x-a)^{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\ \mu,\rho,p}(\omega(c(x-a))^{\alpha}; s, r)\big].$$

Further, in view of (1.7), we complete the proof of (2.2).

Theorem 2.2. *Let* $\alpha, \gamma, \lambda, \rho, \eta \in \mathbb{C};\ \mathrm{Re}(\alpha), \mathrm{Re}(\gamma), \mathrm{Re}(\lambda), \mathrm{Re}(\rho), \mathrm{Re}(\eta) > 0;\ \delta, \mu, p > 0.$ *Then*

$${}_0I^{\eta}_{x}\big[\nu\ E^{1,\delta}_{\alpha,1,\lambda,\mu,\rho,p}(\nu(cx)^{\alpha}; s, r)\big] = \nu\ x^{\eta}\ E^{1,\delta}_{\alpha,\eta+1,\ \lambda,\mu,\rho,p}(\nu(cx)^{\alpha}; s, r). \tag{2.3}$$

Proof. Applying (1.4) to the left-hand side of (2.3) and then using (1.1), we get

$${}_0I^{\eta}_{x}\Big[\nu\ E^{1,\delta}_{\alpha,1,\lambda,\mu,\rho,p}(\nu(cx)^{\alpha}; s, r)\Big]$$
$$= \frac{1}{\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{[(1)_{\delta n}]^{s}\ c^{\alpha(pn+\rho-1)} \nu^{(pn+\rho)}}{\Gamma(\alpha(pn+\rho-1)+1)\ [(\lambda)_{\mu n}]^{r}\ (\rho)_{pn}} \int_{0}^{x} t^{\alpha(pn+\rho-1)}(x-t)^{(\eta-1)}\ dt.$$

After certain simplifications with subsequent use of (1.1), we get the proof of (2.3).

Theorem 2.3. *Let* $\alpha, \beta, \gamma, \lambda, \rho, \nu, \omega \in \mathbb{C};\ \mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho, \nu) > 0;\ \delta, \mu,\ p > 0.$ *Then*

$$\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\ (t-a)^{\nu-1}\big)(x) = (x-a)^{\beta+\nu-1}\ \Gamma(\nu) E^{\gamma,\delta}_{\alpha,\beta+\nu,\ \lambda,\mu,\rho,p}\big(\omega(x-a)^{\alpha}\big). \tag{2.4}$$

Proof. Setting $f(t)=(t-a)^{\nu-1}$ in (1.2), we obtain

$$\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\,(t-a)^{\nu-1}\right)(x)=\int\limits_a^x (x-t)^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,}\big(\omega(x-t)^{\alpha}\big)\,(t-a)^{\nu-1}\,dt.$$

By virtue of (1.1), this is reduced to

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^s\,\omega^{(p\,n+\rho-1)}}{\Gamma(\alpha(p\,n+\rho-1)+\beta)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}}$$

$$\times\int\limits_a^x (x-t)^{\alpha(p\,n+\rho-1)+\beta-1}\,(t-a)^{\nu-1}\,dt.$$

Simplifying this equation, we get

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^s\,((\rho)_{pn})^{-1}\,\omega^{(p\,n+\rho-1)}}{\Gamma(\alpha(p\,n+\rho-1)+\beta)[(\lambda)_{\mu\,n}]^r}B(\alpha(p\,n+\rho)+\beta-1,\nu).$$

As a result of subsequent simplification of this equation, we prove (2.4).

