Chapter 1

Introduction

1.1 The hypergeometric function

The field of Special Functions is being flourished since 19^{th} century because of its wide applications in many areas of science. *The Circle of special functions formed by the "arcs" namely OPS (Orthogonal Polynomial System), Series transformation identities, Generating function theory, Differential Equations, Contour integral theory, q-Theory, Inverse Series Relations, Fractional Calculus etc. having center at "Hypergeometric Function".*

The landmarks of this field is the introduction of the series

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+3)}{c(c+1)(c+3)}\frac{z^3}{3!} + \cdots ,$$

which is called the Gauss series [50, Eq. 1.1.1, p.1]. It is represented by the symbol

$$ {}_2F_1\left[\begin{array}{cc} a, b; & z \\ c; & \end{array}\right] \quad \text{or} \quad {}_2\mathrm{F}_1[\mathrm{a, b; c; z}], \tag{1.1}$$

called the hypergeometric function or Guass function. Here z is a variable and a, b are called numerator parameters and c is called the denominator parameter. If either of a or b is a negative integer $-n$ say, then the series has only a finite number of terms and becomes a polynomial in z, expressed as

$$ {}_2F_1[-n, b; c; z].$$

The term '*hypergeometric*' (from Greek $\dot{\upsilon}\pi\epsilon\rho$, meaning: above or beyond) was first used by the Oxford professor John Wallis 1655 in his book *Arithmetica Infinitorum* [50] to denote any series which was beyond the ordinary geometric series

$$1 + x + x^2 + x^3 + \cdots .$$

The hypergeometric series were also studied by Leonhard Euler, but the systematic treatment was given by Carl Friedrich Gauss (1777-1855). Amongst those who later on contributed to the development of this theory, the names of A. T. Vandemonde (1735-1796), Ernst Kummer (1810-1893), E. Heine (1821-1888), F. H. Jackson (1870-1960), L. J. Rogers (1862-1933), W. N. Bailey (1893-1961) and W. Hahn (1911-1998) are worth mentioning.

The study of Special Functions often requires the use of Gamma function, Beta function and their properties. They are defined as follows.

Definition 1.1. The Gamma function [2, 46] is defined by the integral:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} \mathrm{d}t, \qquad \Re(z) > 0.$$

It satisfies the functional equation

$$\Gamma(z+1) = z\, \Gamma(z).$$

The Stirling's asymptotic formula of Gamma function for large n, is given by [18]

$$\Gamma(\alpha+n) \sim \sqrt{2\pi} e^{-(\alpha+n)}\, (\alpha+n)^{(\alpha+n-1/2)}.$$

Definition 1.2. The Beta function is denoted and defined as [2, 46]

$$B(p,q) = \int_0^1 t^{p-1}\, (1-t)^{q-1} \mathrm{d}t, \quad \Re(p,q) > 0. \tag{1.2}$$

For $\alpha \in \mathbb{C}$, then the Pochhammer symbol is given by [1, 46]

$$(\alpha)_n = \begin{cases} 1, & \text{if} \quad n = 0 \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if} \quad n \in \mathbb{N} \\ \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} & \text{if} \quad n \in \mathbb{C}. \end{cases}$$

The idea of extending the number of parameters in the Gauss function seems to have occurred for the first time, in the work of Clausen (1828) [50]. He introduced a series with three numerator parameters and two denominator parameters. The generalized hypergeometric function or generalized Gauss function is defined as follows.

Definition 1.3. The generalized hypergeometric function is defined by [1, 18, 46]

$$ {}_pF_q[z] = {}_pF_q\left[\begin{array}{ll} a_1, a_2, \ldots, a_p; & z \\ b_1, b_2, \ldots, b_q; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1.3} $$

where $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$, and b_j's are neither zero nor negative integer.
The above series converges absolutely under one of the following conditions:
(i) $|z| < \infty$, if $p \leq q$, (ii) $|z| < 1$, if $p = q+1$,
(iii) $|z| = 1$, if $\Re\left(\sum\limits_{j=1}^{q} b_j - \sum\limits_{i=1}^{p} a_i\right) > 0$.

