Chapter 5

The generalized q-ℓ-Ψ hypergeometric function

5.1 Introduction

In this chapter a q-extension of the generalized ℓ-Hypergeometric function is defined and its properties corresponding to those of Chapter 4, are obtained. Eventually, this q-form extends the functions defined by (3.2) and (3.26) of Chapter 3. Also, the proposed q-function may be viewed as an extension of generalized basic hypergeometric function ${}_r\phi_{s+p}$ in which p tends to infinity together with the summation index.

As a special cases, the work characterizes both the q-exponential functions (1.12) and (1.13) and thereby the q-trigonometric and q-hyperbolic functions. A series transformation formula and an ℓ-analogue of the q-Maclaurin's series are also obtained.

The proposed q-extension is as defined below.

Definition 5.1. For $z \in \mathbb{C}$ and $0 < q < 1$, an extension of the generalized basic hypergeometric function is denoted and defined by

$$\begin{aligned} {}_r\Psi_s^p(\ell : z) &= {}_r\Psi_s^p \left[\begin{matrix} a_1, & a_2, & \ldots, & a_r; & & & & & q; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p : \ell); & & \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \frac{\left[(-1)^n \, q^{\binom{n}{2}}\right]^{1+s+p\ell-r\ell-r}}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}} \frac{z^n}{(q; q)_n}, \end{aligned} \tag{5.1}$$

where $a_i \in \mathbb{C},\ b_j, c_k \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}, \forall i = 1,2,\ldots,r,\ \forall j = 1,2,\ldots,s, \forall k = 1,2,\ldots,p$, and $\ell \in \mathbb{C}$ with $\Re(\ell) \geq 0$, $\Re(1+s+p\ell) \geq \Re(r\ell+r)$.
A convention is adopted here that when $p = 0$, then either $\ell = 0$ or the terms containing ℓ must be treated as absent.

This function will be referred to as the *generalized q-ℓ-Ψ hypergeometric function*, in brief, the *generalized q-ℓ-Ψ function.* Throughout the work, $1+s+p\ell-r\ell-r = \kappa$, $0 < q < 1$ and $i,\ j,\ k$ vary over their respective range as stated in the definition.

Note 5.2. *Evidently,* ${}_r\Psi_s^p(0:z) = {}_r\phi_s[z] = {}_r\Psi_s^0(-:z)$.

In (5.1), the coefficient of z^n will be abbreviated by the notation:

$$\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\ \frac{\left[(-1)^n\ q^{\binom{n}{2}}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}\ (q;q)_n} = \sigma_n. \tag{5.2}$$

5.2 Main Results

5.2.1 Convergence

The series in (5.1) is absolutely convergent under one of the following conditions:
(i) $|z| < \infty$, if $\Re(1+s+p\ell) > \Re(r\ell+r)$,

(ii) $|z| < \left|\frac{(1-q^2)^{\frac{p}{2}}\ \Gamma_{q^2}^p\left(\frac{1}{2}\right)}{(1-q)^{\sum_{k=1}^{p} c_k}\ \left\{\prod_{k=1}^{p}\Gamma_q(c_k)\right\}}\right|^{\Re(\ell)}$, if $\Re(1+s+p\ell) = \Re(r\ell+r)$.
This is proved in

Theorem 5.3. *If $\Re(\ell) \geq 0$ then the generalized q-ℓ-Ψ function is*
(i) an entire function of z for $\Re(1+s+p\ell) > \Re(r\ell+r)$ and
(ii) an analytic function for $\Re(1+s+p\ell) = \Re(r\ell+r)$.

Proof. Here in (5.2) expressing the q-factorial functions $(a_i;q)_n,\ (b_j;q)_n$, and $(c_k;q)_n$ in corresponding q-Gamma functions with the aid of the the formula:

$$(\alpha;q)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}\ (1-q)^n \tag{5.3}$$

and applying the q-analogue of the Stirling's formula [41, Eq.(2.25), p.482] :

$$\Gamma_q(u) \sim (1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-u} \; e^{\frac{\mu q^u}{1-q-q^u}}, \qquad 0<\mu<1 \tag{5.4}$$

for large $|u|$, with $u = a_i+n, \; b_j+n, \; c_k+n$ in turn, one finds

$$\begin{aligned}
\sigma_n &= \frac{\left\{\prod_{j=1}^{s}\Gamma_q(b_j)\right\}}{\left\{\prod_{i=1}^{r}\Gamma_q(a_i)\right\}} \frac{\left\{\prod_{i=1}^{r}\Gamma_q(a_i+n)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_q(b_j+n)\right\}} \frac{\left\{\prod_{k=1}^{p}\Gamma_q^{\ell n}(c_k)\right\}}{\left\{\prod_{k=1}^{p}\Gamma_q^{\ell n}(c_k+n)\right\}} \\
&\quad \times \frac{\left[(-1)^n \, q^{\binom{n}{2}}\right]^{\kappa}(1-q)^r}{(1-q)^{s+1+p\ell n} \; \Gamma_q(n+1)} \\
&\sim \frac{\left\{\prod_{j=1}^{s}\Gamma_q(b_j)\right\}}{\left\{\prod_{i=1}^{r}\Gamma_q(a_i)\right\}} \frac{(1+q)^{\frac{r}{2}} \; \Gamma_{q^2}^r\left(\frac{1}{2}\right)\left\{\prod_{i=1}^{r}(1-q)^{\frac{1}{2}-(a_i+n)} \; e^{\frac{\mu \; q^{a_i+n}}{1-q-q^{a_i+n}}}\right\}}{(1+q)^{\frac{s}{2}} \; \Gamma_{q^2}^s\left(\frac{1}{2}\right)\left\{\prod_{j=1}^{s}(1-q)^{\frac{1}{2}-(b_j+n)} \; e^{\frac{\mu \; q^{b_j+n}}{1-q-q^{b_j+n}}}\right\}} \\
&\quad \times \frac{\left\{\prod_{k=1}^{p}\Gamma_q^{\ell n}(c_k)\right\}}{\left[(1+q)^{\frac{p}{2}} \; \Gamma_{q^2}^p\left(\frac{1}{2}\right)\left\{\prod_{k=1}^{p}(1-q)^{\frac{1}{2}-(c_k+n)} \; e^{\frac{\mu \; q^{c_k+n}}{1-q-q^{c_k+n}}}\right\}\right]^{\ell n}} \\
&\quad \times \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{\kappa} \; (1-q)^{r-s-1-\ell p n}}{(1+q)^{\frac{1}{2}} \; \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)} \; e^{\frac{\mu \; q^{n+1}}{1-q-q^{n+1}}}}.
\end{aligned}$$

Since $0<q<1$,

$$\begin{aligned}
\lim_{n\to\infty}\sup|\sigma_n|^{\frac{1}{n}} &\sim \lim_{n\to\infty}\sup\left|\frac{(1-q)^{\sum_{k=1}^{p}c_k}\left\{\prod_{k=1}^{p}\Gamma_q(c_k)\right\}}{(1-q^2)^{\frac{p}{2}} \; \Gamma_{q^2}^p\left(\frac{1}{2}\right)}\right|^{\Re(\ell)} \left|q^{\kappa(n-1)}\right| \\
&= \begin{cases} 0, & \text{if} \quad \Re(\kappa)>0 \\ \left|\frac{(1-q)^{\sum_{k=1}^{p}c_k}\left\{\prod_{k=1}^{p}\Gamma_q(c_k)\right\}}{(1-q^2)^{\frac{p}{2}} \; \Gamma_{q^2}^p\left(\frac{1}{2}\right)}\right|^{\Re(\ell)}, & \text{if} \quad \Re(\kappa)=0, \end{cases},
\end{aligned}$$

provided $\Re(\ell)\geq 0$. □

5.2.2 Order of the generalized q-ℓ-Ψ function

Theorem 5.4. *If* $\Re(\ell) \geq 0$ *and* $\Re(1+s+p\ell) > \Re(r\ell + r)$ *then the generalized* q-ℓ-Ψ *function is an entire function of order zero.*

Proof. It is known that [8, 40] if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of f is given by

$$\rho(f) = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln |a_n|^{-1}}. \tag{5.5}$$

Now from the definition of the generalized q-ℓ-Ψ function,

$$\begin{aligned}\sigma_n^{-1} \quad = \quad & \frac{\left\{\prod_{j=1}^{s} \Gamma_q(b_j)\right\}}{\left\{\prod_{i=1}^{r} \Gamma_q(a_i)\right\}} \frac{\left\{\prod_{j=1}^{s} \Gamma_q(b_j+n)\right\}}{\left\{\prod_{i=1}^{r} \Gamma_q(a_i+n)\right\}} \frac{\left\{\prod_{k=1}^{p} \Gamma_q^{\ell n}(c_k+n)\right\}}{\left\{\prod_{k=1}^{p} \Gamma_q^{\ell n}(c_k)\right\}} \\ & \times \frac{(1-q)^{s+1+\ell p n}\ \Gamma_q(n+1)}{\left[(-1)^n\ q^{\binom{n}{2}}\right]^{\kappa} (1-q)^r}.\end{aligned}$$