We now show that the operator defined by (1.2) is, in fact, bounded. The proof is presented in the form of the following theorem:

Theorem 2.4. *Let the function ϕ be given by*

$$L(a,b)=\left\{f:\|f\|_1=\int\limits_a^b |f(t)|dt<\infty\right\}$$

in the space $L(a,b)$ of Lebesgue measurable functions on a finite interval $[a,b]$ of the real line $\mathbb{R}$. Then the integral operator $\mathcal{E}^{\omega;\gamma,\delta}_{a+;\,\alpha,\beta,\lambda,\mu,\rho,p}$ is bounded on $L(a,b)$ and

$$\left\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\phi\right\|_1\le\mathfrak{M}\|\phi\|_1, \tag{2.5}$$

where the constant $\mathfrak{M}$, $0<\mathfrak{M}<\infty$, is given by

$$\mathfrak{M}=(b-a)^{\mathrm{Re}(\beta)}\sum_{k=0}^{\infty}\frac{|(\gamma)_{\delta\,k}|^s}{|\Gamma(\alpha\,(pk+\rho-1)+\beta)|\,(\mathrm{Re}\,(\alpha(p\,k+\rho-1)+\beta))}$$

$$\times\frac{|\omega\,((b-a)c)^{\alpha}|^{\mathrm{Re}(pk+\rho-1)}}{|(\lambda)_{\mu\,k}|^r\,|(\rho)_{pk}|}. \tag{2.6}$$

Proof. Using (1.2) and (1.3) and interchanging the order of integration by applying the Dirichlet formula [9], we get

$$\left\|\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}\phi\right\|_1 = \int_a^b \left|\int_a^x (x-t)^{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\,\omega(c(x-t))^{\alpha}; s,r))\ \phi(t)\ dt\right| dx$$

$$\leq \int_a^b \left[\int_t^b (x-t)^{\mathrm{Re}(\beta)-1}\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(c(x-t))^{\alpha}; s,r)\right| dx\right] |\phi(t)|\, dt.$$

Substituting $x-t=u$, using (1.1) and simplifying the above equation, we find

$$= \int_a^b \left[\int_0^{b-t} u^{\mathrm{Re}(\beta)-1}\ \left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(cu)^{\alpha}; s,r)\right| du\right] |\phi(t)|\ dt$$

$$\leq \int_a^b \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s\ (\omega\ c^{\alpha})^{pk+\rho-1}}{\left|\Gamma(\alpha\ (pk+\rho-1)+\beta)\right|\ |(\lambda)_{\mu k}|^r\ |(\rho)_{pk}|}$$

$$\times \left[\int_0^{b-a} u^{\mathrm{Re}(\alpha(pk+\rho-1)+\beta-1)}\right] |\phi(t)|\, dt$$

$$= \int_a^b \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta\, k}|^s\ (\omega\ c^{\alpha})^{pk+\rho-1}|b-a|^{\mathrm{Re}(\alpha(pk+\rho-1)+\beta)}\ |\phi(t)|}{\left|\Gamma(\alpha\ (pk+\rho-1)+\beta)\right|\ |(\lambda)_{\mu k}|^r\ |(\rho)_{pk}|\,\mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)}\, dt$$

$$= (b-a)^{\mathrm{Re}(\beta)}\ \sum_{k=0}^{\infty}\ \frac{|(\gamma)_{\delta\, k}|^s}{|\Gamma(\alpha\ (pk+\rho-1)+\beta)|\ |(\lambda)_{\mu\, k}|^r\ |(\rho)_{pk}|}$$

$$\times \frac{|\omega\ (c(b-a))^{\mathrm{Re}\,(\alpha)}|^{pk+\rho-1}}{\mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)}\ \int_a^b |\phi(t)|\ dt$$

$$= (b-a)^{\mathrm{Re}(\beta)}\ \sum_{k=0}^{\infty}\ \frac{|(\gamma)_{\delta\, k}|^s\ |\omega\ ((b-a)c)^{\alpha}|^{\mathrm{Re}(pk+\rho-1)}}{|\Gamma(\alpha\ (pk+\rho-1)+\beta)|\ \mathrm{Re}\,(\alpha(pk+\rho-1)+\beta)}\frac{\|\phi\|_1}{|(\lambda)_{\mu\, k}|^r\ |(\rho)_{pk}|}$$

$$= \mathfrak{M}\ \|\phi\|_1,$$

where $\mathfrak{M}$ is finite and given by (2.6). This completes the proof of the property of boundedness for the integral operator $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}$, as asserted by Theorem 2.4.

The following theorem incorporates the fractional differential equation for (1.2).

