SUMMARY

SUMMARY OF THE THESIS ENTITLED

A UNIFICATION OF GENERALIZED MITTAG-LEFFLER FUNCTIONS, FAMILY OF BESSEL FUNCTION AND THEIR q-ANALOGUES

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The thesis entitled “A UNIFICATION OF GENERALIZED MITTAG - LEFFLER FUNCTIONS, FAMILY OF BESSEL FUNCTION AND THEIR q-ANALOGUES” carries out generalization of Mittag-Leffler function as defined below.

Definition 1. *For* $\alpha, \beta, \gamma, \lambda \in \mathbb{C},\ \Re(\alpha, \beta, \gamma, \lambda) > 0,\ \delta, \mu > 0,\ r \in \{-1, 0\} \cup \mathbb{N},\ s \in \mathbb{N} \cup \{0\},\ \Re(\alpha) + r\mu - s\delta + 1 > 0$

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n + \beta)\ [(\lambda)_{\mu n}]^r\ n!} z^n, \tag{1}$$

and derive certain properties of it.
The objective of constructing this function is to
(i) include certain existing generalizations of Mittag-Leffler function,
(ii) also include the functions such as Bessel Maitland function, Dotsenko function, Bessel function, generalized Bessel Maitland function, Lommel function etc. especially by means of parameters r, γ, λ (Table-1, 2 below)
(iii) obtain inverse inequality relations and some other inequalities by means of the integer $'s'$.

Chapter 1 introduces the subject matter and lists certain definitions, notations, formulae and results together with certain Fractional Calculus formulae.

Chapter 2 begins with absolute convergence test of series of the function (1). The subsequent properties include (i) order and its type, (ii) asymptotic estimate, (iii) differential equation, (iv) Eigen function property and (v) Mellin-Barnes contour integral representation. Certian mixed recurrence type relations are derived; and the results involving integral transforms namely, Euler-Beta transform, Mellin-Barnes transform, Laplace transform and Whittaker transform are recorded. The special cases such as the generalized hypergeometric function, generalized Laguerre polynomial, Fox H-function etc. are also illustrated. It is noteworthy that the function in (1), besides containing the Shukla and Prajapati's function [13]

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{2}$$

where $\alpha, \beta, \gamma \in \mathbb{C};\ \Re(\alpha,\ \beta,\ \gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$, also includes some other functions namely,
(i) Bessel-Maitland function [5, Eq.(1.7.8), p.19] :

$$J^{\mu}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu + n\mu + 1)} \frac{z^n}{n!},$$

(ii) Dotsenko function [5, Eq.(1.8.9), p.24] :

$${}_2R_1(a, b; c, \omega; \nu; z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\ \Gamma(b + n\frac{\omega}{\nu})}{\Gamma(c + n\frac{\omega}{\nu})} \frac{z^n}{n!},$$

(iii) A particular form ($m=2$) of extension of Mittag-Leffler function due to Saxena and Nishimoto [12] given by

$$E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{Kn}}{\Gamma(\alpha_1 n+\beta_1)\ \Gamma(\alpha_2 n+\beta_2)}\frac{z^n}{n!},$$

where $z,\gamma,\alpha_j,\beta_j\in\mathbb{C},\Re(\alpha_1+\alpha_2)>\Re(K)-1,\Re(K)>0$.
(iv) The Elliptic function [7, Eq.(1), p.211] :

$$K(k)=\frac{\pi}{2}\ {}_2F_1\left(\begin{array}{l}\frac{1}{2},\ \ \frac{1}{2};\ \ k^2\\ 1;\end{array}\right).$$

All these functions are tabulated below as particular cases of (1).

Table-1

Function	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$
Mittag-Leffler	0	1	α	1	1	1	-	-
Wiman	0	1	α	β	1	1	-	-
Prabhakar	0	1	α	β	γ	1	-	-
Shukla and Prajapati	0	1	α	β	γ	q	-	-
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-
Dotsenko	-1	1	ω/ν	c	a	1	b	ω/ν
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2
Elliptic	-1	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1

Some of the main results are stated below.

Theorem 1. *Let* $\Re(\alpha,\beta,\gamma,\lambda)>0,\Re(\alpha)+r\mu-s\delta+1>0,\ \delta,\mu>0,\ r\in\{-1,0\}\cup\mathbb{N},\ s\in\mathbb{N}\cup\{0\}$. *Then* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$ *is an entire function of order* $\varrho=\dfrac{1}{\Re(\alpha)+r\mu-s\delta+1}$ *and type* $\sigma=\dfrac{1}{\varrho}\left(\frac{\delta^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha)}\mu^{r\mu}}\right)^{\varrho}$.

In the notations

$$\theta=zD,\ D=\frac{d}{dz},\ \Upsilon_j^{(a,b;m)}=\prod_{j=0}^{a-1}\left[\left(\theta+\frac{b+j}{a}-1\right)\right]^m,\ \Delta_j^{(a,b;m)}=\prod_{j=0}^{a-1}\left[\left(\theta+\frac{b+j}{a}\right)\right]^m,$$

$$\Theta_j^{(a,b;m)}=\prod_{j=0}^{a-1}\left[\left(-\theta+\frac{b+j}{a}-1\right)\right]^m,\ \Omega_{\Theta;\Upsilon}=P^{-1}\ D\ \Theta_m^{(\delta,\gamma;-s)}\Upsilon_k^{(\mu,\lambda;r)}\Upsilon_j^{(\alpha,\beta;1)},\ P=\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}},$$

the following differential equation is obtained.