Hence from (5.4),

$$\begin{aligned}
&\ln|\sigma_n|^{-1} \\
&\sim \sum_{j=1}^{s} \ln|\Gamma_q(b_j)| - \sum_{i=1}^{r} \ln|\Gamma_q(a_i)| + (1+s-r)\ \ln(1-q) + \Re(\ell p n)\ \ln(1-q) \\
&\quad + \sum_{j=1}^{s} \ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(b_j+n)}\ e^{\frac{\mu\ q^{b_j+n}}{1-q-q^{b_j+n}}} \right| \\
&\quad - \sum_{i=1}^{r} \ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(a_i+n)}\ e^{\frac{\mu\ q^{a_i+n}}{1-q-q^{a_i+n}}} \right| \\
&\quad + \Re(\ell)\ n \sum_{k=1}^{p} \ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(c_k+n)}\ e^{\frac{\mu\ q^{c_k+n}}{1-q-q^{c_k+n}}} \right| \\
&\quad + \ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right) (1-q)^{\frac{1}{2}-(n+1)}\ e^{\frac{\mu\ q^{n+1}}{1-q-q^{n+1}}} \right| \\
&\quad - \Re(\ell)\ n \sum_{k=1}^{p} \ln|\Gamma_q(c_k)| - \frac{n(n-1)}{2}(\kappa) \ln q \\
&= D_1 - D_2 + D_3 + D_4 + \sum_{j=1}^{s} B_{j,n} - \sum_{i=1}^{r} A_{i,n} + \Re(\ell)\ n \sum_{k=1}^{p} C_{k,n} \\
&\quad + E_{1,n} - \Re(\ell)\ n\ D_5 - E_{q,n} \quad \text{(say)}.
\end{aligned} \tag{5.6}$$

Evidently, for $m = 1, 2, 3, 4,$

$$\lim_{n\to\infty} \frac{D_m}{n \ln n} = 0.$$

Also

$$\lim_{n\to\infty} \frac{\Re(\ell)\ n\ D_5}{n \ln n} = 0.$$

Next,

$$\lim_{n\to\infty} \frac{B_{j,n}}{n \ln n} = \lim_{n\to\infty} \frac{\ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(b_j+n)}\ e^{\frac{\mu\ q^{b_j+n}}{1-q-q^{b_j+n}}} \right|}{n \ln n} = 0,$$

Similarly,

$$\lim_{n\to\infty} \frac{A_{i,n}}{n \ln n} = 0, \quad \lim_{n\to\infty} \frac{E_{1,n}}{n \ln n} = 0.$$

Now, it can be verified that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\Re(\ell)\ n\ C_{k,n} - E_{q,n}}{n \ln n} \\ &= \lim_{n\to\infty} \frac{\Re(\ell)\ n \ln \left| (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c_k+n)}\ e^{\frac{\mu\ q^{c_k+n}}{1-q-q^{c_k+n}}} \right|}{n \ln n} \\ &\quad - \lim_{n\to\infty} \frac{n(n-1)(\kappa)\ln q}{2\ n \ln n} \\ &= -\infty, \end{aligned}$$

as $\frac{n-1}{\ln n} \to \infty$ when $n \to \infty$. Hence, from (5.6), one finds

$$\lim_{n\to\infty} \frac{\ln |\sigma_n|^{-1}}{n \ln n} = -\infty.$$

Thus in view of the definition of order of function,

$$\rho\left({}_r\Psi_s^p(\ell : z)\right) = \lim_{n\to\infty} \frac{n \ln n}{\ln |\sigma_n|^{-1}} = 0.$$

□

Remark 5.5. As remarked in previous Chapters 2,3,4 ([4, Theorem 1.1, p.1]), the generalized q-ℓ-Ψ function has infinitely many zeros by virtue of being entire function and $\rho\left({}_r\Psi_s^p(\ell : z)\right)$ is finite and not a positive integer.

5.2.3 q-Difference Equation

The difference equation of the generalized q-ℓ-Ψ function occurs for $\ell \in \mathbb{N} \cup \{0\}$ which is obtained by means of already introduced operator (Chapter 3, Definition 3.4):

$$ {}_p\Delta_\alpha^{\theta_q}(f(z;q)) = \begin{cases} \sum\limits_{n=1}^{\infty} a_{n,q}\ (\alpha;q)_{n-1}^{p}\ \left(\Theta_q^{\alpha-1}\right)^{pn}\ z^n, & \text{if} \quad \mathrm{p} \in \mathbb{N} \\ f(z;q), & \text{if} \quad \mathrm{p} = 0 \end{cases} \tag{5.7} $$

wherein

$$ f(z;q) = \sum_{n=1}^{\infty} a_{n,q}\ z^n,\ 0 \neq z \in \mathbb{C},\ p \in \mathbb{N} \cup \{0\},\ \alpha \in \mathbb{C}, $$

$$ \Theta_q^{\alpha} \ := \ \left(q^{\alpha}\ \theta_q - q^{\alpha} + 1\right) \equiv \left(\alpha\ \theta_q - \alpha + 1\right), \tag{5.8} $$

and

$$ \theta_q f(z) = f(z) - f(zq). \tag{5.9} $$

One more operator needs to be defined. It is given as

Definition 5.6. Let $f(z;q) = \sum\limits_{n=0}^{\infty} a_{n,q}\ z^n$, $0 \neq z \in \mathbb{C},\ p, r, s \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta, \gamma \in \mathbb{C}$. Define

$$ {}_{(\beta,s)}\Lambda_{(\gamma:\ell,p)}^{(r,\kappa,\theta_q)}\ f(z;q) = \left[\left\{\prod_{k=1}^{p}\ {}_\ell\Delta_{\gamma_k}^{\theta_q}\right\}\left\{\prod_{j=1}^{s}\Theta_q^{\beta_j-1}\right\}\theta_q\right](-q)^{\kappa}\ f\left(\frac{z}{q^{\kappa}};q\right), \tag{5.10} $$

where the operators $\Theta_q^{\beta-1}$ and ${}_\ell\Delta_c^{\theta_q}$ are as defined in (5.8) and (5.7) respectively.

These operators help in deriving the difference equation as follows.

Theorem 5.7. *For $\ell \in \mathbb{N} \cup \{0\}, a_i, z \in \mathbb{C}$, and $b_j, c_k \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, the function $w = {}_r\Psi_s^p(\ell : z)$ satisfies the difference equation*

$$ \left[{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)} - z\left\{\prod_{i=1}^{r}\Theta_q^{a_i}\right\}\right] w \ = \ 0, \tag{5.11} $$

where the operators Θ_q^{α} and ${}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}$ are as defined in (5.8) and (5.10).

Just as the differential equations derived for the functions in the preceding chapters, here also the following lemma is proved which first permits to apply the infinite order derivative operator ${}_{(b,s)}\Lambda^{(r,\kappa,\theta_q)}_{(c:\ell,p)}$ on w.

Lemma 5.8. *If* $\ell \in \mathbb{N}\cup\{0\}, a_i, z \in \mathbb{C},\ b_j, c_k \in \mathbb{C}\setminus\{0,-1,-2,\ldots\},\ w = {}_r\Psi^p_s(\ell;q;z) = \sum_{n=0}^{\infty}\sigma_n z^n$ *and* ${}_{(b,s)}\Lambda^{(r,\kappa,\theta_q)}_{(c:\ell,p)} = \sum_{n=0}^{\infty} f_n((a,r),(b,s),(c:\ell,p);q,z)$ *then the operator* ${}_{(b,s)}\Lambda^{(r,\kappa,\theta_q)}_{(c:\ell,p)}$ *is applicable to the generalized q-ℓ-Ψ function provided that the series*

$$\sum_{n=0}^{\infty}\sigma_n\ f_n((a,r),(b,s),(c:\ell,p);q,z)$$

converges (cf. [49, Definition 11, p.20]).