Theorem 2.5. *If* $0<\eta<1,\ 0\leq\nu\leq 1,\ \omega,\xi\in\mathbb{C},\ R(\alpha)=R(\delta)-1>0$ *and* $\min\{\mathrm{Re}(\beta,\gamma,\lambda,\mu,\rho)\}>0,$ *then*

$$\left(D^{\eta,\nu}_{0+}\ y\right)(x) = \xi\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0+}\right)(x) + f(x) \tag{2.7}$$

with the initial condition

$$\Big(I_0 +^{(1-\nu)(1-\eta)}\ y\Big)(0+) = C,$$

has a solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\,\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\, x^{\eta+\beta}\, E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\big(\omega(a\,x^{\alpha})\big) + \frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}\, f(t)\, dt, \tag{2.8}$$

where C is an arbitrary constant.

Proof. Applying the Laplace transformation to each side of (2.7) and using relations(1.2) and (1.11), we conclude, by virtue of the Laplace convolution theorem, that

$$\begin{aligned} s^{\eta}\,Y(s) - C\,s^{\nu\,(1-\eta)} &= \xi\,\mathcal{L}\big[x^{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega\,x^{\alpha})\big](s)\mathcal{L}(1)(s) + F(s) \\ &= \xi\,s^{-\beta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^{s}\,(\omega\,(as)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu\,n}]^{r}(\rho)_{pn}} + F(s), \end{aligned}$$

which immediately yields

$$Y(s) = C\,s^{\nu\,(1-\eta)-\eta} + \xi\,s^{-\beta-\eta-1}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta\,n}]^{s}\,(\omega\,(as)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu\,n}]^{r}\,(\rho)_{pn}} + F(s)\,s^{-\eta}. \tag{2.9}$$

Further, by taking the inverse Laplace transform of each side of equation (2.9), we get

$$\begin{aligned} y(x) &= C\,\mathcal{L}^{-1}\big(s^{\nu(1-\eta)-\eta}\big)(x) \\ &\quad + \xi\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,(\omega\,(a)^{\alpha})^{pn+\rho-1}}{[(\lambda)_{\mu n}]^{r}\,(\rho)_{pn}}\,\mathcal{L}^{-1}\big(s^{-\alpha(pn+\rho-1)-\beta-\eta-1}\big)(x) + \mathcal{L}^{-1}(s^{-\eta}\,F(s)) \\ &= C\,\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + \xi\,x^{\eta+\beta}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,(\omega\,a^{\alpha})^{pn+\rho-1}\,[(\rho)_{pn}]^{-1}\,x^{\alpha(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)[(\lambda)_{\mu n}]^{r}} \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}\,f(t)\,dt \\ &= C\,\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\,\nu)} + \xi\,x^{\eta+\beta}\,E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}\big(\omega(a\,x^{\alpha})\big) + \frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}\,f(t)\,dt, \end{aligned}$$

which completes the proof of Theorem 2.5 under the various already indicated parametric constraints.

Theorem 2.6. *If* $0 < \eta < 1,\ 0 \leq \nu \leq 1,\ \omega, \xi \in \mathbb{C},\ R(\alpha) = R(\delta) - 1 > 0$ *and* $\min\{R(\beta, \gamma, \lambda, \mu, \rho)\} > 0$, *then*

$$\left(D_{0+}^{\eta,\nu}\, y\right)(x) = \xi \left(\mathcal{E}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0+}^{\gamma,\delta}\right)(x) + x^{\beta}\ E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(ax)^{\alpha}); s, r\big) \tag{2.10}$$

with the initial condition

$$\left(I_0 +^{(1-\nu)(1-\eta)}\ y\right)(0+) = C$$

has a solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\ \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)} + (\xi+1)\ x^{\eta+\beta}\ E_{\alpha,\beta\eta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(ax)^{\alpha}); s, r\big), \tag{2.11}$$

where C is an arbitrary constant.