Theorem 2. *Let* $\alpha, \mu, \delta \in \mathbb{N}$ *then* $y = E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r)$ *satisfies the equation*

$$\left[\Upsilon^{(\mu,\lambda;r)}_{k}\ \Upsilon^{(\alpha,\beta;1)}_{j}\ \theta - z\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}}\ \Delta^{(\delta,\gamma;s)}_{m}\right] y = 0. \tag{3}$$

And the eigen function property is given as

Theorem 3. *If* $\alpha, \mu, \delta \in \mathbb{N}$ *then* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r)$ *is an eigen function with respect to the operator* $\Omega_{\Theta;\Upsilon}$. *That is,*

$$\Omega_{\Theta;\Upsilon}\left(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z; s, r)\right) = \zeta\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z; s, r). \tag{4}$$

Mixed Relation:

Theorem 4. *For* $\alpha,\ \beta,\ \gamma,\ \lambda\ \in \mathbb{C},\ k \in \mathbb{N},\ \Re(\alpha, \beta, \gamma, \lambda) > 0, \delta,\ \mu > 0,$ *we get*

$$\begin{aligned} & E^{\gamma,\delta}_{\alpha,\beta+k,\lambda,\mu}(z; s, r) - E^{\gamma,\delta}_{\alpha,\beta+k+1,\lambda,\mu}(z; s, r) \\ = \ & \alpha^2 z^2\ \ddot{E}^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z; s, r) + \alpha z\left[\alpha + 2(\beta + k)\right] \dot{E}^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z; s, r) \\ & +(\beta^2 + 2\beta k + k^2 - 1)E^{\gamma,\delta}_{\alpha,\beta+k+2,\lambda,\mu}(z; s, r), \end{aligned} \tag{5}$$

where,

$$\dot{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r) = \frac{d}{dz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r), \quad \ddot{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r) = \frac{d^2}{dz^2}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r).$$

Double series representation:

$$^{*}E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z; s, r) \ = \ \sum_{i,j=0}^{\infty}\frac{1}{(i+j)_{\rho}}\frac{(-1)^i}{i!\ j!}\ ^{*}E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z; s, r). \tag{6}$$

Relationship with Wright function:

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r) \ = \ \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s}\ {}_s\Psi_{r+1}\left[\begin{array}{ll} [(\gamma,\delta)]^s; & z \\ (\beta,\alpha),\quad [(\lambda,\mu)]^r; & \end{array}\right].$$

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Bessel Function Family

In (1) replacing z by $-\frac{z^2}{4}$ and multiplying the series by $\left(\frac{z}{2}\right)^{\xi}$, one gets

$$\left(\frac{z}{2}\right)^{\xi}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r\right) = \left(\frac{z}{2}\right)^{\xi}\ \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!}\left(-\frac{z^2}{4}\right)^n, \tag{7}$$

in which $\alpha, \beta, \gamma, \lambda \in \mathbb{C},\ \Re(\alpha, \beta, \gamma, \lambda) > 0,\ \xi, \delta, \mu > 0, r, s \in \mathbb{N} \cup \{0\}$.

For suitable choices of ξ, this gives the Bessel function of first kind $J_{\nu}(z)$, the generalized Bessel-Maitland function $J^{\sigma}_{\nu,\eta}(z)$, Lommel function $S_{\eta,\nu}(z)$ and Struve function $H_{\nu}(z)$.
All these functions are tabulated below as particular cases of (7).

Table-2

Function	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	$\boldsymbol{\xi}$
Bessel	0	0	1	$\nu+1$	-	-	-	-	ν
Generalized Bessel-Maitland	1	0	σ	$\nu+\eta+1$	-	-	$\eta+1$	1	$\nu+2\eta$
Lommel	1	1	1	$\frac{\eta-\nu+3}{2}$	1	1	$\frac{\eta+\nu+3}{2}$	1	$\eta+1$
Struve	1	1	1	$3/2$	1	1	$3/2+\nu$	1	$\nu+1$

The function (7) turns out to be entire function and satisfies Theorem 1 with the type $\sigma = \dfrac{1}{\varrho}\left(\frac{\delta^{s\delta}}{4\ \{\Re(\alpha)\}^{\Re(\alpha)}\mu^{r\mu}}\right)^{\varrho}$.

Here considering the following operators in their indicated notations:

$$D = \frac{d}{dz},\ \theta = zD,\ \theta_\xi f(z) = z^{\xi/2} f(z^{1/2})\ \theta\left(z^{-\xi/2} f(z^{1/2})\right),$$

$$\mathfrak{D}_\xi^i f(z) = z^{\xi/2-i} f(z^{1/2})\ D^i\left(z^{-\xi/2} f(z^{1/2})\right),\ \Upsilon_{\xi,j}^{(a,b;m)} = \prod_{j=0}^{a-1}\left[\left(\theta_\xi + \frac{b+j}{a} - 1\right)\right]^m,$$

$$\Delta_{\xi,j}^{(a,b;m)} = \prod_{j=0}^{a-1}\left[\left(\theta_\xi + \frac{b+j}{a}\right)\right]^m,\ \Theta_{\xi,j}^{(a,b;m)} = \prod_{j=0}^{a-1}\left[\left(-\theta_{-\xi} + \frac{b+j}{a} - 1\right)\right]^m$$

and

$$\Omega_{\xi,\Theta}^{\Upsilon} = -4P^{-1}\ \mathfrak{D}_\xi\ \Theta_{\xi,m}^{(\delta,\gamma;-s)}\ \Upsilon_{\xi,k}^{(\mu,\lambda;r)}\ \Upsilon_{\xi,j}^{(\alpha,\beta;1)},\quad P = \frac{\delta^{s\delta}}{\alpha^\alpha\ \mu^{r\mu}}, \tag{8}$$

the differential equation is obtained which is stated below.

Theorem 5. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $y_\xi = \left(\frac{z}{2}\right)^\xi\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}; s, r\right)$ satisfies the equation*

$$\left[\Upsilon_{\xi,k}^{(\mu,\lambda;r)}\ \Upsilon_{\xi,j}^{(\alpha,\beta;1)}\ \theta_\xi + \frac{z^2}{4}\ P\ \Delta_{\xi,m}^{(\delta,\gamma;s)}\right] y_\xi = 0. \tag{9}$$

And the eigen function property is given by

Theorem 6. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $\left(\frac{z}{2}\right)^\xi\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}; s, r\right)$ is an eigen function with respect to the operator $\Omega_{\xi,\Theta}^{\Upsilon}$ as defined by (8).*
That is,

$$\Omega_{\xi,\Theta}^{\Upsilon}\left(\left(\frac{z}{2}\right)^\xi\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\zeta\frac{z^2}{4}; s, r\right)\right) = \zeta\ \left(\frac{z}{2}\right)^\xi\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\zeta\frac{z^2}{4}; s, r\right). \tag{10}$$

In Chapter 3, q-extension of the results of Chapter 2 are provided. In view of two q-exponential functions, the two q-analogues of the function (1) are defined.

Definition 2. *If* $\alpha,\beta,\gamma,\lambda \in \mathbb{C}$ *with* $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\mu > 0,\ r \in \{-1,0\}\cup\mathbb{N},\ s \in \mathbb{N}\cup\{0\}$ *then*

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n, \tag{11}$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ *with* $\Re(p) > 0$.