1.1.1 The confluent hypergeometric series

If $p = 1$ and $q = 1$ in the definition of the generalized hypergeometric function (1.3) then it reduces to the function

$$ {}_1F_1\left[\begin{array}{ll} a; & z \\ c; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.4} $$

where $c \neq 0, -1, -2, \ldots$ and $|z| < \infty$.
Alternatively, this function occurs from the hypergeometric function (1.1) if z is replaced by $\frac{z}{b}$ and then b is allowed to tend to infinity. Due to this limiting process, the function (1.4) is also termed as the confluent hypergeometric function [46, Ch. 7,p.123]. In fact,

$$ \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n\, n!} \frac{(b)_n}{b^n} z^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n\, n!} z^n = {}_1F_1\left[\begin{array}{ll} a; & z \\ c; & \end{array}\right]. \tag{1.5} $$

In the above process the parameter: b may be replaced by parameter: a in which case one obtains ${}_1F_1(b; c; z)$.
The function ${}_2F_1$ is analytic in the unit disk, whereas the confluent function ${}_1F_1$ is an entire function. This function is introduced by Kummer (1837), hence is also known as the Kummer's (confluent hypergeometric) function.

Amongst the well-known properties of the hypergeometric function, few of them are listed below as theorems.

1.1.2 Differential equation

Theorem 1.4. *The function* $w = {}_1F_1[a;c;z]$ *satisfies the differential equation [18, Ch. 4.2, Eq.(2)]*

$$[\theta(\theta + c - 1) - z(\theta + a)]\, w \;=\; 0,$$

where $\theta = z\frac{\mathrm{d}}{\mathrm{dz}}$.

This theorem can be extended for the generalized hypergeometric function (1.3) as follows.

Theorem 1.5. *The function* $w = {}_pF_q[z]$ *is a solution of the differential equation*

$$\left[\theta \prod_{j=1}^{q}(\theta + b_j - 1) - z \prod_{i=1}^{p}(\theta + a_i)\right] w \;=\; 0,$$

when $p \leq q$ *or* $p = q+1$ *and* $|z| < 1$. *Also* b_j*'s are neither zero nor negative integer.*

1.1.3 Integral representation

Theorem 1.6. *If* $z \in \mathbb{C}$ *and* $\Re(c) > \Re(a) > 0$, *then [46, Ch. 7, Eq. 9, p.124]*

$${}_1F_1[a;c;z] \;=\; \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,e^{zt}\,\mathrm{dt}.$$

For the function ${}_pF_q[z]$, the above theorem is extended as follows.

Theorem 1.7. *If* $p \leq q+1$, $\Re(b_1) > \Re(a_1) > 0$, *no one of* $b_1, b_2, \ldots, b_q$ *is zero or a negative integer, and* $|z| < 1$, *then*

$$\begin{aligned}
&{}_pF_q\left[\begin{array}{ll} a_1, a_2, \ldots, a_p; & z \\ b_1, b_2, \ldots, b_q; & \end{array}\right] \\
&= \frac{\Gamma(b_1)}{\Gamma(a_1)\,\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}\,(1-t)^{b_1-a_1-1}\;{}_{p-1}F_{q-1}\left[\begin{array}{ll} a_2, a_3, \ldots, a_p; & zt \\ b_2, b_3, \ldots, b_q; & \end{array}\right] \mathrm{dt}.
\end{aligned}$$

1.1.4 Contiguous function relations

Gauss defined the functions *contiguous* to ${}_2F_1[a, b; c; z]$ by increasing or decreasing one of the parameters by unity. In the notations

$$\begin{aligned} F &= {}_2F_1[a, b; c; z], \\ F(a+) &= {}_2F_1[a+1, b; c; z], \\ F(a-) &= {}_2F_1[a-1, b; c; z], \end{aligned}$$

and similarly, $F(b+)$, $F(b-)$, $F(c+)$, $F(c-)$, the contiguous functions relations are listed below [46, Sec. 33, p. 50].

1. $(a-b)F = aF(a+) - bF(b+)$.
2. $(a-c+1)F = aF(a+) - (c-1)F(c-)$.
3. $[(1-z)a + (a+b-c)]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+)$.