Proof. Substituting

$$f(t) = t^{\beta}\ E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(at)^{\alpha}); s, r\big)$$

in Theorem 2.5, we get

$$y(x) = C\ \frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\ \nu)} + \xi\ x^{\eta+\beta}\ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(ax)^{\alpha}); s, r\big)$$

$$+ \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\ t^{\beta}\ E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(ax)^{\alpha}); s, r\big)\ dt. \tag{2.12}$$

Here,

$$\int_0^x (x-t)^{\eta-1}\ t^{\beta}\ E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\big((\omega(ax)^{\alpha}); s, r\big)\ dt$$

$$= \int_0^x (x-t)^{\eta-1}\ t^{\beta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\ n}]^s (\omega\ (at)^{\alpha})^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta+1)[(\lambda)_{\mu\ n}]^r\ (\rho)_{pn}}\ dt$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\ n}]^s (\omega\ (a)^{\alpha})^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta+1)[(\lambda)_{\mu\ n}]^r\ (\rho)_{pn}} \int_0^x (x-t)^{\eta-1}\ t^{\alpha(pn+\rho-1)+\beta}\ dt. \tag{2.13}$$

We take $t = xu$, then $dt = xdu$ and we have $u \to 0$ as $t \to 0$, and $u \to 1$ as $t \to x$. Therefore,

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (\omega\ a^{\alpha})^{pn+\rho-1}\ x^{\alpha(pn+\rho-1)+\eta+\beta}}{\Gamma(\alpha(pn+\rho-1)+\beta)\ [(\lambda)_{\mu n}]^r\ (\rho)_{pn}} \int_0^1 (1-u)^{\eta-1}\ u^{\alpha(pn+\rho-1)+\beta} dt$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s \, (\omega\, a^{\alpha})^{pn+\rho-1} \, x^{\alpha(pn+\rho-1)+\eta+\beta} \, \Gamma(\eta)\, \Gamma(\alpha(pn+\rho-1)+\beta)}{\Gamma(\alpha(pn+\rho-1)+\beta)\, [(\lambda)_{\mu\, n}]^r \, (\rho)_{pn} \, \Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)}$$

$$= \Gamma(\eta) \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s \, (\omega\, (ax)^{\alpha})^{pn+\rho-1} \, x^{(\eta+\beta)}}{\Gamma(\alpha(pn+\rho-1)+\beta+\eta+1)\, [(\lambda)_{\mu\, n}]^r \, (\rho)_{pn}}$$

$$= x^{(\eta+\beta)} \, \Gamma(\eta) \, E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu,\rho,p} \big(\omega(ax)^{\alpha}; s, r\big).$$

Using this result in (2.12), we get (2.11).

This completes the proof of Theorem 2.6.

Theorem 2.7 (Mellin transform of the operator $\big(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f\big)(x)$). *Let* $\alpha, \beta, \gamma, \lambda, \rho, \omega \in \mathbb{C},$ $\mathrm{Re}(\alpha, \beta, \gamma, \lambda, \rho) > 0;$ $\delta, \mu > 0,$ $p \in \mathbb{N},$ *and* $\mathrm{Re}(1 - S - \alpha\rho + \alpha - \beta) > 0.$ *Then*

$$M\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f)(x); S\Big\} = \frac{[\Gamma(\lambda)]^r \Gamma(\rho)\, p}{2\pi i\, [\Gamma(\gamma)]^s \Gamma(1-S)} H^{r+3,s+1}_{s+1,r+3}$$
$$\times \left[-wt^{\alpha} \,\middle|\, \begin{matrix} [(1-\gamma,\delta)]^s, (0,p) \\ (0,1), (1 - S - \alpha\rho + \alpha - \beta,\ \alpha\, p), [(1-\lambda,\mu)]^r, (1-\rho,p) \end{matrix} \right]$$
$$\times M\big\{t^{\beta} f(t);\ S\big\}. \tag{2.14}$$

Proof. By the definition of the Mellin transform, we find

$$M\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+} f)(x); S\Big\} = \int_0^{\infty} x^{S-1} \int_0^{x} (x-t)^{\beta-1} \, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^{\alpha}; s, r\big) f(t)\, dt\, dx.$$