Definition 3. *If* $\alpha,\beta,\gamma,\lambda,z \in \mathbb{C}$ *with* $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\mu > 0,\ r \in \{-1,0\}\cup\mathbb{N},\ s \in \mathbb{N}\cup\{0\}, |z| < \left|(1-q)^{s\delta-\alpha-r\mu-1}\right|$ *and* $\alpha^2 + r\mu^2 + 1 = s\delta^2$ *then*

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{[\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n. \tag{12}$$

The following table enlists particular cases of (11) and (12).

Table-3

q-Function of	r	s	α	β	γ	δ	λ	μ	Particular case of
Mittag-Leffler	0	1	α	1	1	1	-	-	(11)
Wiman	0	1	α	β	1	1	-	-	(11)
Prabhakar	0	1	α	β	γ	1	-	-	(11)
Shukla and Prajapati	0	1	α	β	γ	q	-	-	(11)
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-	(11)
Dotsenko	-1	1	ω/ν	c	a	1	b	ω/ν	(12)
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2	(11)
Elliptic	-1	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	(12)

The function in (11) turns out to be an entire function of order zero under the conditions mentioned in following theorem.

Theorem 7. *Let* $\Re(\alpha,\beta,\gamma,\lambda) > 0, \Re(\alpha^2+r\mu^2-s\delta^2+1) > 0, \delta,\mu > 0,\ r \in \{-1,0\}\cup\mathbb{N},\ s \in \mathbb{N}\cup\{0\}$ *and* $0 < q < 1$. *Then* $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ *is an entire function of* z *of oder zero.*

On the other hand, the function defined by (12) is analytic function as the series converges absolutely for $|z| < \left|(1-q)^{s\delta-\alpha-r\mu-1}\right|$.
Next, assume the following notations for the indicated oprator expressions.

$$\Lambda_q f(x) = f(x) - f(xq^{-1}), \quad \Theta f(x) = f(x) - f(xq),$$

$$\mathcal{D}_q\ f(x) = (1-q)\ D_q f(x) := (1-q)\ \frac{f(x)-f(xq)}{x-xq} = \frac{f(x)-f(xq)}{x},$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi_{u,v}^{(a,b,c;m)}$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi_{u,v}^{(a,b,c;m)}.$$

$$\frac{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[(\Lambda_q + c^{-u}q^{1-(b+v)/a} - 1)]^m}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Omega_{u,v}^{(a,b,c;m)},$$

and

$$\Delta_q = \mathcal{D}_q \; \Omega_{j,i}^{(\delta,\gamma,\zeta;-s)} \; \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \; \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}.$$

Here the operators $\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}, \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}, \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$ are not commutative with the operator $\mathcal{D}_q$.
It is shown that both the functions (11) and (12) satisfy the difference equations of the following forms.

Theorem 8. *Let $\alpha, \mu, \delta \in \mathbb{N}$, then $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ satisfies the difference equation*

$$\begin{aligned}&\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \; \Phi_{h,m}^{(\alpha,\beta,\sigma;1)} \; \Theta\right] E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)\\ &- \left[(-1)^p \; z \; \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\right] E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zq^p; s, r|q) = 0\end{aligned} \tag{13}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Theorem 9. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $Y = e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ satisfies the equation*

$$\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \; \Phi_{h,m}^{(\alpha,\beta,\sigma;1)} \; \Theta - z \; \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\right] Y = 0, \tag{14}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

The eigen function property is given as

Theorem 10. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ is an eigen function with respect to the operator Δ_q. That is,*

$$\Delta_q \; e_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(cz; s, r|q) = c \; e_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(cz; s, r|q). \tag{15}$$

The function $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z; s, r|q)$ defined by (11) does not possess this property.
The following theorem states the Mellin-Barnes contour integral formula for (11).

Theorem 11. *Let $\alpha \in \mathbb{R}_+; \beta,\ \gamma,\ \lambda \in \mathbb{C}$ with $\Re(\beta,\gamma,\lambda) > 0$ and $\delta,\ \mu > 0$. Then the function $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ is expressible as the Mellin - Barnes q-integral given by*

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q) = \frac{1}{2\pi i} \int_L \frac{(-1)^{-pS}\ q^{-pS(-S-1)/2}\ \Gamma_q(S)[\Gamma_q(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r}\, d_qS, \quad (16)$$

where $|argz| < \pi$. The contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S = -n$ to the left; and the poles at $S = (\gamma+n)/\delta$ to the right of the path for all $n \in \mathbb{N}\cup\{0\}$.

Further,

Theorem 12. *The mixed relation for $k \in \mathbb{N}$ with $k \geq 2$, is given by*

$$\begin{aligned} D_q\Big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+k,\,\lambda,\,\mu}(z^\alpha; s, r|q)\Big) &= (1-q^{k-1})\ D_q\Big(z^\beta E^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^\alpha; s, r|q)\Big) \\ &\quad + q^{k-1}\ (1-q)\ D_q^2\Big(z^{\beta+1} E^{\gamma,\delta}_{\alpha,\,\beta+k+1,\,\lambda,\,\mu}(z^\alpha; s, r|q)\Big). \quad (17) \end{aligned}$$

q-Bessel Function Family

Definition 4. *If $\alpha, \beta, \gamma, \lambda, z \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\mu > 0,\ r, s \in \mathbb{N}\cup\{0\}$ then*

$$\left(\frac{z}{2}\right)^\xi\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(-\frac{z^2}{4}; s, r|q\right) = \sum_{n=0}^{\infty} \frac{(-1)^{(p+1)n}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{2^{2n+\xi}\ \Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^{2n+\xi}, \quad (18)$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.

The q-analogues of those functions listed above in Chapter 2 in Table-2 are all yielded by the function (18).
They are tabulated below together with the indicated substitutions.