Proof. Here

$$\begin{aligned}
&{}_{(b,s)}\Lambda^{(r,\kappa,\theta_q)}_{(c:\ell,p)}\ w\\
&= \left[\left\{\prod_{k=1}^{p}\ {}_\ell\Delta^{\theta_q}_{c_k}\right\}\left\{\prod_{j=1}^{s}\Theta_q^{b_j-1}\right\}\theta_q\right]\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad\times\frac{\left[(-1)^n\ q^{n(n-1)2}\right]^{\kappa}(-q)^{\kappa}}{(c_1,c_2,\ldots,c_p;q)^{\ell n}_n(q;q)_n}\ \left(\frac{z}{q^{\kappa}}\right)^n\\
&= \left[\left\{\prod_{k=1}^{p}\ {}_\ell\Delta^{\theta_q}_{c_k}\right\}\left\{\prod_{j=1}^{s}\Theta_q^{b_j-1}\right\}\theta_q\right]\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad\times\frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)^{\ell n}_n}\ \frac{z^n}{(q;q)_n}\\
&= \left[\left\{\prod_{k=1}^{p}\ {}_\ell\Delta^{\theta_q}_{c_k}\right\}\left\{\prod_{j=1}^{s}\Theta_q^{b_j-1}\right\}\right]\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad\times\frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)^{\ell n}_n}\ \frac{z^n-z^nq^n}{(q;q)_n}\\
&= \left[\left\{\prod_{k=1}^{p}\ {}_\ell\Delta^{\theta_q}_{c_k}\right\}\left\{\prod_{j=1}^{s}(q^{b_j-1}\theta_q-q^{b_j-1}+1)\right\}\right]\sum_{n=1}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad\times\frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)^{\ell n}_n}\ \frac{z^n}{(q;q)_{n-1}}\\
&= \left\{\prod_{k=1}^{p}\ {}_\ell\Delta^{\theta_q}_{c_k}\right\}\left\{\prod_{j=1}^{s-1}(q^{b_j-1}\theta_q-q^{b_j-1}+1)\right\}\sum_{n=1}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad\times\frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)^{\ell n}_n}\ \frac{(q^{b_s-1}\theta_q-q^{b_s-1}+1)\ z^n}{(q;q)_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \prod_{k=1}^{p} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \left\{ \prod_{j=1}^{s-1} (q^{b_j-1}\theta_q - q^{b_j-1}+1) \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n} \\
&\times \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{\left(q^{b_s-1}(z^n - z^n q^n) - z^n(q^{b_s-1}-1)\right)}{(q;q)_{n-1}} \\
&= \left\{ \prod_{k=1}^{p} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \left\{ \prod_{j=1}^{s-1} (q^{b_j-1}\theta_q - q^{b_j-1}+1) \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n} \\
&\times \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{z^n\,(1-q^{b_s+n-1})}{(q;q)_{n-1}} \\
&= \left\{ \prod_{k=1}^{p} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \left\{ \prod_{j=1}^{s-1} (q^{b_j-1}\theta_q - q^{b_j-1}+1) \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_{s-1};q)_n} \\
&\times \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(b_s;q)_{n-1}\;(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{z^n}{(q;q)_{n-1}}.
\end{aligned}
$$

Now applying the operators $(q^{b_j-1}\theta_q - q^{b_j-1}+1)$ for $j=1,2,\ldots,s-1$, one finds that

$$
\begin{aligned}
&{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\; w \\
&= \left\{ \prod_{k=1}^{p} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{z^n}{(q;q)_{n-1}} \\
&= \left\{ \prod_{k=1}^{p-1} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} {}_{\ell}\Delta_{c_p}^{\theta_q} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \\
&\times \frac{z^n}{(q;q)_{n-1}} \\
&= \left\{ \prod_{k=1}^{p-1} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \\
&\times \frac{(c_p;q)_{n-1}\;\left(q^{c_p-1}\theta_q - q^{c_p-1}+1\right)^{\ell n}\; z^n}{(q;q)_{n-1}}.
\end{aligned}
$$

From (3.10),

$$
\left(q^{c_k-1}\theta_q - q^{c_k-1}+1\right)^{\ell n}\; z^n = (1-q^{n+c_k-1})^{\ell n}\; z^n,
$$

where $k=1,2,\ldots,p$. Thus

$$
\begin{aligned}
&{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\; w \\
&= \left\{ \prod_{k=1}^{p-1} {}_{\ell}\Delta_{c_k}^{\theta_q} \right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\, q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}
\end{aligned}
$$

$$\times \frac{(c_p;q)_{n-1}^{\ell}\ (1-q^{n+c_p-1})^{\ell n}\ z^n}{(q;q)_{n-1}}$$

$$= \left\{\prod_{k=1}^{p-1} {}_{\ell}\Delta_{c_k}^{\theta_q}\right\} \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_{p-1};q)_n^{\ell n}\ (c_p;q)_{n-1}^{\ell n-\ell}}$$

$$\times \frac{z^n}{(q;q)_{n-1}}.$$

Proceeding similarly, by applying ${}_{\ell}\Delta_{c_k}^{\theta_q}$ for $k = 1, 2, \ldots, p-1$, one gets

$$\begin{aligned}
&{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\ w \\
&= \sum_{n=1}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_{n-1}} \frac{\left[(-1)^{n+1}\ q^{(n-1)(n-2)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_{n-1}^{\ell n-\ell}} \frac{z^n}{(q;q)_{n-1}} \\
&= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n} \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{z^{n+1}}{(q;q)_n} \\
&= z \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n} \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \frac{z^n}{(q;q)_n} \qquad (5.12) \\
&= \sum_{n=0}^{\infty} f_n((a,r),(b,s),(c,p:\ell);q,z).
\end{aligned}$$

To complete the proof, it remains to show that

$$\sum_{n=0}^{\infty} \sigma_n\ f_n((a,r),(b,s),(c,p:\ell);q,z)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n^2\ \left\{\prod_{i=1}^{r}(1-a_iq^n)\right\}}{(b_1,b_2,\ldots,b_s;q)_n^2} \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{2\kappa}\ z^{n+1}}{(c_1,c_2,\ldots,c_p;q)_n^{2\ell n}\ (q;q)_n^2}$$

is convergent.
For that, take

$$\sigma_n\ f_n((a,r),(b,s),(c,p:\ell);q,z) = \Omega_n,$$

then

$$|\Omega_n| = \frac{\left|(a_1,a_2,\ldots,a_r;q)_n^2\ \left\{\prod_{i=1}^{r}(1-a_iq^n)\right\}\ (q^{n(n-1)/2})^{2(\kappa)}\ z^{n+1}\right|}{|(b_1,b_2,\ldots,b_s;q)_n^2\ (c_1,c_2,\ldots,c_p;q)_n^{2\ell n}\ (q;q)_n^2|}$$

$$= \frac{\left|\left\{\prod_{j=1}^{s}\Gamma_q^2(b_j)\right\}\right|}{\left|\left\{\prod_{i=1}^{r}\Gamma_q^2(a_i)\right\}\right|} \frac{\left|\left\{\prod_{i=1}^{r}\Gamma_q^2(a_i+n)\right\}\right| \; \left|\left\{\prod_{i=1}^{r}(1-a_iq^n)\right\}\right|}{\left|\left\{\prod_{j=1}^{s}\Gamma_q^2(b_j+n)\right\}\right| \; \left|\left\{\prod_{k=1}^{p}\Gamma_q^{2\ell n}(c_k+n)\right\}\right|}$$
$$\times \frac{\left|\left\{\prod_{k=1}^{p}\Gamma_q^{2\ell n}(c_k)\right\}\right| \; \left|(q^{n(n-1)/2})^{2\kappa}\ z^{n+1}\right|}{\left|(1-q)^{2(s+1+\ell pn-r)}\right| \; \left|\Gamma_q^2(n+1)\right|}.$$

With the aid of the q-Stirling's formula (5.4),

$$|\Omega_n| \sim \frac{\left|\left\{\prod_{j=1}^{s}\Gamma_q^2(b_j)\right\}\right|}{\left|\left\{\prod_{i=1}^{r}\Gamma_q^2(a_i)\right\}\right|} \frac{\left|(1+q)^{\frac{r}{2}}\ \Gamma_{q^2}^r\left(\frac{1}{2}\right)\left\{\prod_{i=1}^{r}(1-q)^{\frac{1}{2}-(a_i+n)}\ e^{\frac{\mu\ q^{a_i+n}}{1-q-q^{a_i+n}}}\right\}\right|^2}{\left|(1+q)^{\frac{s}{2}}\ \Gamma_{q^2}^s\left(\frac{1}{2}\right)\left\{\prod_{j=1}^{s}(1-q)^{\frac{1}{2}-(b_j+n)}\ e^{\frac{\mu\ q^{b_j+n}}{1-q-q^{b_j+n}}}\right\}\right|^2}$$
$$\times \frac{\left|\left\{\prod_{i=1}^{r}(1-a_iq^n)\right\}\right| \; \left|\left\{\prod_{k=1}^{p}\Gamma_q^{2\ell n}(c_k)\right\}\right|}{\left|(1+q)^{\frac{p}{2}}\ \Gamma_{q^2}^p\left(\frac{1}{2}\right)\left\{\prod_{k=1}^{p}(1-q)^{\frac{1}{2}-(c_k+n)}\ e^{\frac{\mu\ q^{c_k+n}}{1-q-q^{c_k+n}}}\right\}\right|^{2n\ell}}$$
$$\times \frac{\left|(q^{n(n-1)/2})^{2\kappa}\ (1-q)^{-2(s+1+\ell pn-r)}\ z^{n+1}\right|}{\left|(1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)}\ e^{\frac{\mu\ q^{n+1}}{1-q-q^{n+1}}}\right|^2}.$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty}\sup|\Omega_n|^{\frac{1}{n}} \\
&\sim \lim_{n\to\infty}\sup \frac{\left|\left\{\prod_{k=1}^{p}\Gamma_q^{2\ell}(c_k)\right\}\right|\ |z|}{\left|\Gamma_{q^2}^{2\ell p}\left(\frac{1}{2}\right)\right|} \left|\left\{\prod_{i=1}^{r}(1-a_iq^n)^{\frac{1}{n}}\right\}\right|\ q^{2(n-1)\kappa} \\
&= 0,
\end{aligned}$$