Changing the order of integration, which can be done under given conditions, we conclude that

$$M\Big\{(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\Big\} = \int_0^{\infty} f(t) \int_t^{\infty} x^{S-1} (x-t)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\big(\omega(x-t)^{\alpha}; s, r\big)\, dx\ dt.$$

If we set $x = t + u$, then this integral takes the form

$$M\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\Big\} = \int_0^{\infty} f(t) \int_t^{\infty} (t+u)^{S-1} u^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega u^{\alpha})\, du\ dt.$$

To evaluate the u-integral, we express the Mittag-Leffler function in terms of its Mellin–Barnes contour integral by using relation (1.14). As a result, this expression can be represented in the form

$$M\Big\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+} f)(x); S\Big\} = \int_0^{\infty} f(t) \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p (-\omega)^{\rho-1}}{2\pi i [\Gamma(\gamma)]^s}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi)\,\Gamma(1+p\xi)\,[\Gamma(\gamma+\delta\xi)]^s\,(-\omega)^{p\xi}}{\Gamma(\beta+\alpha\rho-\alpha+\alpha p\xi)\,[\Gamma(\lambda+\mu\xi)]^r\Gamma(\rho+p\xi)}$$

$$\times \int_0^{\infty} (t+u)^{s-1} u^{\alpha(p\xi+\rho-1)+\beta-1} du\, d\xi\, dt.$$

If we evaluate the u-integral with the help of the formula

$$\int_0^{\infty} x^{\nu-1}(x+a)^{-\rho} dx = \frac{\Gamma(\nu)\Gamma(\rho-\nu)}{\Gamma(\rho)}, \quad \mathrm{Re}(\rho) > \mathrm{Re}(\nu) > 0,$$

then the right-hand side of this equation can be simplified to

$$\frac{[\Gamma(\lambda)]^r\Gamma(\rho)p}{2\pi i[\Gamma(\gamma)]^s\Gamma(1-s)}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi)\Gamma(1+p\xi)[\Gamma(\gamma+\delta\xi)]^s\Gamma(1-s-\alpha(p\xi+\rho-1)-\beta)}{[\Gamma(\lambda+\mu\xi)]^r\Gamma(\rho+p\xi)}$$

$$\times (-\omega t^{\alpha})^{p\xi+\rho-1} d\xi \int_0^{\infty} t^{\beta+s-1} f(t)\, dt.$$

By using the definition of the H-function, we obtain the desired result.

For $s=1,\ r=0,\ \rho=1,\ p=1,$ and $\delta=q$ Theorem 2.7 is reduced to the following corollary:

Corollary 2.1.

$$M\left\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x); S\right\} = \frac{1}{\Gamma(\gamma)\Gamma(1-S)} H^{2,1}_{1,2}\left[-wt^{\alpha}\ \middle|\ \begin{matrix}(1-\gamma,q)\\(0,1)(1-S-\beta,\,\alpha)\end{matrix}\right] M\{t^{\beta}f(t); S\},$$

where $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0;\ q\in(0,1)\cup\mathbb{N},\ \mathrm{Re}(1-S-\beta)>0,$ *and* $H^{2,1}_{1,2}(\cdot)$ *is the* H*-function defined by (1.15).*

Theorem 2.8 (Laplace transform of the operator $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;0_+}f\right)(x)$**).**