For the generalized hypergeometric function ${}_pF_q[z] = F$, the contiguous function relations are as follows.

1. $(a_1 - a_k)F = a_1F(a_1+) - a_kF(a_k+); \quad k = 2, 3, \ldots, p.$
2. $(a_1 - b_k + 1)F = a_1F(a_1+) - (b_k - 1)F(b_k-); \quad k = 1, 2, \ldots, q.$

Following is the well known formula for ${}_2F_1[1]$.

Theorem 1.8. *If* $\Re(c-a-b) > 0$ *and* $c \neq 0, -1, -2, \ldots$ *then [46, Thm. 18, p.49]*

$$ {}_2F_1\left[\begin{matrix} a, & b; & 1 \\ & c; & \end{matrix}\right] = \frac{\Gamma(c)\ \Gamma(c-a-b)}{\Gamma(c-a)\ \Gamma(c-b)}. \tag{1.6}$$

Among various summation theorems occurring hitherto in the literature (e.g. [46, 50]), the following will be referred to in later chapter.

Theorem 1.9. *If* n *is a non-negative integer then [46, Ex. 4, p.106]*

$$ {}_3F_2\left[\begin{matrix} -2n, & a, & 1-b-2n; & 1 \\ & 1-a-2n, & b; & \end{matrix}\right] = \frac{(2n)!\ (a)_n\ (b-a)_n}{n!\ (a)_{2n}\ (b)_n}. \tag{1.7}$$

Also, the following series transformations will be referred to in later chapter.

1.1.5 The Ramanujan's theorem

Theorem 1.10. *If z is non integral then [46, Ex. 5, p.106]*

$$ {}_1F_1\left[\begin{matrix} a; & x \\ b; & \end{matrix}\right] {}_1F_1\left[\begin{matrix} a; & -x \\ b; & \end{matrix}\right] = {}_2F_3\left[\begin{matrix} & a, & b-a; & \frac{x^2}{4} \\ b, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}; & \end{matrix}\right]. $$

1.1.6 The Kummer's First formula

Theorem 1.11. *If b is neither zero nor a negative integer then [46, Thm. 42, p.125]*

$$ e^{-z}\, {}_1F_1\left[\begin{matrix} a; & z \\ b; & \end{matrix}\right] = {}_1F_1\left[\begin{matrix} b-a; & -z \\ b; & \end{matrix}\right]. $$

1.2 Sikkema's function

In 1953, P. C. Sikkema [49, p.6] considered the entire (integral) function

$$ f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!^n} \tag{1.8} $$

of order zero. This function plays an *important role* in the generalization of the hypergeometric function carried out in this Thesis.

1.3 The Bessel function

The Bessel functions are named after Friedrich Wilhelm Bessel (the outstanding astronomer of the nineteenth century, 1784-1846). However, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessel functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end. In 1764, Leonhard Euler employed Bessel functions of both zero and integral orders in an analysis of vibrations of a stretched membrane, an investigation which was further developed by Lord Rayleigh in 1878, where he demonstrated that the Bessel functions are particular cases of Laplace functions. (see `www.uma.ac.ir/files/site1/a_akbari_994c8e8/bessel.pdf`)

The Bessel functions are important in pure mathematics in connection with

problems in number theory, integral transforms, the evaluation of integrals, the theory of differential equations, etc. However, the most extensive treatise on the theory of Bessel functions is that of G. N. Watson in 1922.
In context of definition of generalized hypergeometric function ${}_pF_q[z]$, the Bessel function is defined as follows.

Definition 1.12. For n not a negative integer, the Bessel function [46, 59] is denoted and defined as

$$\begin{aligned} J_n(z) &= \frac{(z/2)^n}{\Gamma(1+n)}\, {}_0F_1\left[\begin{matrix} -; & -\frac{z^2}{4} \\ 1+n; & \end{matrix}\right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\; z^{n+2k}}{2^{n+2k}\; k!\; \Gamma(1+n+k)}. \end{aligned} \tag{1.9}$$

For n a negative integer, $J_n(z) := (-1)^n\; J_{-n}(z)$.
The immediate result from the definition of the Bessel function is

$$J_n(-z) = (-1)^n\; J_n(z).$$

The following properties are frequently occurring in the literature.