provided $\ell \in \mathbb{N}\cup\{0\}$ and $(1+s+p\ell-r\ell-r)>0$. □

Proof. (of Theorem 5.7) From (5.12),

$$\begin{aligned}
&{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\ w \\
&= \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\frac{z^{n+1}}{(q;q)_n} \qquad (5.13)
\end{aligned}$$

$$
\begin{aligned}
&= z \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r-1}; q)_{n+1}\ (a_r; q)_n\ (1 - a_r q^n)}{(b_1, b_2, \ldots, b_s; q)_n} \\
&\quad \times \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}} \frac{z^n}{(q;q)_n} \\
&= z \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r-1}; q)_{n+1}\ (a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \\
&\quad \times \frac{(a_r z^n - a_r z^n q^n - a_r z^n + z^n)\ \left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}\ (q;q)_n} \\
&= z \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r-1}; q)_{n+1}\ (a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \\
&\quad \times \frac{(a_r \theta_q - a_r + 1)\ z^n\ \left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}\ (q;q)_n} \\
&= z \Theta_q^{a_r} \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r-1}; q)_{n+1}\ (a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}\ z^n}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}\ (q;q)_n}. \qquad (5.14)
\end{aligned}
$$

Following the same procedure for the remaining parameters $a_1, a_2, \ldots, a_{r-1}$, one finally obtains

$$
\begin{aligned}
&{}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\ w \\
&= z \left\{ \prod_{i=1}^{r} \Theta_q^{a_i} \right\} \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1, c_2, \ldots, c_p; q)_n^{\ell n}} \frac{z^n}{(q;q)_n}.
\end{aligned}
$$

Hence (5.11) follows. □

5.2.4 Contiguous function relations

The contiguous function relations for the basic hypergeometric series have been derived by Swarttouw [52]. Here an attempt is made to obtain some contiguous function relations involving the generalized q-ℓ-Ψ function.
Using the notation (cf. [46, Sec. 48, p.80])

$$
\Psi = \sum_{n=0}^{\infty} \sigma_n z^n
$$

that is,

$$
\Psi[z] = {}_r\Psi_s^p \left[\begin{array}{llllllllll} a_1, & a_2, & \ldots, & a_r; & & & & & q; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p : \ell); & & \end{array} \right],
$$

the contiguous functions may be defined as

$$\begin{aligned}\Psi(a_1+) &:= {}_r\Psi_s^p\left[\begin{matrix} a_1 q, & a_2, & \ldots, & a_r; & & & & & q; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p : \ell); & & \end{matrix}\right],\\ \Psi(a_1-) &:= {}_r\Psi_s^p\left[\begin{matrix} a_1 q^{-1}, & a_2, & \ldots, & a_r; & & & & & q; & z \\ b_1, & b_2, & \ldots, & b_s; & (c_1, & c_2, & \ldots, & c_p : \ell); & & \end{matrix}\right]\end{aligned}$$

and similarly the functions $\Psi(b_j+), \Psi(b_j-), \Psi(c_k+), \Psi(c_k-)$.
The following series representations are evident.

$$\begin{aligned}\Psi(a_i+) &= \sum_{n=0}^{\infty} \tfrac{1-a_i q^n}{1-a_i}\ \sigma_n\ z^n, & \Psi(a_i-) &= \sum_{n=0}^{\infty} \tfrac{1-a_i q^{-1}}{1-a_i q^{n-1}}\ \sigma_n\ z^n,\\ \Psi(b_j+) &= \sum_{n=0}^{\infty} \tfrac{1-b_j}{1-b_j q^n}\ \sigma_n\ z^n, & \Psi(b_j-) &= \sum_{n=0}^{\infty} \tfrac{1-b_j q^{n-1}}{1-b_j q^{-1}}\ \sigma_n\ z^n,\\ \Psi(c_k+) &= \sum_{n=0}^{\infty} \tfrac{(1-c_k)^{\ell n}}{(1-c_k q^n)^{\ell n}}\ \sigma_n\ z^n, & \Psi(c_k-) &= \sum_{n=0}^{\infty} \tfrac{(1-c_k q^{n-1})^{\ell n}}{(1-c_k q^{-1})^{\ell n}}\ \sigma_n\ z^n.\end{aligned} \tag{5.15}$$

If $zD_q = \nabla_q$, where

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \tag{5.16}$$

then

$$\nabla_q \Psi = \sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)} \sigma_n\ z^n. \tag{5.17}$$

From this,

$$\begin{aligned}\left(a_i\ \nabla_q + \frac{1-a_i}{1-q}\right)\Psi &= a_i \sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)} \sigma_n z^n + \frac{1-a_i}{1-q} \sum_{n=0}^{\infty} \sigma_n z^n\\ &= \sum_{n=0}^{\infty} \left[a_i - a_i q^n + 1 - a_i\right] \frac{\sigma_n z^n}{1-q}\\ &= \frac{1-a_i}{1-q} \sum_{n=0}^{\infty} \frac{1-a_i q^n}{1-a_i} \sigma_n z^n\\ &= \frac{1-a_i}{1-q} \Psi(a_i+).\end{aligned} \tag{5.18}$$

Here, choose $i = 1$ to get

$$\left(a_1\ \nabla_q + \frac{1-a_1}{1-q}\right)\Psi = \frac{1-a_1}{1-q} \Psi(a_1+). \tag{5.19}$$

Next, using (5.17),

$$\begin{aligned}\left(b_jq^{-1}\,\nabla_q+\frac{1-b_jq^{-1}}{1-q}\right)\Psi &= b_jq^{-1}\sum_{n=0}^{\infty}\frac{(1-q^n)}{(1-q)}\sigma_n z^n+\frac{1-b_jq^{-1}}{1-q}\sum_{n=0}^{\infty}\sigma_n z^n\\ &= \sum_{n=0}^{\infty}\left[b_jq^{-1}-b_jq^{n-1}+1-b_jq^{-1}\right]\frac{\sigma_n z^n}{1-q}\\ &= \frac{1-b_jq^{-1}}{1-q}\Psi(b_j-). \qquad (5.20)\end{aligned}$$

Now multiplying (5.17) by a_i and subtracting from (5.18), one gets

$$\begin{aligned}(1-a_i)\Psi &= (1-a_i)\Psi(a_i+)-a_i\sum_{n=0}^{\infty}(1-q^n)\sigma_n z^n\\ \Psi &= (1-a_i)\Psi(a_i+)+a_i\Psi[zq];\end{aligned}$$

whereas on multiplying (5.17) by b_jq^{-1} and subtracting from (5.20) gives

$$\Psi=(1-b_jq^{-1})\Psi(b_j-)+b_j\Psi[zq]. \qquad (5.21)$$

Now multiplying (5.18) and (5.19) by a_1 and a_i respectively, and then eliminating ∇_q from them, one obtains for $i=2,3,\ldots,r$,

$$(a_i-a_1)\Psi = a_i(1-a_1)\Psi(a_1+)-a_1(1-a_i)\Psi(a_i+). \qquad (5.22)$$

Once again multiplying (5.18) and (5.20) by b_jq^{-1} and a_i respectively, and then eliminating ∇_q, yields

$$(b_jq^{-1}-a_i)\Psi = b_jq^{-1}(1-a_i)\Psi(a_i+)-a_i(1-b_jq^{-1})\Psi(b_j-). \qquad (5.23)$$

Further, the contiguous relations (5.22) and (5.23) when added, yield the identity:

$$\begin{aligned}(b_jq^{-1}-a_i)\Psi &= (b_jq^{-1}-a_1)(1-a_i)\Psi(a_i+)+a_i(1-a_1)\Psi(a_i+)\\ &\quad -a_i(1-b_jq^{-1})\Psi(b_j-), \qquad (5.24)\end{aligned}$$

where $i=2,3,\ldots,r$.