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;\,0_+}f)(x); P\right\} = \frac{[\Gamma(\lambda)]^r\,\Gamma(\rho)\,\omega^{\rho-1}}{[\Gamma(\gamma)]^s\,P^{\beta+\alpha\rho-\alpha}}\ {}_{s+1}\psi_{r+1}\left[\begin{matrix}[(\gamma,q)]^s, & (1,1); & \omega^p/P^{p\alpha}\\ [(\lambda,\mu)]^r, & (\rho,p); & \end{matrix}\right] F(P),$$

where $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0;\ \mathrm{Re}(p)>|\omega|^{1/\mathrm{Re}(\alpha)},$ *and* $F(P)$ *is the Laplace transform of* $f(t),$

defined by

$$L\{f(t);p\} = F(P) = \int_0^\infty e^{-Pt} f(t)\,dt,$$

where $\mathrm{Re}(p) > 0$ *and the integral is convergent.*

Proof. By the definition of Laplace transform, we conclude that

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\omega;\,0_+} f)(x);P\right\} = \int_0^\infty e^{-Pt} \int_0^x (x-t)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega(x-t)^\alpha] f(t)\,dt\,dx.$$

Changing the order of integration, which is possible under the conditions imposed in the theorem, we find

$$L\left\{(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\,\omega;\,0_+} f)(x);P\right\} = \int_0^\infty f(t)dt \int_t^\infty e^{-Pt}(x-t)^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega(x-t)^\alpha]\,dx.$$

If we set $x = t + u$, then

$$L\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p\omega;0_+} f)(x);P\right\} = \int_0^\infty e^{Pt} f(t)dt \int_0^\infty e^{-Pu} u^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}[\omega u^\alpha]\,du.$$

Thus, by virtue of definition (1.1), we obtain

$$= \sum_{k=0}^\infty \frac{[(\gamma)_{\delta k}]^s\ \omega^{(pk+\rho-1)}}{\Gamma(\alpha k+\beta)\ [(\lambda)_{\mu k}]^r\ (\rho)_{pk}} \int_0^\infty e^{Pt} f(t)dt \int_0^\infty e^{-Pu} u^{\beta+\alpha(pk+\rho-1)-1} du$$

$$= \sum_{k=0}^\infty \frac{[(\gamma)_{\delta k}]^s\ \omega^{(pk+\rho-1)}}{P^{\beta+\alpha(pk+\rho-1)}[(\lambda)_{\mu k}]^r\ (\rho)_{pk}} \int_0^\infty e^{-Pt} f(t)dt$$

$$= \frac{[\Gamma(\lambda)]^r\ \Gamma(\rho)\,\omega^{\rho-1}}{[\Gamma(\gamma)]^s\ P^{\beta+\alpha\rho-\alpha}}\ {}_{s+1}\psi_{r+1}\left[\begin{matrix}[(\gamma,q)]^s, & (1,1); & \omega^p/P^{p\alpha} \\ \\ [(\lambda,\mu)]^r, & (\rho,p); & \end{matrix}\right] F(p),$$

where $F(P)$ is the Laplace transform of $f(t)$.

For $s = 1,\ r = 0,\ \rho = 1,\ p = 1$, and $\delta = q$ Theorem 2.8 is reduced to the following corollary:

Corollary 2.2.

$$L\left\{(\mathcal{E}^{\gamma,q}_{\alpha,\beta,\omega;0_+} f)(x);P\right\} = \frac{1}{\Gamma(\gamma)} P_1^{-\beta}\psi_0\left[\begin{matrix}(\gamma,q); & \omega/P^\alpha \\ -; & \end{matrix}\right] F(p),$$

where $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(P) > |\omega|^{1/\mathrm{Re}(\alpha)}$, *and* $F(P)$ *is the Laplace transform of* $f(t)$, *defined by*

$$L\{f(t);P\} = F(P) = \int_0^\infty e^{-P\,t} f(t)\,dt,$$

where, in turn, $\mathrm{Re}(P) > 0$ *and the integral is convergent.*

3. Properties

In this section, some properties of the functions $E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p)$ and $E_t(c,-\eta,\gamma,\delta,\lambda,\mu,\rho,p)$ are established. We first consider the function

$$f(t) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (ct)^{(pn+\rho-1)}}{\Gamma(\rho)\ ((\rho)_{pn})^2\ [(\lambda)_{\mu n}]^r},$$

where $\gamma \in \mathbb{C},\ \delta > 0,$ and c is an arbitrary constant.