1.3.1 The generating function relation

Theorem 1.13. *For $t \neq 0$ and for all finite $|z|$ [46, 59],*

$$\sum_{n=-\infty}^{\infty} J_n(z)\; t^n = \exp\left(\frac{z}{2}\left(t - t^{-1}\right)\right).$$

1.3.2 The Bessel's integral

Theorem 1.14. *The Bessel's integral is given by [46, 59]*

$$J_n(z) \;=\; \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - z\sin\theta\right) \mathrm{d}\theta,$$

for integral n.

1.3.3 Differential recurrence relation

Theorem 1.15. *A differential recurrence relation for the Bessel function is [46, 59]*

$$2\ J_n'(z) \quad = \quad J_{n-1}(z) - J_{n+1}(z).$$

1.3.4 Jacobi's expansion in series of Bessel coefficients

Theorem 1.16. *If $n \in \mathbb{Z}$ then ([46, Ex.3, p.120], [59, Sec. 2.22, p.22])*

$$\exp(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta + 2i\ \sum_{n=0}^{\infty} J_{2n+1}(z)\ \sin(2n+1)\theta.$$

Following are the well-known properties which will be extended in Chapter 6.

1. $\sum\limits_{n=-\infty}^{\infty} J_n(z)\ t^n = \exp\left(\frac{z}{2}\left(t - t^{-1}\right)\right).$

2. $J_n(z) = \frac{1}{\pi}\int\limits_0^{\pi} \cos\left(n\theta - z\sin\theta\right) \mathrm{d}\theta.$

3. $2\ J_n'(z) = J_{n-1}(z) - J_{n+1}(z).$

4. $2^k \dfrac{d^k}{dz^k}\ J_n(z) = \sum\limits_{m=0}^{k} (-1)^{k-m}\binom{k}{m} J_{n+k-2m}(z).$

5. $J_0^2(z) + 2\sum\limits_{n=1}^{\infty}\ J_n^2(z) = 1.$

6. $\cos z = J_0(z) + 2\sum\limits_{n=1}^{\infty}(-1)^n\ J_{2n}(z).$

7. $\sin z = 2\sum\limits_{n=0}^{\infty}(-1)^n\ J_{2n+1}(z).$

8. $\cos(z\sin\theta) = J_0(z) + 2\sum\limits_{n=1}^{\infty} J_{2n}(z)\ \cos 2n\theta.$

9. $\sin(z\sin\theta) = 2\sum\limits_{n=0}^{\infty} J_{2n+1}(z)\ \sin(2n+1)\theta.$

10. $[1 + (-1)^n]\ J_n(z) = \frac{2}{\pi}\int\limits_0^{\pi} \cos n\theta\ \cos(z\sin\theta)\mathrm{d}\theta.$

11. $[1 - (-1)^n]\ J_n(z) = \frac{2}{\pi}\int\limits_0^{\pi} \sin n\theta\ \sin(z\sin\theta)\mathrm{d}\theta.$

12. $J_{2n}(z) = \frac{1}{\pi}\int\limits_0^\pi \cos 2n\theta \ \cos(z\sin\theta)\mathrm{d}\theta.$

13. $J_{2n+1}(z) = \frac{1}{\pi}\int\limits_0^\pi \sin(2n+1)\theta \ \sin(z\sin\theta)\mathrm{d}\theta.$

14. $\int\limits_0^\pi \cos(2n+1)\theta \ \cos(z\sin\theta)\mathrm{d}\theta = 0.$

15. $\int\limits_0^\pi \sin 2n\theta \ \sin(z\sin\theta)\mathrm{d}\theta = 0.$

16. $|J_n(z)| \leq \frac{\left|\frac{z}{2}\right|^n}{n!} \ \exp\left(\left|\frac{z^2}{4}\right|\right).$

17. $J_n(z_1+z_2) = \sum\limits_{m=-\infty}^{\infty} J_m(z_1) \ J_{n-m}(z_2).$

18. $J_n(2z) = \sum\limits_{m=0}^{n} J_m(z) \ J_{n-m}(z) + 2\sum\limits_{m=1}^{\infty} (-1)^m \ J_m(z) \ J_{n+m}(z).$