Table-4

q-Function of	r	s	α	β	γ	δ	λ	μ	ξ
Bessel	0	0	1	$\nu+1$	-	-	-	-	ν
Generalized Bessel-Maitland	1	0	σ	$\nu+\eta+1$	-	-	$\eta+1$	1	$\nu+2\eta$
Lommel	1	1	1	$\frac{\eta-\nu+3}{2}$	1	1	$\frac{\eta+\nu+3}{2}$	1	$\eta+1$
Struve	1	1	1	$3/2$	1	1	$3/2+\nu$	1	$\nu+1$

The function in (18) turns out to be an entire function of order zero under the conditions mentioned in Theorem 7. Next, assuming the following notations for the indicated operator expressions:

$$\delta_q f(x) = f(xq^{1/2}),\ \ \Theta f(x) = f(x) - f(xq),\ \ \Big(\Theta + q^{\xi/2}\delta_q - 1\Big) = \Phi_\xi,$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta+c^{-u}q^{1-(b+v)/a}q^{\xi/2}\delta_q-1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}}=\Phi_{\xi;u,v}^{(a,b,c;m)},$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta+c^{-u}q^{(b+v)/a}q^{\xi/2}\delta_q-1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}}=\Psi_{\xi;u,v}^{(a,b,c;m)},$$

one finds the difference equation as follows.

Theorem 13. *Let $\alpha,\mu,\delta\in\mathbb{N}$ then $w=\left(\frac{z}{2}\right)^{\xi}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4};s,r|q\right)$ satisfies the equation*

$$\begin{aligned}&q^{s\xi}\ \Phi_{\xi;\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{\xi;h,m}^{(\alpha,\beta,\sigma;1)}\ \Phi_{\xi}\ \left(\frac{z}{2}\right)^{\xi}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}q^{s};s,r|q\right)\\ &+\frac{(-1)^p\ q^{(r+2)(\xi+1)+s}\ z^2}{4}\ \Psi_{\xi,j,i}^{(\delta,\gamma,\zeta;s)}\ \left(\frac{z}{2}\right)^{\xi}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4}q^{p+r+2};s,r|q\right)=0,\end{aligned} \tag{19}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

The following theorem states the Mellin-Barnes contour integral formula for (18).

Theorem 14. *Let $\alpha\in\mathbb{R}_+;\beta,\ \gamma,\ \lambda\in\mathbb{C}$ with $\Re(\beta,\gamma,\lambda)>0$ and $\delta,\ \mu>0$. Then the function $\left(\frac{z}{2}\right)^{\xi}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4};s,r|q\right)$ is expressible as the Mellin - Barnes q-integral given by*

$$\left(\frac{z}{2}\right)^{\xi}E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\left(-\frac{z^2}{4};s,r|q\right)=\frac{\left(\frac{z}{2}\right)^{\xi}}{2\pi i}\int_L\frac{(-1)^{(p+1)S}\ q^{-pS(-S-1)/2}\ \Gamma_q(S)\ [\Gamma_q(\gamma-\delta S)]^s\ z^{-2S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r}d_qS,$$

where $|argz|<\pi$. The contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S=-n$ to the left; and the poles at $S=(\gamma+n)/\delta$ to the right of the path for all $n\in\mathbb{N}\cup\{0\}$.

In Chapter 4, the Mittag-Leffler (M-L) type operator in the space L(a,b) of Lebesgue measurable (real or complex) functions is introduced as follows.

$$\left(\mathcal{E}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\,0_+}^{\gamma,\,\delta}f\right)(x)=\int_a^x(x-t)^{\beta-1}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}\Big(\omega(x-t)^{\alpha};s,r\Big)f(t)\ dt, \tag{20}$$

where, $\alpha,\beta,\gamma,\lambda,\omega\in\mathbb{C}$; $\Re(\alpha,\beta,\gamma,\lambda)>0$; $\delta,\mu>0$, and $x>a$.
For this operator the following result is obtained.

Theorem 15. *Let the function ϕ be in the space $L(a,b)$ of Lebesgue measurable functions on a finite interval $[a,b]$ of the real line $\mathbb{R}$ given by*

$L(a,b) = \left\{f : \|f\|_1 = \int\limits_a^b |f(t)|\, dt < \infty\right\}.$
Then the integral operator $\mathcal{E}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\,0_+}$ *is bounded on* $L(a,b)$ *and*

$$\|\mathcal{E}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\,0_+}\phi\|_1 \;\leq\; \mathfrak{M}\,\|\phi\|_1, \tag{21}$$

where the constant $\mathfrak{M}$ $(0 < \mathfrak{M} < \infty)$ *given by*

$$\mathfrak{M} \;=\; (b-a)^{\Re(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta k}|^s}{|\Gamma(\alpha k+\beta)|\,(\Re(\alpha k+\beta))} \times \frac{|\omega\,(b-a)^{\alpha}|^k}{|(\lambda)_{\mu k}|^r\; k!}. \tag{22}$$

Transforms of proposed operator (20) are also obtained; the following are illustrations amongst them.

Theorem 16. *Mellin transform of the operator* $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\,0_+}f\right)(x)$
Let $\alpha,\beta,\gamma,\lambda,\omega \in \mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda) > 0;\ \delta,\mu > 0,\ \Re(1-S-\beta) > 0$ *then*

$$\mathcal{M}\left\{(\mathcal{E}^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;\,0_+}f)(x);S\right\} = \frac{[\Gamma(\lambda)]^r}{2\pi\;[\Gamma(\gamma)]^s\;\Gamma(1-S)} \times H^{r+3,s+1}_{s+1,r+3}\left[-wt^{\alpha}\,\middle|\,\begin{matrix}[(1-\gamma,\,\delta)]^s, & (0,1) & & \\ (0,1), & (1-S-\beta,\alpha), & [(1-\lambda,\mu)]^r, & (0,1)\end{matrix}\right] \times\mathcal{M}\{t^{\beta}f(t);S\}.$$

Theorem 17. *Laplace transform of the operator* $\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0_+}f\right)(x)$ *given by*

$$\mathcal{L}\left\{(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0_+}f)(x);P\right\} = \frac{[\Gamma(\lambda)]^r}{[\Gamma(\gamma)]^s\;P^{\beta}}\;{}_{s+1}\psi_{r+1}\left[\begin{matrix}[(\gamma,q)]^s, & (1,1); & \omega/P^{\alpha}\\ [(\lambda,\mu)]^r, & (1,1); & \end{matrix}\right]F(P),$$

where $(\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0); \Re(p) > |\omega|^{1/\Re(\alpha)}$ *and F(P) is the Laplace transform of* $f(t)$*, defined by*

$$L\{f(t);P\} = F(P) = \int_0^{\infty} e^{-Pt}f(t)\,dt,$$

where $\Re(P) > 0$ *and the integral is convergent.*

With the aid of Riemann-Liouville fractional integral operator, the Kober fractional integral operator and fractional differential operator of arbitrary order, some properties are derived. One of them is

Theorem 18. *Let* $a \in \mathbb{R}_+ = [0,\infty), \alpha,\beta,\gamma,\lambda,\eta \in \mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda,\eta) > 0;\ \delta,\ \mu > 0$ *for* $x > a$*, then*

$$\left(I^{\eta}_{a+}(t-a)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(t-a)^{\alpha};s,r)\right)(x) = (x-a)^{(\eta+\beta-1)}\;E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega(x-a)^{\alpha};s,r),$$

and

$$\left(D^{\eta}_{a+}(t-a)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(c(t-a))^{\alpha};s,r)\right)(x)=(x-a)^{(\beta-\eta-1)}E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega(x-a)^{\alpha};s,r).$$

The following thorem states the integro-differential equation.