Next, one more q-contiguous function relation is obtained for $\ell\in\mathbb{N}\cup\{0\}$ which is given by

$$ {}_2\Psi^1_{2+\ell}\left[\begin{matrix} a,0; & & q; & \frac{zq}{(1-cq^{-1})^\ell}\\ b,c,c,\ldots,c,0; & (c:\ell); & & \end{matrix}\right]$$

$$= \frac{a-b}{1-b}\ {}_2\Psi^1_{2+\ell}\left[\begin{matrix} a, 0; & & q; & \frac{zq}{(1-cq^{-1})^\ell} \\ bq, c, c, \ldots, c, 0; & (c:\ell); & & \end{matrix}\right]$$

$$+\frac{1-a}{1-b}\ {}_2\Psi^1_{2+\ell}\left[\begin{matrix} aq, 0; & & q; & \frac{zq}{(1-cq^{-1})^\ell} \\ bq, c, c, \ldots, c, 0; & (c:\ell); & & \end{matrix}\right]. \qquad (5.25)$$

This may be proved as follows. With

$${}_1\Psi^1_1 = \sum_{n=0}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell n}} \frac{z^n}{(q;q)_n},$$

$$\begin{aligned}
\theta_q\ {}_1\Psi^1_1(c-) &= \sum_{n=1}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (q^{c-1};q)_n^{\ell n}} \frac{z^n}{(q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_{n-1}^{\ell n}\ (1-q^{c-1})^{\ell n}} \frac{z^n}{(q;q)_{n-1}} \\
&= z\sum_{n=0}^{\infty} \frac{1-aq^n}{1-bq^n} \frac{(a;q)_n\ (-1)^{n+1}\ q^{n(n+1)/2}}{(b;q)_n\ (c;q)_n^{\ell}\ (c;q)_n^{\ell n}\ (1-q^{c-1})^{\ell n+\ell}} \frac{z^n}{(q;q)_n} \\
&= \frac{-z}{(1-q^{c-1})^\ell}\sum_{n=0}^{\infty} \frac{1-aq^n}{1-bq^n} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell}\ (c;q)_n^{\ell n}} \frac{z^n\ q^n}{(1-q^{c-1})^{\ell n}\ (q;q)_n}.
\end{aligned}$$

Now, writing

$$\frac{1-aq^n}{1-bq^n} = 1 - \frac{aq^n - bq^n}{1-bq^n},$$

one further gets

$$\begin{aligned}
&\theta_q\ {}_1\Psi^1_1(c-) \\
&= \frac{-z}{(1-q^{c-1})^\ell}\sum_{n=0}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell}\ (c;q)_n^{\ell n}} \frac{z^n\ q^n}{(1-q^{c-1})^{\ell n}\ (q;q)_n} \\
&\quad +\frac{z(a-b)}{(1-b)(1-q^{c-1})^\ell}\sum_{n=0}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(bq;q)_n\ (c;q)_n^{\ell}\ (c;q)_n^{\ell n}} \frac{z^n\ q^{2n}}{(1-q^{c-1})^{\ell n}\ (q;q)_n} \\
&= \frac{-z}{(1-q^{c-1})^\ell}\ {}_2\Psi^1_{2+\ell}\left[\begin{matrix} a, 0; & & q; & \frac{zq}{(1-cq^{-1})^\ell} \\ b, c, c, \ldots, c, 0; & (c:\ell); & & \end{matrix}\right] \\
&\quad +\frac{z(a-b)}{(1-b)(1-q^{c-1})^\ell}\ {}_2\Psi^1_{2+\ell}\left[\begin{matrix} a, 0; & & q; & \frac{z}{(1-cq^{-1})^\ell} \\ bq, c, c, \ldots, c, 0; & (c:\ell); & & \end{matrix}\right]. \qquad (5.26)
\end{aligned}$$

Also

$$\theta_q\ {}_1\Psi^1_1 = \sum_{n=1}^{\infty} \frac{(a;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(b;q)_n\ (c;q)_n^{\ell n}} \frac{z^n}{(q;q)_{n-1}}$$

$$= \frac{(1-a)z}{(1-b)} \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}\ (-1)^n\ q^{n(n-1)/2}}{(bq;q)_{n-1}\ (1-c)^{\ell n}\ (cq;q)_{n-1}^{\ell n}} \frac{z^{n-1}}{(q;q)_{n-1}}$$

$$= \frac{(1-a)z}{(1-b)\ (1-c)^{\ell}} \sum_{n=0}^{\infty} \frac{(aq;q)_n\ (-1)^{n+1}\ q^{n(n+1)/2}}{(bq;q)_n\ (1-c)^{\ell n}\ (cq;q)_n^{\ell n+\ell}} \frac{z^n}{(q;q)_n}$$

$$= \frac{-(1-a)z}{(b-1)\ (1-c)^{\ell}} \sum_{n=0}^{\infty} \frac{(aq;q)_n\ (-1)^n\ q^{n(n-1)/2}}{(bq;q)_n\ (cq;q)_n^{\ell}\ (cq;q)_n^{\ell n}} \frac{z^n\ q^n}{(q;q)_n\ (1-c)^{\ell n}}$$

$$= \frac{(1-a)z}{(b-1)\ (1-c)^{\ell}}\ {}_2\Psi_{2+\ell}^{1}\left[\begin{matrix} aq, 0; & q; & \frac{zq}{(1-c)^{\ell}} \\ bq, cq, cq, \ldots, cq, 0; & (cq:\ell); & \end{matrix}\right].$$

Hence

$$\theta_q\ {}_1\Psi_1^1(c-) = \frac{(1-a)z}{(b-1)\ (1-cq^{-1})^{\ell}}\ {}_2\Psi_{2+\ell}^{1}\left[\begin{matrix} aq, 0; & q; & \frac{zq}{(1-cq^{-1})^{\ell}} \\ bq, c, c, \ldots, c, 0; & (c:\ell); & \end{matrix}\right] \quad (5.27)$$

Elimination of $\theta_q\ {}_1\Psi_1^1(c-)$ from (5.26) and (5.27) yields (5.25).

Remark 5.9. 1. By taking $\ell = 0$ in (5.25), it reduces to the q-contiguous function relation of basic hypergeometric function given as follows.

$${}_1\phi_1\left[\begin{matrix} a; & q, & zq \\ b; & & \end{matrix}\right] = \frac{a-b}{1-b}\ {}_1\phi_1\left[\begin{matrix} a; & q, & zq \\ bq; & & \end{matrix}\right] + \frac{1-a}{1-b}\ {}_1\phi_1\left[\begin{matrix} aq; & q, & zq \\ bq; & & \end{matrix}\right].$$

2. In (5.25) taking $q \to 1$ one gets the contiguous function relation (4.16):

$$\begin{aligned} &{}_1H_{1+\ell}^{1}\left[\begin{matrix} a; & & \frac{z}{(1-c)^{\ell}} \\ b, c, c, \ldots, c; & (c:\ell); & \end{matrix}\right] \\ &= (1-ab^{-1})\ {}_1H_{1+\ell}^{1}\left[\begin{matrix} a; & & \frac{z}{(1-c)^{\ell}} \\ b+1, c, c, \ldots, c; & (c:\ell); & \end{matrix}\right] \\ &\quad + ab^{-1}\ {}_1H_{1+\ell}^{1}\left[\begin{matrix} a+1; & & \frac{z}{(1-c)^{\ell}} \\ b+1, c, c, \ldots, c; & (c:\ell); & \end{matrix}\right] \end{aligned}$$

obtained in Chapter 4.

5.2.5 Eigen function property

In order to obtain the generalized q-ℓ-Ψ function as an eigen function, the following operators are reconsidered from Chapter 3. They are listed below.

$$\text{(i)}\quad I_q^{\alpha}(f(z;q)) = \frac{z^{-\alpha}}{1-q}\, I_q\left(z^{\alpha-1} f(z;q)\right), \quad \Re(\alpha) > 0, \tag{5.28}$$

where [23, Eq.(1.11.1), p.19]

$$I_q(g(z)) = \int_0^z g(t)\ d_q t = z\ (1-q) \sum_{n=0}^{\infty} g(zq^n)\ q^n, \tag{5.29}$$

and $f(z;q) = \sum_{n=0}^{\infty} a_{n,q}\ z^n,\ |z| < R, z \neq 0, R > 0,\ \alpha_i, \beta_j, \gamma_k \in \mathbb{C},\ m,n,p \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}$. Using (5.28) and the operator (5.10),

$$\text{(ii)}\ {}_{(\alpha,m)}\mathcal{E}_{(\beta,n)}^{(\gamma:\ell,p)}(f(z;q)) = \left[\left\{\prod_{i=1}^{m} I_q^{\alpha_i}\right\}\ z^{-1}\left({}_{(\beta,n)}\Lambda_{(\gamma:\ell,p)}^{(m,\kappa,\theta_q)}\right)\right] f(z;q). \tag{5.30}$$

In this notation, the eigen function property is obtained in

Theorem 5.10. *If $\ell \in \mathbb{N} \cup \{0\}$, $a_i \in \mathbb{C}$, $b_j, c_k \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$ then the generalized q-ℓ-Ψ function is an eigen function with respect to the operator ${}_{(a,r)}\mathcal{E}_{(b,s)}^{(c:\ell,p)}$ defined in (5.30). That is,*

$${}_{(a,r)}\mathcal{E}_{(b,s)}^{(c:\ell,p)}\left(\ {}_r\Psi_s^p(\ell:\lambda z)\right) = \lambda\ {}_r\Psi_s^p(\ell:\lambda z), \quad \lambda \in \mathbb{C}. \tag{5.31}$$

Proof. The applicability of this operator to the generalized q-ℓ-Ψ function follows from Lemma 5.8.