1.4 The basic hypergeometric series

The q-Special functions were believed to be born in 1748 when Euler put forward the infinite product [50]

$$(q;q)_\infty^{-1} \ = \ \prod_{k=0}^{\infty}\left(1-q^{k+1}\right)^{-1}$$

as a generating function for p(n), the number of partitions of a positive integer n. But it was dormant about hundred years until Heine converted a simple observation:

$$\lim_{q\to 1}\frac{1-q^a}{1-q} \ = \ a \tag{1.10}$$

into a systematic theory of ${}_2\phi_1$ basic hypergeometric series parallel to the theory of Gauss' ${}_2F_1$ hypergeometric series.

The work of Heine was ignored to a quite a long time. While writing the 2nd edition of "*Handbuch die Kungelfuctionen*", Heine decided to include some of his work on basic hypergeometric series. This material was printed in smaller type. Surprisingly, his inclusion of this material led to some later work due to L. J. Roger [23] which indicated that there was a very close connection between Heine's work on basic hypergeometric series and spherical harmonics.

Using the concept of (1.10), Heine then defined the basic analogue of the Gauss function ${}_2F_1[a,b;c;z]$ as the infinite series:

$$1+\frac{(1-q^a)(1-q^b)z}{(1-q^c)(1-q)}+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})z^2}{(1-q^c)(1-q^{c+1})(1-q)}+\cdots,$$

where $0<q<1$, so that as $q\to 1$, this series tends to the Gauss series, ${}_2F_1[a,b;c;z]$.

A q-analogue of factorial function $(a)_n$ is defined by [23, Eq.(1.2.15) and (1.2.30), p.3,6]

$$(a;q)_n=\begin{cases}1, & \text{if} \quad n=0\\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if} \quad n\in\mathbb{Z}_{>0}\\ \left[(1-aq^{-1})(1-aq^{-2})\dots(1-aq^{-n})\right]^{-1}, & \text{if} \quad n\in\mathbb{Z}_{<0}\\ \frac{(a;q)_\infty}{(aq^n;q)_\infty} & \text{if} \quad n\in\mathbb{C},\end{cases}$$

where $a\in\mathbb{C}$ in general, and

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1.$$

For $a(\equiv q^a)=q$,

$$(q;q)_n=(1-q)(1-q^2)\cdots(1-q^n)$$

is q-analogue of $n!$. In fact,

$$\lim_{q\to 1}\frac{(q;q)_n}{(1-q)^n}=n!$$

In these terminologies, the generalized basic hypergeometric function is defined as follows [23, Eq. (1.2.22), p.4].

Definition 1.17. The generalized basic hypergeometric function is denoted by ${}_r\phi_s[z]$ and defined as

$$\begin{aligned}{}_r\phi_s[z] &= {}_r\phi_s\left[\begin{array}{cccccc} a_1, & a_2, & \dots, & a_r; & q, & z\\ b_1, & b_2, & \dots, & b_s; & & \end{array}\right]\\ &=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(b_1,b_2,\dots,b_s;q)_n}\left[(-1)^n\, q^{\binom{n}{2}}\right]^{1+s-r}\frac{z^n}{(q;q)_n},\end{aligned} \tag{1.11}$$

where $a_i,b_j\in\mathbb{C},\forall i=1,2,\dots,r;\ \forall j=1,2,\dots,s,\ b_j\neq 0,-1,-2,\dots$.

This function turns out to be an analytic function at each point z interior to the unit disk $|z|=1$ when $r=1+s$. Whereas for $r\le s$, it becomes an entire function.

If $r = 1$ and $s = 0$ then (1.11) gets reduced to [23]

$$ {}_1\phi_0\left[\begin{array}{ccc} 0; & q; & z \\ -; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1 \tag{1.12} $$

which is a q-analogue of the exponential function e^z denoted by $e_q(z)$. Another q-analogue of e^z is denoted and defined by [23]

$$ E_q(z) = {}_0\phi_0\left[\begin{array}{ccc} -; & q; & -z \\ -; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}z^n}{(q;q)_n} = (-z;q)_\infty. \tag{1.13} $$

The following are definitions and properties of the q-analogues of $\sin x$ and $\cos x$.