Theorem 19. *If* $0<\eta<1,\ 0\leq\nu\leq 1,\ \omega,\xi\in\mathbb{C},\ \Re(\alpha)>max\{0,\Re(\delta)-1\}$ *and* $min\{\Re(\beta,\gamma,\lambda,\mu)\}>0$ *then*

$$\left(D^{\eta,\,\nu}_{0+}\,y\right)(x)\;=\;\xi\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)(x)\;+\;f(x) \tag{23}$$

with the initial condition

$$\left(I^{(1-\nu)(1-\eta)}_{0+}\,y\right)(0+)\;=\;C,$$

has solution in the space $L(0,\infty)$ *given by*

$$y(x)=C\,\frac{x^{\eta-\nu(1-\eta)-1}}{\Gamma(\eta-\nu+\eta\nu)}+\xi\,x^{\eta+\beta}\,E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}\left(\omega x^{\alpha}\right)+\frac{1}{\Gamma(\eta)}\int_{0}^{x}(x-t)^{\eta-1}\,f(t)\,dt, \tag{24}$$

where C *is arbitrary constant.*

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The q-analogues of the results of Chapter 4 are incorporated in Chapter 5.
The following operators are defined.

$$\left({}_{q}\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;a+}f\right)(x)=\int_{a}^{x}(x-|tq)_{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-tq^{\beta})_{\alpha};s,r|q)\,f(t)\,d_{q}t, \tag{25}$$

where $\alpha,\beta,\gamma,\lambda,\delta,\mu\in\mathbb{N},\ \alpha^2+r\mu^2-s\delta^2+1>0,\ \omega\in\mathbb{C}$ and $x>a$,
and

$$\left({}_{q}\mathfrak{e}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;a+}f\right)(x)=\int_{a}^{x}(x-|tq)_{\beta-1}\,e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-tq^{\beta})_{\alpha};s,r|q)\,f(t)\,d_{q}t, \tag{26}$$

where, $\alpha,\beta,\gamma,\lambda,\delta,\mu\in\mathbb{N},\ \left|(1-q)^{\alpha+r\mu-s\delta+1}\right|<1,\ \omega\in\mathbb{C}$ and $x>a$.
The function $f(t)$ is chosen suitably so as to ensure the existence of these operators. Then with the aid of q-analogue of Riemann-Liouville fractional integral operator, q-analogue of the Kober fractional integral operator and fractional q-differential operator of arbitrary order, some properties are studied. One such result is stated as

Theorem 20. *Let $a \in [0,\infty)$ and $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N},\ \eta \in \mathbb{R}_{>0}$ then for $x > a$*

$$\begin{aligned}&\left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x)\\ =\ & [x-|a]_{\beta+\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega\ [x-|aq^{\beta+\eta-1}]_{\alpha};s,r|q),\end{aligned} \quad (27)$$

and

$$\begin{aligned}&\left({}_qD^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha};s,r|q)\right)(x)\\ =\ & [x-|a]_{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_{\alpha};s,r|q).\end{aligned} \quad (28)$$

The q-integro-differential equations are also derived. The one is stated below.

Theorem 21. *If $0<\eta<\ 1,\ 0\le\nu\le 1,\ \omega,\xi\in\mathbb{C},\ \alpha > max\{0,\delta-1\}$ then*

$$\left({}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\ \left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)\ (x)\ +\ f(x) \quad (29)$$

with the initial condition

$$\left({}_qI_0\ {+}^{(1-\nu)(1-\eta)}\ \ y\right)(0+)\ =\ C, \quad (30)$$

has solution

$$\begin{aligned} y(x) \ =\ & C\ \frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\ x^{\beta+\eta}\\ & \times(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\ E^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\ (\omega\ (xq^{\eta+1})^{\alpha};\ s,\ r|q)\\ & +\frac{(1-q)1-\eta\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int\limits_0^x\ f(t)(x-|tq)_{\eta-1}\ d_qt,\end{aligned} \quad (31)$$

in the space $L(0,\infty)$ wherein C is arbitrary constant.

In Chapter 6, the well known Konhauser Polynomial [6]

$$Z^{\mu}_m(x;k)=\frac{\Gamma(km+\mu+1)}{\Gamma(m+1)}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{x^{kn}}{\Gamma(kn+\mu+1)},\ \Re(\mu)>-1, \quad (32)$$

is extended with the aid of the function (1) by taking $\gamma=-m$, a negative integer, replacing β by $\beta+1$ and z by real variable x. This leads one to a generalized structure of Konhauser polynomial in the form:

$$B^{(\alpha,\beta,\lambda,\mu)}_{m^*}(x^k;s,r)\ =\ \frac{\Gamma(\alpha m+\beta+1)}{(m!)^s}\sum_{n=0}^{m^*}\frac{[(-m)_{\delta n}]^s\ x^{kn}}{\Gamma(\alpha n+\beta+1)[(\lambda)_{\mu n}]^r\ n!}, \quad (33)$$

in which $\alpha,\ \beta,\ \lambda>0,\ m,\ \delta,\ \mu,\ k,\ s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$, and $m^*=[\frac{m}{\delta}]$ denotes the integral part of $\frac{m}{\delta}$.

This polynomial is referred to as Generalized Konhauser polynomial, briefly GKP.