Now for $z \neq 0$,

$$\begin{aligned}
&{}_{(a,r)}\mathcal{E}_{(b,s)}^{(c:\ell,p)}\ {}_r\Psi_s^p(\ell:\lambda z)\\
&= \left[\left\{\prod_{i=1}^{r} I_q^{a_i}\right\}\ z^{-1}\left({}_{(b,s)}\Lambda_{(c:\ell,p)}^{(r,\kappa,\theta_q)}\right)\right] \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\quad \times \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\ \frac{\lambda^n\ z^n}{(q;q)_n}.
\end{aligned}$$

In view of (5.12),

$$\begin{aligned}
&{}_{(a,r)}\mathcal{E}_{(b,s)}^{(c:\ell,p)}\left(\ {}_r\Psi_s^p(\ell:\lambda z)\right)\\
&= \left\{\prod_{i=1}^{r} I_q^{a_i}\right\}\ z^{-1} \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\ \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\ \frac{\lambda^{n+1}\ z^{n+1}}{(q;q)_n}\\
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\}\ I_q^{a_r} \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\ \frac{\left[(-1)^n\ q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\ \frac{\lambda^{n+1}\ z^n}{(q;q)_n}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\} \frac{z^{-a_r}}{1-q}\, I_q \left[\sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\, \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \right.\\
&\quad \left. \times \frac{\lambda^{n+1}\, z^{a_r+n-1}}{(q;q)_n}\right]\\
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\} \frac{z^{-a_r}}{1-q} \left[\sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\, \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \right.\\
&\quad \left. \times \frac{\lambda^{n+1}}{(q;q)_n}\, z(1-q) \sum_{k=0}^{\infty} (zq^k)^{a_r+n-1}\, q^k\right]\\
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\} \frac{z^{-a_r}}{1-q} \left[\sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\, \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}} \right.\\
&\quad \left. \times \frac{\lambda^{n+1}\, z\, (1-q)\, z^{a_r+n-1}}{(q;q)_n} \sum_{k=0}^{\infty} q^{k(a_r+n)}\right]\\
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\} \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_{n+1}}{(b_1,b_2,\ldots,b_s;q)_n}\, \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\\
&\quad \times \frac{\lambda^{n+1}}{(q;q)_n}\, \frac{z^n}{1-q^{a_r+n}}\\
&= \left\{\prod_{i=1}^{r-1} I_q^{a_i}\right\} \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{r-1};q)_{n+1}\, (a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\, \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\\
&\quad \times \frac{\lambda^{n+1}\, z^n}{(q;q)_n}.
\end{aligned}
$$

Applying in this manner the operator $I_q^{a_i}$, for $i = 1, 2, \ldots, r-1$, one ultimately obtains

$$
\begin{aligned}
{}_{(a,r)}\mathcal{E}_{(b,s)}^{(c:\ell,p)}\left(\, {}_r\Psi_s^p(\ell : \lambda z)\right) \;=\; &\sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{r-1};q)_n\, (a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\\
&\times \frac{\left[(-1)^n\, q^{n(n-1)/2}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}}\, \frac{\lambda^{n+1}\, z^n}{(q;q)_n}
\end{aligned}
$$

which is (5.31). □

5.3 Special cases

5.3.1 The q-ℓ-Ψ exponential function

In (5.1), if the parameter $a_1 = 0$, and the parameters $a_2, a_3, \ldots, a_r$, all b_j's and $c_2, c_3, \ldots, c_p$ are absent then it takes the form

$$
{}_1\Psi_0^1\left[\begin{array}{lll} 0; & & q;\;\; z \\ -; & (c_1:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(c_1;q)_n^{\ell n}\,(q;q)_n}, \quad (\Re(\ell) \geq 0). \tag{5.32}
$$

Also by taking the parameter values $a_1 = b_1 = 0$, and treating a_i, b_j, c_k to be absent for $i = 2, 3, \ldots, r$, $j = 2, 3, \ldots, s$, $k = 2, 3, \ldots, p$ and z is replaced by $-z$ then (5.1) reduces to

$$
{}_1\Psi_1^1\left[\begin{array}{lll} 0; & & q;\;\; -z \\ 0; & (c_1:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\,z^n}{(c_1;q)_n^{\ell n}\,(q;q)_n}, \quad (\Re(\ell) \geq 0). \tag{5.33}
$$

These two functions (5.32) and (5.33) enable one to define extensions of q-exponential functions (1.12) and (1.13). The function (5.32) is taken up here whereas the function (5.33) will be studied in Subsection 5.3.4. The function (5.32) characterizes the q-exponential function; in fact for $c_1 \equiv q^{c_1} = q$, it would reduce to

$$
{}_1\Psi_0^1\left[\begin{array}{lll} 0; & & q;\;\; z \\ -; & (1:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^{\ell n+1}}. \tag{5.34}
$$

This function is termed here as q-ℓ-Ψ exponential function which is denoted and defined as follows.

Definition 5.11. The q-ℓ-Ψ exponential function is denoted and defined by

$$
e_{\Psi}^{\ell}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^{\ell n+1}}, \tag{5.35}
$$

for $\Re(\ell) \geq 0$ and $|z| < \left|\sqrt{\frac{1+q}{1-q}}\;\Gamma_{q^2}\left(\frac{1}{2}\right)\right|^{\Re(\ell)}$.

Remark 5.12. Obviously, $e_{\Psi}^{0}(z;q) = e_q(z)$, $|z| < 1$ and $e_{\Psi}^{\ell}(0;q) = 1$.

Eventually, when the parameters a_1, a_2, b and c_2 are absent, the difference equation obtained in Theorem 3.8 would reduce to the form

$$
\left(\{{}_{\ell}\Delta_1^{\theta_q}\}\theta_q - z\right) w = 0 \tag{5.36}
$$

which is satisfied by $w = e^{\ell}_{\Psi}(z;q)$. This can be verified as follows.

$$\begin{aligned}
\left[\{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q\right] w &= \{{}_{\ell}\Delta_1^{\theta_q}\} \sum_{n=1}^{\infty} \frac{1}{(q;q)_n^{\ell n+1}}\ (z^n - z^n q^n) \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell}\ (\theta_q)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell}\ (1-q^n)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell}\ z^n}{(q;q)_{n-1}^{\ell n}\ (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{z^n}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= z\ w.
\end{aligned}$$

Note 5.13. *The case* $\ell = 0$ *in (5.36) yields the equation*

$$(\theta_q - z)\ w = 0,$$

where

$$w = e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \quad |z| < 1$$

is the q*-exponential function (1.12).*

In order to derive eigen function property for the q-ℓ-Ψ exponential function, the following differential operator will be used.

Definition 5.14. Let $f(z;q) = \sum_{n=0}^{\infty} a_{n,q} z^n$, $|z| < R, R > 0,\ p \in \mathbb{N} \cup \{0\}$. Define an operator

$$_p\Omega_{\alpha}^{\mathbb{D}_q^z} = \begin{cases} \sum_{n=1}^{\infty} a_{n,q}\ (\alpha;q)_{n-1}^{p}\ ((1-q)\ \mathbb{D}_q^z)^{pn}\ z^n, & \text{if} \quad p \in \mathbb{N} \\ f(z;q), & \text{if} \quad p = 0 \end{cases}, \tag{5.37}$$

where $\mathbb{D}_q^z$ is the q-analogue of hyper-Bessel differential operators (2.8) given as

$$(\mathbb{D}_q^z)^n = \underbrace{D_q z D_q \ldots D_q z D_q}_{n \text{ derivatives}},$$

in which D_q is a usual q-differential operator:

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}. \tag{5.38}$$

With these operators, the eigen function property is obtained in

Theorem 5.15. *The q-ℓ-Ψ exponential function is an eigen function of the operator*

$${}_\ell\mathcal{D}_M^{(z;q)} = \left(\left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \; \theta_q \right), \tag{5.39}$$

where ${}_\ell\Omega_1^{\mathbb{D}_q^z}$ is as defined in (5.37). That is,

$${}_\ell\mathcal{D}_M^{(z;q)} \left(e_\Psi^\ell(\lambda z; q) \right) = \lambda \; e_\Psi^\ell(\lambda z; q), \;\; \lambda \in \mathbb{C}. \tag{5.40}$$

Proof. The left hand side of (5.40) is

$$\begin{aligned} {}_\ell\mathcal{D}_M^{(z;q)} \left(e_\Psi^\ell(\lambda z; q) \right) &= \left(\left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \; \theta_q \right) \left[\sum_{n=0}^{\infty} \frac{\lambda^n \; z^n}{(q;q)_n^{\ell n+1}} \right] \\ &= \left(\left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \right) \left[\sum_{n=1}^{\infty} \frac{\lambda^n \; z^n}{(q;q)_n^{\ell n} \; (q;q)_{n-1}} \right] \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n}{(q;q)_n^{\ell n} \; (q;q)_{n-1}} (q;q)_{n-1}^\ell \; \left((1-q)\mathbb{D}_q^z\right)^{\ell n} \; z^n. \end{aligned} \tag{5.41}$$