Definition 1.18. Let $|z| < 1$. Define [23, 24]

$$ \begin{aligned} \sin_q(z) &= \frac{e_q(iz) - e_q(-iz)}{2i}, \quad \cos_q(z) = \frac{e_q(iz) + e_q(-iz)}{2}, \\ \mathrm{Sin}_q(z) &= \frac{E_q(iz) - E_q(-iz)}{2i}, \quad \mathrm{Cos}_q(z) = \frac{E_q(iz) + E_q(-iz)}{2}. \end{aligned} $$

It is not difficult to verify that

1. $e_q(iz) = \cos_q(z) + i\ \sin_q(z)$,
2. $E_q(iz) = \mathrm{Cos}_q(z) + i\ \mathrm{Sin}_q(z)$,
3. $\sin_q(z)\ \mathrm{Sin}_q(z) + \cos_q(z)\ \mathrm{Cos}_q(z) = 1$,
4. $\sin_q(z)\ \mathrm{Cos}_q(z) - \mathrm{Sin}_q\ \cos_q(z) = 0$.

Following are the basic definitions in q-theory.
A $q-$Gamma function is defined as [23, 29]

$$ \Gamma_q(\alpha) = \frac{(q;q)_\infty\ (1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}, $$

where $\alpha \neq 0, -1, -2, \ldots$.
Also

$$ (\alpha;q)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}\ (1-q)^n, $$

holds true.
For large $|z|$, the q-Gamma function [41, Eq.(2.25), p.482]

$$ \Gamma_q(z) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-z}\ e^{\frac{\theta q^z}{1-q-q^z}}, $$

where $0 < \theta < 1$.
The difference operator θ_q is defined by

$$\theta_q f(z) = f(z) - f(zq)$$

and q-derivative operator is denoted and defined by [23, Ex.(1.12), p.22]

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}.$$

The q-integral of function f is given by [23, Eq.(1.11.1), p.19]

$$I_q f(z) = \int_0^z f(t) d_q t = z\ (1-q) \sum_{n=0}^{\infty} f(zq^n)\ q^n.$$

1.4.1 Difference equation

Theorem 1.19. *If the difference operator* $\theta_q f(z) = f(z) - f(zq)$ *then* $w = {}_2\phi_1[a, b; c; q; z]$ *is a solution of the difference equation*

$$\left[\theta_q(\theta_q + q^{1-c} - 1) - z\ q^{a+b-c+1}(\theta_q + q^a - 1)(\theta_q + q^b - 1)\right] w \ = \ 0.$$

1.4.2 q-Contiguous function relations

If one and only one of the parameters of the basic hypergeometric series is multiplied by q or divided by q, the resulting function is said to be contiguous to the ${}_r\phi_s[z]$.
In the notations:

$$\begin{aligned} \phi &= {}_r\phi_s(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; q; z) \\ \phi(a_1+) &= {}_r\phi_s(a_1 q, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; q; z) \\ \phi(b_1-) &= {}_r\phi_s(a_1, a_2, \ldots, a_p; b_1 q^{-1}, b_2, \ldots, b_q; q; z), \end{aligned}$$

the q-contiguous function relations are [52]:

1. $(a_1 - a_k)\phi = (1 - a_k)a_1\ \phi(a_k+) - (1 - a_1)a_k\ \phi(a_1+) \quad k = 2, 3, \ldots, r$
2. $(a_1 - b_k q^{-1})\phi = (1 - b_k q^{-1})a_1 \phi(b_k-) - (1 - a_1)b_k q^{-1} \phi(a_1+) \quad k = 1, 2, \ldots, s.$

1.4.3 The q-binomial coefficient

The q-binomial coefficient is denoted and defined as [23]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}\,(q;q)_k}.$$

As $q \to 1$, this approaches to $\binom{n}{k}$.