For this polynomial, several properties involving inequalities are obtained which yield as particular cases, the generating function relations and finite summation formulae. The differential equation obtained is stated here as

Theorem 22. *If* $\alpha,\ \beta,\ \lambda,\ m,\ \delta,\ \mu,\ k,\ s\ \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}$ *and the operator* Θ *is defined by* $\Theta f(x) = x\frac{d}{dx}f(x)$ *then* $U = B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r)$ *satisfies the equation*

$$\Bigg[\left\{\prod_{j=0}^{\alpha-1}\left(\frac{1}{k}\Theta+\frac{\beta+j}{\alpha}-1\right)\right\}\left\{\prod_{i=0}^{\mu-1}\left(\frac{1}{k}\Theta+\frac{\lambda+i}{\mu}-1\right)^{r}\right\}\Theta$$
$$-\frac{\delta^{s\delta}\ k}{\alpha^{\alpha}\ \mu^{r\mu}}\ x^{k}\ \left\{\prod_{l=0}^{\delta-1}\left(\frac{1}{k}\Theta+\frac{-m+l}{\delta}\right)^{s}\right\}\Bigg]U = 0. \tag{34}$$

The presence of parameter s yields an *unusual* inverse series relations involving the polynomial (33). In fact for $s = 1$, the usual inverse series relation occurs whereas for other values of s, the series relations involve the inequality.
If the real valued functions $f(x, n; s)$ and $g(x, n; s),\ s \in \mathbb{N}\setminus\{1\}$ are such that $f(x, n; s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r)$ and $g(x, n; s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r)$, then one finds the following inequality relations.

Theorem 23. *Let* $f(x, n; s)$ *and* $g(x, n; s)$ *be real valued functions,* $\alpha, \beta, \lambda > 0$, *and* $\mu, k \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}$. *If* s *is odd positive integer and* $m, (n-$*a non negative integer) are even positive integers, then*

$$f(x, n; s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r) \tag{35}$$

implies

$$x^{kn} > \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^{r}\,n!}{(mn!)^{s}}\sum_{j=0}^{mn}\frac{[(-mn)_j]^{s}}{\Gamma(\alpha j+\beta+1)\ j!}\ f(x, j; s); \tag{36}$$

and

$$x^{kn} < \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^{r}\,n!}{(mn!)^{s}}\sum_{j=0}^{mn}\frac{[(-mn)_j]^{s}}{\Gamma(\alpha j+\beta+1)\ j!}\ g(x, j; s), \tag{37}$$

implies

$$g(x, n; s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r). \tag{38}$$

Towards the converse of these inequality relations, the following are obtained.

Theorem 24. *Let* $f(x, n; s)$ *and* $g(x, n; s)$ *be real valued functions,* $\alpha, \beta, \lambda > 0$, *and* $\mu, k \in \mathbb{N},\ \ r \in \mathbb{N}\cup\{0\}$. *If either* s *is an even positive integer or* $s, m, (n-$*a non negative integer) are all odd positive integers, then*

$$x^{kn} > \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^{r}\,n!}{(mn!)^{s}}\sum_{j=0}^{mn}\frac{[(-mn)_j]^{s}}{\Gamma(\alpha j+\beta+1)\ j!}\ f(x, j; s) \tag{39}$$

implies

$$f(x, n; s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r); \tag{40}$$

and

$$g(x,n;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r) \tag{41}$$

implies

$$x^{kn} < \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r\,n!}{(mn!)^s}\sum_{j=0}^{mn}\frac{[(-mn)_j]^s}{\Gamma(\alpha j+\beta+1)\ j!}\ g(x,j;s). \tag{42}$$

For $s=1$, one obtains the following inverse series relations for the polynomial (33).

Theorem 25. *For* $\alpha,\beta,\lambda>0, m,\mu,k\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$,

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r) \;=\; \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{j=0}^{[n/m]}\frac{(-n)_{mj}\ x^{kj}}{\Gamma(\alpha j+\beta+1)\,[(\lambda)_{\mu j}]^r\ j!} \tag{43}$$

if and only if

$$\frac{x^{kn}}{n!} \;=\; \frac{\Gamma(\alpha n+\beta+1)\,[(\lambda)_{\mu n}]^r}{(mn)!}\sum_{j=0}^{mn}\frac{(-mn)_j}{\Gamma(\alpha j+\beta+1)}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r), \tag{44}$$

and for $n\neq ml,\ l\in\mathbb{N}$,

$$\sum_{j=0}^{n}\frac{(-n)_j}{\Gamma(\alpha j+\beta+1)}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r)=0. \tag{45}$$

Also, one of the series inequalities involving the polynomial (33) with *positive terms* is obtained as.

Theorem 26. *If* $\alpha,\beta,\lambda,\sigma,x>0,\ \delta,\mu,m,k,s\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\},\ 0<t<1$, *and* $p=\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}}$ *as before, then the following series inequalities hold.*

$$\sum_{m=0}^{\infty}\frac{B_{m^*}^{(\alpha,\beta,\lambda,m)}(x^k;s,r)}{(\beta+1)_{\alpha m}}\ t^{ms} \le e^{ts}\ {}_0F_{\alpha+r\mu}\left[\begin{matrix} -; & \frac{x^k}{\alpha^{\alpha}\ \mu^{\mu}}\ (-t)^{s\delta} \\ \Delta(\alpha;\beta+1),\quad \Delta(\mu;\lambda)^r, & \end{matrix}\right], \tag{46}$$

$$\sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{(\beta+1)_{\alpha m}}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \le (1-t)^{-s\sigma}\ {}_{s\delta}F_{\alpha+r\mu}\left[\begin{matrix} \Delta(\delta;\sigma)^s; & px^k\left(\frac{-t}{1-t}\right)^{s\delta} \\ \Delta(\alpha;\beta+1),\quad \Delta(\mu;\lambda)^r; & \end{matrix}\right], \tag{47}$$

and

$$\sum_{m=0}^{\infty}\frac{((\sigma)_m)^s}{\Gamma(\alpha m+\beta+1)}B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)\ t^{ms} \le (1-t)^{-s\sigma}E_{\alpha,\,\beta+1,\,\lambda,\,\mu}^{\sigma,\delta}\left(x^k\left(\frac{-t}{1-t}\right)^{s\delta};s,r\right). \tag{48}$$

The generating function relation follows at once when $s = 1$. One more finite series inequality involving GKP is given in

Theorem 27. *If* $\beta, \lambda > 0,\ \delta, \mu, k, m, s, w \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ -1 < w\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} - \left(\frac{x}{k}\right)^{\frac{k}{s\delta}}\right) < 0$ *then*

$$B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r) \leq \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \left(\begin{array}{c} \beta + km \\ kj \end{array}\right) \frac{(kj)!}{j!} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} - 1\right)^j \times B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r). \quad (49)$$

The q-extension of (33) is defined in Chapter 7. Throughout this work, $(\alpha; q)_n$ will be abbreviated as $[\alpha]_n$. In parallel to the two q-exponential functions, there are two q-forms of (33) are defined.