Now it can be easily verified that for $n = 1$,

$$\begin{aligned} (1-q)^\ell \left(\mathbb{D}_q^z\right)^\ell \; z &= (1-q)^\ell \; \underbrace{D_q z D_q \ldots D_q z D_q}_{\ell \text{ derivatives}} \; z \\ &= (1-q)^\ell \; \underbrace{D_q z D_q \ldots D_q z D_q}_{\ell-1 \text{ derivatives}} \; z \; \frac{z - zq}{z(1-q)} \\ &= \ldots = (1-q)^\ell. \end{aligned}$$

For $n = 2$,

$$\begin{aligned} (1-q)^{2\ell} \left(\mathbb{D}_q^z\right)^{2\ell} \; z^2 &= (1-q)^{2\ell} \; \underbrace{D_q z D_q \ldots D_q z D_q}_{2\ell \text{ derivatives}} \; z^2 \\ &= (1-q)^{2\ell} \; \underbrace{D_q z D_q \ldots D_q z D_q}_{2\ell-1 \text{ derivatives}} \; z \; \frac{z^2 - z^2 q^2}{z(1-q)} \\ &= (1-q)^{2\ell} \; \frac{(1-q^2)}{(1-q)} \underbrace{\{D_q z D_q \ldots D_q z D_q\}}_{2\ell-1 \text{ derivatives}} \; z^2 \end{aligned}$$

$$
\begin{aligned}
&= \ldots = (1-q)^{2\ell} \frac{(1-q^2)^{2\ell}}{(1-q)^{2\ell}} z \\
&= (1-q^2)^{2\ell} z.
\end{aligned}
$$

In general for $n \in \mathbb{N}$,

$$
(1-q)^{\ell n} \left(\mathbb{D}_q^z\right)^{\ell n} z^n = (1-q^n)^{\ell n} z^{n-1}. \tag{5.42}
$$

Hence using (5.42) in (5.41), one gets

$$
\begin{aligned}
{}_\ell\mathcal{D}_M^{(z;q)}\left(e_\Psi^\ell(\lambda z;q)\right) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} (q;q)_{n-1}^{\ell}\ (1-q^n)^{\ell n}\ z^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n\ z^{n-1}}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{n+1}\ z^n}{(q;q)_n^{\ell n+1}} \\
&= \lambda\ e_\Psi^\ell(\lambda z;q).
\end{aligned}
$$

□

Note 5.16. *If* $f(z;q) = \sum_{n=0}^{\infty} a_{n,q}\ z^n$ *and* $g(z;q) = \sum_{n=0}^{\infty} b_{n,q}\ z^n$, $|z| < R$ *then for* $\alpha, \beta \in \mathbb{R}$ *and* $p \in \mathbb{N} \cup \{0\}$,

$$
{}_p\mathcal{D}_M^{(z;q)}\left(\alpha\ f(z;q) + \beta\ g(z;q)\right) = \alpha\ {}_p\mathcal{D}_M^{(z;q)}(f(z;q)) + \beta\ {}_p\mathcal{D}_M^{(z;q)}(g(z;q)). \tag{5.43}
$$

It is interesting to see that in view of Definition 5.11,

$$
\begin{aligned}
e_\Psi^\ell(iz;q) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{((q;q)_n)^{\ell n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(i)^{2n}\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}} + \sum_{n=0}^{\infty} \frac{(i)^{2n+1}\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}} + i \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}.
\end{aligned} \tag{5.44}
$$

These infinite series are resembling with those of q-cosine and q-sine series. They are reconsidered in next Subsection.

5.3.2 The q-ℓ-Ψ trigonometric functions

The first and second series on the right hand side of (5.44) extend the q-cosine and q-sine functions [23, Ex. 1.14, p.23] respectively which are denoted here by $\cos_\Psi^\ell(z;q)$ and $\sin_\Psi^\ell(z;q)$. In fact,

$$\Re(e_\Psi^\ell(iz;q))) = \Re\left({}_1\Psi_0^1\left[\begin{array}{ccc} 0; & q; & iz \\ -; & (1:\ell); & \end{array}\right]\right) := \cos_\Psi^\ell(z;q), \tag{5.45}$$

and

$$\Im(e_\Psi^\ell(iz;q))) = \Im\left({}_1\Psi_0^1\left[\begin{array}{ccc} 0; & q; & iz \\ -; & (1:\ell); & \end{array}\right]\right) := \sin_\Psi^\ell(z;q), \tag{5.46}$$

which imply that

$$e_\Psi^\ell(iz;q) = \cos_\Psi^\ell(z;q) + i\sin_\Psi^\ell(z;q). \tag{5.47}$$

Remark 5.17. It is noteworthy that $\cos_\Psi^0(z;q) = \cos_q(z)$, and $\sin_\Psi^0(z) = \sin_q(z)$.

Further,

$$\begin{aligned}
\frac{1}{2}\left[e_\Psi^\ell(iz;q) + e_\Psi^\ell(-iz;q)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{(q;q)_n^{\ell n+1}} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2}\Bigg[1 + \frac{iz}{((q;q)_1)^{\ell+1}} + \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \dots \\
&\qquad +1 + \frac{-iz}{(q;q)_1^{\ell+1}} + \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \dots\Bigg] \\
&= \frac{1}{2}\left[2\left(1 - \frac{z^2}{(q;q)_2^{2\ell+1}} + \frac{z^4}{(q;q)_4^{4\ell+1}} - \dots\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\, z^{2n}}{(q;q)_{2n}^{2\ell n+1}} \\
&= \cos_\Psi^\ell(z;q)
\end{aligned}$$

and likewise,

$$\begin{aligned}
\frac{1}{2i}\left[e_\Psi^\ell(iz;q) - e_\Psi^\ell(-iz;q)\right] &= \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{(q;q)_n^{\ell n+1}} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2i}\Bigg[1 + \frac{iz}{((q;q)_1)^{\ell+1}} + \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \dots \\
&\qquad -1 - \frac{-iz}{(q;q)_1^{\ell+1}} - \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \dots\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2i}\left[2i\left(\frac{z}{(q;q)_1^{\ell+1}} - \frac{z^3}{(q;q)_3^{3\ell+1}} + \dots\right)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}} \\
&= \sin_\Psi^\ell(z;q).
\end{aligned}
$$

Hence from Remark 5.12,

$$
\begin{aligned}
\cos_\Psi^\ell(0;q) &= \frac{1}{2}\left[e_\Psi^\ell(0;q) + e_\Psi^\ell(0;q)\right] = 1, \\
\sin_\Psi^\ell(0;q) &= \frac{1}{2i}\left[e_\Psi^\ell(0;q) - e_\Psi^\ell(0;q)\right] = 0.
\end{aligned}
$$

Remark 5.18. The operator (5.39) yields:

1. ${}_\ell\mathcal{D}_M^{(z;q)}(\cos_\Psi^\ell(z;q)) = -\sin_\Psi^\ell(z;q),$
2. ${}_\ell\mathcal{D}_M^{(z;q)}(\sin_\Psi^\ell(z;q)) = \cos_\Psi^\ell(z;q).$

Just as the functions $\sin z$ and $\cos z$ are solutions of the equation $\frac{d^2y}{dz^2}+y=0$, these generalized functions are also the solutions of a differential equation. This is shown in

Theorem 5.19. *The q-ℓ-Ψ cosine and sine functions are solutions of the differential equation*

$$\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 \nu + \nu = 0.$$

Proof. From Theorem 5.15,

$${}_\ell\mathcal{D}_M^{(z;q)}(e_\Psi^\ell(iz;q)) = i(e_\Psi^\ell(iz;q))).$$

Hence,

$$
\begin{aligned}
\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2(e_\Psi^\ell(iz;q)) &= {}_\ell\mathcal{D}_M^{(z;q)}(\ {}_\ell\mathcal{D}_M^{(z;q)}(e_\Psi^\ell(iz;q))) \\
&= {}_\ell\mathcal{D}_M^{(z;q)}(i(e_\Psi^\ell(iz;q))) \\
&= -e_\Psi^\ell(iz;q).
\end{aligned}
$$

Now using (5.47), this may be written as

$$\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2(\cos_\Psi^\ell(z;q) + i\sin_\Psi^\ell(z;q)) = -(\cos_\Psi^\ell(z;q) + i\sin_\Psi^\ell(z;q)).$$

By making an appeal to the property (5.43) and comparing real and imaginary parts, one finds that

$$\left({}_{\ell}\mathcal{D}_{M}^{(z;q)}\right)^{2}(\cos_{\Psi}^{\ell}(z;q))+\cos_{\Psi}^{\ell}(z;q)=0,$$

where $\nu=\cos_{\Psi}^{\ell}(z;q)$ and with $\nu=\sin_{\Psi}^{\ell}(z;q)$,

$$\left({}_{\ell}\mathcal{D}_{M}^{(z;q)}\right)^{2}(\sin_{\Psi}^{\ell}(z;q))+\sin_{\Psi}^{\ell}(z;q)=0.$$