1.4.4 The q-binomial theorem

Theorem 1.20. *If $|z| < 1$, then [23, Eq. (1.3.12), p.8]:*

$${}_1\phi_0 \left[\begin{matrix} \alpha; & q; & z \\ -; & & \end{matrix} \right] = \frac{(\alpha z;q)_\infty}{(z;q)_\infty}.$$

1.4.5 q-Gauss summation formula

Theorem 1.21. *The q-Gauss summation formula is given by [23, Eq. (1.5.1), p.10]*

$${}_2\phi_1 \left[\begin{matrix} a, & b; & q; & c/ab \\ & c; & & \end{matrix} \right] = \frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty} \qquad (|b| < 1,\ |c/ab| < 1). \tag{1.14}$$

1.4.6 The product formula

Theorem 1.22. *For $|z| < 1$, [23, Eq.(1.3.13), p.8]*

$${}_1\phi_0 \left[\begin{matrix} a; & q; & z \\ -; & & \end{matrix} \right] {}_1\phi_0 \left[\begin{matrix} b; & q; & az \\ -; & & \end{matrix} \right] = {}_1\phi_0 \left[\begin{matrix} ab; & q; & z \\ -; & & \end{matrix} \right].$$

1.4.7 q-Taylor's series

In [31], Jackson introduced the following q-Taylor series,

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n}\, D_q^n f(a)[x-a]_n$$

where D_q is usual q-differential operator and for $n \geq 1$

$$[x-a]_n = (x-a)(x-aq)\cdots(x-aq^{n-1}),$$

and for $n = 0$, $[x-a]_0 = 1$.

1.5 q-Analogue of Sikkema's function

A q-analogue of Sikkema's function (1.8) may be given by

$$\sum_{n=1}^{\infty} \frac{z^n}{(q;q)_n^n}. \tag{1.15}$$

1.6 q-Bessel function

At the beginning of the 20$^{\text{th}}$ centaury F. H. Jackson, (at the time when Charley Chaplain was in the British Royal Navy), introduced in a series of papers, basic analogues of the Bessel functions.
In the basic hypergeometric series notation, the most general forms read [23]

$$J_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (x/2)^\nu \; {}_2\phi_1 \left[\begin{array}{cccc} 0, & 0; & q; & -z^2/4 \\ & q^{\nu+1}; & & \end{array} \right], \tag{1.16}$$

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (x/2)^\nu \; {}_0\phi_1 \left[\begin{array}{ccc} -; & q; & -\frac{z^2 q^{\nu+1}}{4} \\ & q^{\nu+1}; & \end{array} \right], \tag{1.17}$$

where $0 < q < 1$. The above notations for the q-Bessel functions are due to M. E. H. Ismail [27] and the relation between these two analogues is:

$$J_\nu^{(2)}(z;q) = (-z^2/4;q)_\infty \, J_\nu^{(1)}(z;q); \quad |z| < 2.$$

The work carried out in this thesis is related to the q-Bessel function (1.16).
The basic properties of this function are stated below.

1.6.1 The generating function relation

Theorem 1.23. *For the function defined in (1.16), the generating function is given by [45, Eq. 1.5, p.59]*

$$\sum_{n=-\infty}^{\infty} t^n \; J_n^{(1)}(z;q) \;\; = \;\; e_q\left(\frac{zt}{2}\right) \; e_q\left(\frac{-z}{2t}\right).$$

1.6.2 The q-Bessel's integral

Theorem 1.24. *If* $|z| < 1$ *then for (1.16), the q-Bessel's integral is (cf. [53, Thm. 3.13, p.38])*

$$J_\nu^{(1)}(z;q) \;\; = \;\; \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \, \frac{1}{\left(\frac{ze^{i\theta}}{2}, \frac{-ze^{-i\theta}}{2}; q\right)_\infty} \, \mathrm{d}\theta.$$

Other results of this function are stated in Chapter 7.

The next two pages show the images of some contributors in the development of Special Functions' theory whose works form the bases of this entire work.

Carl Friedrich Gauss [1777-1855]

Friedrich Bessel [1784-1846]

Carl Jacobi [1804-1851]

Ernst Kummer [1810-1893]

Heinrich Eduard Heine [1821-1888]

Carl Johannes Thomae [1840-1921]

Srinivasa Ramanujan [1887-1920]

Wolfgang Hahn [1911-1998]