Definition 5. *For* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$, *the integral part of* $\frac{m}{\delta}$, *define*

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{sk\delta n(m+(\delta nk-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r} \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n}. \quad (50)$$

Definition 6. *For* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$, *the integral part of* $\frac{m}{\delta}$, *define*

$$b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}. \quad (51)$$

Henceforth the polynomials in (50) and (51) will be referred to as q-GKP.
By taking $m \to \infty$ in (50), one gets

$$\lim_{m\to\infty} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q). \quad (52)$$

This is taken up in

Theorem 28. *Let*

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{[q^{\beta+1}]_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{[q^{\beta+1}]_{\alpha n}\ [[q^\lambda]_{\mu n}]^r} \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n}. \quad (53)$$

Then $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ *approaches as limit* $m \to \infty$ *in any bounded domain to the entire function*

$$B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) = \frac{[q^{\beta+1}]_{\infty}}{[(q^k; q^k)_\infty]^s} \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{[q^{\beta+1}]_{\alpha n}\ [[q^\lambda]_{\mu n}]^r} \times \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k; q^k)_n}. \quad (54)$$

Theorem 29. *Let* $\alpha, \beta, \lambda, m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\},\ m^* = [\frac{m}{\delta}]$ *then* $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ *satisfies the equation*

$$\begin{aligned}&\left[\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,g}^{(\alpha,\beta+1,\sigma;1)}\ \Theta\right] B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)\\ &-x^k\ q^{s(k\delta(k\delta-1)/2)+sk\delta m}\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2}; s, r|q) = 0,\end{aligned} \tag{55}$$

where χ *is* $(\delta k)^{th}$ *root of unity,* η *is* μ^{th} *root of unity,* σ *is* α^{th} *root of unity.*

Theorem 30. *Let* $\alpha, \beta, \lambda, m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\},\ m^* = [\frac{m}{\delta}]$ *then the difference equation satisfied by* $W = b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ *is*

$$\left[\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,g}^{(\alpha,\beta+1,\sigma;1)}\ \Theta - x^k\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\right] W = 0, \tag{56}$$

where χ *is* $(\delta k)^{th}$ *root of unity,* η *is* μ^{th} *root of unity,* σ *is* α^{th} *root of unity.*

If the real valued functions $F(x, n; s|q),\ G(x, n; s|q),\ f(x, n; s|q),\ g(x, n; s|q)$, where $s \in \mathbb{N}\setminus\{1\}$ are such that

$$F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q),\ G(x, n; s|q) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q),$$

$$f(x, n; s|q) < b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q),\quad g(x, n; s|q) > b_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$$

then there hold the following inequality relations.

Theorem 31. *Let* $F(x, n; s|q)$ *and* $G(x, n; s|q)$ *be real valued functions,* $\alpha, \beta, \lambda > 0$, *and* $\mu, k \in \mathbb{N},\ r \in \mathbb{N}\cup\{0\}$. *If* s *is odd positive integer and* $m, (n-$*a non negative integer) are even positive integers, then*

$$F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) \tag{57}$$

implies

$$\begin{aligned}x^{kn}\ \ >\ \ &\frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1}; q)_{\alpha n}\ [(q^{\lambda}; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s}\\ &\times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}}F(x, j; s|q);\end{aligned} \tag{58}$$

and

$$\begin{aligned}x^{kn}\ \ <\ \ &\frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1}; q)_{\alpha n}\ [(q^{\lambda}; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s}\\ &\times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}}G(x, j; s|q)\end{aligned} \tag{59}$$

implies

$$G(x, n; s|q)\ \ >\ \ B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q). \tag{60}$$

Towards the converse of these inequality relations, one can obtain the following theorem.

Theorem 32. *Let $F(x,n;s|q)$ and $G(x,n;s|q)$ be real valued functions, $\alpha,\beta,\lambda>0$, and $\mu,k\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$. If either s is an even positive integer or $s,m,(n-$a non negative integer) are all odd positive integers, then*

$$x^{kn} > \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}F(x,j;s|q) \tag{61}$$

implies

$$F(x,n;s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q); \tag{62}$$

and

$$G(x,n;s|q) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)m \tag{63}$$

implies

$$x^{kn} < \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-skmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \times\sum_{j=0}^{mn}\frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}G(x,j;s|q)m. \tag{64}$$

For $s=1$, the polynomial (50) yields the following inverse series relation.

Theorem 33. *For $\alpha,\beta,\lambda>0$, $m,\mu,k\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$,*

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) = \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n}\sum_{j=0}^{\left[\frac{n}{m}\right]}\frac{q^{k(mj(mj-1)/2+mjn)}\ q^{mj(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^{\lambda};q)_{\mu j}]^r} \times\frac{(q^{-nk};q^k)_{mj}\ x^{kj}}{(q^k;q^k)_j} \tag{65}$$

if and only if

$$\frac{x^{kn}}{(q^k;q^k)_n} = \frac{q^{-mn(\alpha(\beta+1)+r\mu\lambda)}\ q^{-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}{(q^k;q^k)_{mn}} \times\sum_{j=0}^{mn}\frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \tag{66}$$

and for $n\neq ml$, $l\in\mathbb{N}$,

$$\sum_{j=0}^{n}\frac{q^{kj}\ (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q),=0. \tag{67}$$

Similar type of results for inequality relations and inverse series relation, one can obtain for (51).
Next, the following inequalities contain q-GKP.

Theorem 34. *If* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\},\ 0 < st < 1$ *then the following series inequality holds.*

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{\alpha m}}\, t^{ms} \leq \Big(e_{q^k}(t)\Big)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} \times B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q). \tag{68}$$

Theorem 35. *If* $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}, 0 < t < 1,\ 0 < st < 1$ *then*

$$\sum_{m=0}^{\infty} \frac{q^{skm(m-1)/2}\ b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{\alpha m}}\, t^{ms} \leq \Big(E_{q^k}(t)\Big)^s \sum_{n=0}^{\infty} \frac{(-tq^{-k})^{s\delta n}\ x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}. \tag{69}$$

Theorem 36. *If* $\beta,\ \lambda \in \mathbb{R}_{>0},\ m,\ \delta,\ \mu,\ k,\ s \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$, *then*

$$B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) \leq (q^{\beta+1}; q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k; q^k)_j} \times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(-j-s+1)}; q^k\right)_j \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{k(m-j)}}. \tag{70}$$

$$b_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) \leq \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j q^{skj(j+1)/2-skmj}\ (q^{\beta+1}; q)_{km}}{(q^{\beta+1}; q)_{k(m-j)}\ (q^k; q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} \times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(1-s)}; q^k\right)_j b_{(m-j)^*}^{(\alpha,\beta,\lambda,\mu)}(y^k; s, r|q). \tag{71}$$

References

[1] R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Comb. Phil. Soc. **66** (1969), no. 3, 365–370.