Thus, by virtue of (1.4), the fractional integral operator of order ν is given by

$$I^\nu\ f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (c\xi)^{(pn+\rho-1)}}{\Gamma(\rho)\ ((\rho)_{pn})^2\ [(\lambda)_{\mu n}]^r}\ d\xi$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ c^{(pn+\rho-1)}}{\Gamma(\rho)\ ((\rho)_{pn})^2\ [(\lambda)_{\mu n}]^r} \int_0^t \xi^{pn+\rho-1}(t-\xi)^{\nu-1}\ d\xi.$$

After certain simplifications, in view of (1.1), we can write

$$I^\nu\ f(t) = t^\nu \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (ct)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+\nu+1)\ ((\rho)_{pn})\ [(\lambda)_{\mu\, n}]^r} \tag{3.1}$$

$$= t^\nu\ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.2}$$

We denote the function (3.2) by $E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p)$, i.e.,

$$E_t(c,\nu,\gamma,\delta,\lambda,\mu,\rho,p) = t^\nu\ E^{\gamma,\delta}_{1,\nu+1,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.3}$$

Thus, by using (1.7), the fractional differential operator of order η can be represented as follows:

$$D^\eta\ f(t) = D^k \left[I^{k-\eta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (ct)^{(pn+\rho-1)}}{\Gamma(\rho)\ ((\rho)_{pn})^2}\ [(\lambda)_{\mu\, n}]^r \right].$$

Applying (3.1), after certain simplifications, in view of (1.1), we get

$$D^\eta\ f(t) = t^{-\eta} \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s\ (ct)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+(1-\eta))\ ((\rho)_{pn})\ [(\lambda)_{\mu n}]^r} = t^{-\eta}\ E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu,\rho,p}(ct;s,r). \tag{3.4}$$

We denote function (3.4) by $E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)$, i.e.,

$$E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) = t^{-\eta}\, E^{\gamma,\delta}_{1,1-\eta,\lambda,\mu,\rho,p}(ct; s, r). \tag{3.5}$$

Theorem 3.1. *Let* $\gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0,\ \delta > 0,$ *let* c *be an arbitrary constant, and let the fractional integral and differential operator be of order* σ*. Then*

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = E_t(c, \sigma + \nu, \gamma, \delta, \lambda, \mu, \rho, p) \tag{3.6}$$

and

$$D^\sigma\big(E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)\big) = E_t(c, \nu - \sigma, \gamma, \delta, \lambda, \mu, \rho, p). \tag{3.7}$$

Proof. It follows from (1.4) that

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1}\, E_\xi(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)\, d\xi.$$

By using (3.3), we represent this equation in the form

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1}\, \xi^\nu \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\, (c\xi)^{(pn+\rho-1)}}{\Gamma(1(pn+\rho-1)+\nu+1)\, ((\rho)_{pn})[(\lambda)_{\mu\, n}]^r}\, d\xi.$$

Thus, substituting $\xi = xt,$ after necessary simplifications followed by the application of relation (3.3), we obtain relation (3.6).

From (1.7), by using (3.6), we get

$$D^\sigma\big(E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)\big) = D^k\big\{t^{k-\sigma+\nu}\, E^{\gamma,\delta}_{1,k-\eta+\nu+1,\lambda,\mu,\rho,p}(ct; s, r)\big\}.$$

In view of (1.1) and (3.3), we get (3.7).

In the light of Theorem 2.7, we prove the following theorem:

Theorem 3.2. *Let* $\eta \in \mathbb{C},\ \mathrm{Re}(\eta) > 0,\ \delta > 0,$ *let* c *be an arbitrary constant, and let the fractional integral and differential operator be of order* σ*. Then*

$$I^\sigma E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) = E_t(c, \sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p),$$

$$D^\sigma\big(E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)\big) = E_t(c, -\sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p).$$

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Dear Author,

I am glad to inform you that the second revised version of your paper No. MS/JRP/13/13 entitled “Extended Mittag-Leffler function and its properties”, written jointly with B. I. Dave, is now accepted for publication in the Mathematics Student.

Thanking you

(J. R. Patadia)