[2] W. A. Al-Salam and A. Verma, *q-konhauser polynomials*, Pacific Journal of Mathematics **1** (1983), no. 108, 1–7.

[3] M. H. Annaby and Z. S. Mansour, *q-fractional calculus and equations*, Springer-Verlag Berlin Heidelberg (2012).

[4] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, Cambridge (1990).

[5] H. J. Haubold, A. M. Mathai, and R. K. Saxena, *The h-function: Theory and applications*, Centre for Mathematical Sciences, Pala Campus, Kerala, India **37** (2008).

[6] J. Konhauser, *Biorthogonal polynomials suggested by the laguerre polynomials*, Pac. J. Math. **21** (1967), no. 9, 303–314.

[7] Y. L. Luke, *The special functions and their approximations*, Academic Press, New York, London **1** (1969).

[8] G. Mittag-Leffler, *Sur la nouvelle fonction* $e_\alpha(x)$, C. R. Acad. Sci., Paris **137** (1903), 554–558.

[9] D. S. Moak, *The q-analogue of the laguerre polynomials*, J. Math. Anal. Appl. **81** (1981), 20–47.

[10] T. R. Prabhakar and S. Rekha, *Some results on the polynomials* $l_n^{\alpha,\beta}(x)$, Rocky Mountain J. Math. **8** (1978), no. 4, 751–754.

[11] E. D. Rainville, *Special functions*, Macmillan Co., New York (1960).

[12] R. K. Saxena and K. N. Nishimoto, *Fractional calculus of generalized mittag-leffler functions*, J. Fract. Calc. **37** (2010), 43–52.

[13] A. K. Shukla and J. C. Prajapati, *On a generalization of mittag-leffler function and its properties*, J. Math. Anal. Appl. **336** (2007), no. 2, 797–811.

[14] H. M. Srivastava and Z. Tomovski, *Fractional calculus with an integral operator containing a generalized mittag-leffler function in the kernel*, Applied Math. Computation **211** (2009), no. 1, 198–210.

[15] A. Wiman, *Über de fundamental satz in der theoric der funktionen* $e_\alpha(x)$, Acta Math. **29** (1905), 191–201.

LIST OF PUBLISHED / ACCEPTED PAPERS

1. Prajapti, J., Dave, B. and Nathwani, B., *On a unification of generalized Mittag-Leffler Function and family of Bessel Functions*, Advances in Pure Mathematics **3**(2013), 127–137.

2. Prajapati, J. and Nathwani, B., *Recurrence relation of a unified generalized Mittag-Leffler-Function*, Palestine Journal of Mathematics **3**(1)(2014), 94–98.
   MR ♯ 3109941

3. Prajapati, J. and Nathwani, B., *Fractional calculus of a unified Mittag-Leffler function*, Ukrainian Mathematical Journal **66**(2015), 1267–1280.
   MR ♯ 3334434

4. B. V. Nathwani, and B. I. Dave, *Generalized Mittag-Leffler function and its properties*, The Mathematics Student (To appear).

LIST OF PAPERS PRESENTED IN CONFERENCES

1. Presented a paper entitled ”Mittag-Leffler type Generalized Konhauser Polynomial and its properties” in International Conference on ”Special Functions and their applications (ICSFA 2015)(XIV Annual Conference of the Society for Special Functions and their applications) and Symposium on Fractional Calculus and their application in Special Functions” organized by Department of Mathematics, Amity Institute of Applied Sciences, Amity University, UP, Noida, India held on September 10-12, 2015.

2. Presented a paper entitled ”Unification of Mittag-Leffler function and family of Bessel Function” in National Conference on âĂIJCurrent Developments in Analysis and its applicationsâĂİ organized by Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara on March 14-15, 2015.

3. Presented a paper entitled ”Unification of q-ML function and family of q-Bessel FunctionâĂİ in International conference on âĂIJThe works of Sriniwas Ramanujan and their applications in Science and EngineeringâĂİ and International Symposium on âĂIJNew trends in applications of Mathematics in Science and Engineering specially in Fractional Calculus, Bio-informatics and Special Functions (ICSFA-2014)” at Department of Applied Sciences and Civil Engineering, RJIT, BSF Academy, Tekanpur, Gwalior(MP) on 22-23, December 2014.

4. Presented paper on ”Certain Properties of Unified Mittag-Leffler Function” in ”Regional Indian Science Congress Conference-2012” organized by The Maharaja Sayajirao University of Baroda, Vadodara held on September 16, 2012.

5. Presented a paper entitled ”A q-Extension of Generalized Mittag-Leffler Function and Family of Bessel Function” in International Conference on ”Special Functions and their Applications (ICSFA-2012) and Symposium on Life and Works of Ramanujan” organized by Department of Applied Mathematics and Humanities, Saradar Vallabhbhai National Institute of Technology, Surat during June 27-29, 2012 and received **M. I. Qureshi best paper presentation award**.

6. Presented a paper entitled ”Unification of Generalized Mittag-Leffler Function and Certain Hypergeometric functions and its Properties” in International Conference on ”Special Functions and their Applications (ICSFA-2011) and Symposium on Works of Ramanujan” organized by Department of Mathematics and Statistics, J. N. Vyas University, Jodhpur during July 28-30, 2011.

LIST OF COMMUNICATED PAPERS

1. B. V. Nathwani, and B. I. Dave, *An extended q- Mittag-Leffler function and its properties.*

2. Nathwani, B. and Dave, B., *Fractional q-calculus of an extended q- Mittag-Leffler function.*

3. B. V. Nathwani, and B. I. Dave, *Inequalities involving Mittag-Leffler type Konhauser polynomial.*

4. B. V. Nathwani, and B. I. Dave, *Inequalities involving Mittag-Leffler type q-Konhauser polynomial.*