□

5.3.3 The q-ℓ-Ψ hyperbolic functions

Again from the definition of q-ℓ-Ψ exponential function (5.35), one can observe that

$$\begin{aligned} e_{\Psi}^{\ell}(z;q) &= \sum_{n=0}^{\infty}\frac{z^{n}}{(q;q)_{n}^{\ell n+1}} \\ &= \sum_{n=0}^{\infty}\frac{z^{2n}}{(q;q)_{2n}^{2\ell n+1}}+\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}. \end{aligned} \tag{5.48}$$

If the first series (with even powers of z) on the right hand side is denoted by (cf. [14, Eq. (3.15), p.487])

$$\mathcal{E}(e_{\Psi}^{\ell}(z;q))=\mathcal{E}\left({}_{1}\Psi_{0}^{1}\left[\begin{array}{ccc}0; & q; & z\\ -; & (1:\ell); & \end{array}\right]\right)=\cosh_{\Psi}^{\ell}(z;q) \tag{5.49}$$

which is called here the hyperbolic q-ℓ-Ψ cosine function and the second series (with odd powers of z) on the right hand side is denoted by (cf. [14, Eq. (3.16), p.487])

$$\mathcal{O}(e_{\Psi}^{\ell}(z;q))=\mathcal{O}\left({}_{1}\Psi_{0}^{1}\left[\begin{array}{ccc}0; & q; & z\\ -; & (1:\ell); & \end{array}\right]\right)=\sinh_{\Psi}^{\ell}(z;q) \tag{5.50}$$

which is called the hyperbolic q-ℓ-Ψ sine function then from (5.48) one gets

$$e_{\Psi}^{\ell}(z;q)=\cosh_{\Psi}^{\ell}(z;q)+\sinh_{\Psi}^{\ell}(z;q). \tag{5.51}$$

Remark 5.20. $\cosh_{\Psi}^{0}(z;q)=\cosh_{q}(z)$, and $\sinh_{\Psi}^{0}(z;q)=\sinh_{q}(z)$.

Also,

$$\begin{aligned}
\frac{1}{2}\left[e^{\ell}_{\Psi}(z;q)+e^{\ell}_{\Psi}(-z;q)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{(q;q)^{\ell n+1}}+\sum_{n=0}^{\infty}\frac{(-z)^n}{(q;q)^{\ell n+1}}\right]\\
&= \frac{1}{2}\Bigg[1+\frac{z}{(q;q)_1^{\ell+1}}+\frac{z^2}{(q;q)_2^{2\ell+1}}+\dots\\
&\quad +1+\frac{-z}{(q;q)_1^{\ell+1}}+\frac{z^2}{(q;q)_2^{2\ell+1}}+\dots\Bigg]\\
&= \frac{1}{2}\left[2\left(1+\frac{z^2}{(q;q)_2^{2\ell+1}}+\frac{z^4}{(q;q)_4^{4\ell+1}}+\dots\right)\right]\\
&= \sum_{n=0}^{\infty}\frac{z^{2n}}{(q;q)_{2n}^{2\ell n+1}}\\
&= \cosh^{\ell}_{\Psi}(z;q).
\end{aligned} \tag{5.52}$$

Likewise,

$$\begin{aligned}
\frac{1}{2}\left[e^{\ell}_{\Psi}(z;q)-e^{\ell}_{\Psi}(-z;q)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{(q;q)^{\ell n+1}}-\sum_{n=0}^{\infty}\frac{(-z)^n}{(q;q)^{\ell n+1}}\right]\\
&= \frac{1}{2}\Bigg[1+\frac{z}{(q;q)_1^{\ell+1}}+\frac{z^2}{(q;q)_2^{2\ell+1}}+\dots\\
&\quad -1-\frac{-z}{(q;q)_1^{\ell+1}}-\frac{z^2}{(q;q)_2^{2\ell+1}}-\dots\Bigg]\\
&= \frac{1}{2}\left[2\left(\frac{z}{(q;q)_1^{\ell+1}}+\frac{z^3}{(q;q)_3^{3\ell+1}}+\dots\right)\right]\\
&= \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}\\
&= \sinh^{\ell}_{\Psi}(z;q).
\end{aligned} \tag{5.53}$$

In particular at $z=0$,

$$\begin{aligned}
\cosh^{\ell}_{\Psi}(0;q) &= \frac{1}{2}\left[e^{\ell}_{\Psi}(0;q)+e^{\ell}_{\Psi}(0;q)\right]=1,\\
\sinh^{\ell}_{\Psi}(0;q) &= \frac{1}{2}\left[e^{\ell}_{\Psi}(0;q)-e^{\ell}_{\Psi}(0;q)\right]=0.
\end{aligned}$$

The differential equation satisfied by the functions (5.49) and (5.50) is given in

Theorem 5.21. *The hyperbolic q-ℓ-Ψ cosine and sine functions are solutions of the differential equation*

$$\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 \nu - \nu = 0.$$

Proof. From (5.52), (5.43) and (5.40),

$$\begin{aligned}
&\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 (\cosh_\Psi^\ell(z;q)) - \cosh_\Psi^\ell(z;q)\\
&= \left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 \left(\frac{e_\Psi^\ell(z;q)+e_\Psi^\ell(-z;q)}{2}\right) - \left(\frac{e_\Psi^\ell(z;q)+e_\Psi^\ell(-z;q)}{2}\right)\\
&= \frac{1}{2}\left[e_\Psi^\ell(z;q)+e_\Psi^\ell(-z;q)-e_\Psi^\ell(z;q)-e_\Psi^\ell(-z;q)\right]\\
&= 0.
\end{aligned}$$

Similarly in view of (5.53), (5.43) and (5.40),

$$\begin{aligned}
&\left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 \sinh_\Psi^\ell(z;q) - \sinh_\Psi^\ell(z;q)\\
&= \left({}_\ell\mathcal{D}_M^{(z;q)}\right)^2 \left(\frac{e_\Psi^\ell(z;q)-e_\Psi^\ell(-z;q)}{2}\right) - \left(\frac{e_\Psi^\ell(z;q)-e_\Psi^\ell(-z;q)}{2}\right)\\
&= \frac{1}{2}\left[e_\Psi^\ell(z;q)-e_\Psi^\ell(-z;q)-e_\Psi^\ell(z;q)+e_\Psi^\ell(-z;q)\right]\\
&= 0.
\end{aligned}$$

Then $\nu = \cosh_\Psi^\ell(z;q)$ and $\nu = \sinh_\Psi^\ell(z;q)$ are solutions. □

5.3.4 The q-ℓ-Ψ Exponential function

Here the function (5.33) given by

$${}_1\Psi_1^1\left[\begin{matrix} 0; & & q; & -z\\ 0; & (c_1:\ell); & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\ z^n}{(c_1;q)_n^{\ell n}\ (q;q)_n}, \quad (\Re(\ell)\geq 0), \tag{5.54}$$

will be studied; in particular the properties analogous to those obtained in Subsections 5.3.2 and 5.3.3 will be derived. This function characterizes the q-Exponential function $E_q(z)$. In fact, for $c_1(\equiv q^{c_1}) = 1$, the above series would reduce to

$${}_1\Psi_1^1\left[\begin{matrix} 0; & & q; & -z\\ 0; & (q:\ell); & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}}$$

which enables one to define the q-ℓ-Ψ Exponential function as follows:

Definition 5.22. The q-ℓ-Ψ Exponential function is denoted and defined by

$$E_{\Psi}^{\ell}(z;q) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}}, \tag{5.55}$$

for all $z \in \mathbb{C}$ and $\Re(\ell) \geq 0$.

Remark 5.23. Obviously, $E_{\Psi}^{0}(z;q) = E_q(z)$, and $E_{\Psi}^{\ell}(0;q) = 1$.

The reducibility of the difference equation of Theorem 5.7 to the difference equation for the function (5.55) occurs in the form

$$\left({}_{(0,1)}\Lambda_{(1,\ell)}^{(1,1,\theta_q)} + z \right) w = 0, \tag{5.56}$$

where $w = E_{\Psi}^{\ell}(z;q)$. This can be verified as follows.

$$\begin{aligned}
\left({}_{(0,1)}\Lambda_{(1,\ell)}^{(1,1,\theta_q)} \right) w &= \{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q\ (-q)\ E_q^{\ell}\left(\frac{z}{q};q\right) \\
&= \{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q \sum_{n=0}^{\infty} \frac{-q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n+1}}\ z^n \\
&= -\{{}_{\ell}\Delta_1^{\theta_q}\} \sum_{n=0}^{\infty} \frac{q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n+1}}\ (1-q^n) z^n \\
&= -\sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ (\theta_q)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= -\sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ (1-q^n)^{\ell n}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= -\sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}\ (q;q)_{n-1}^{\ell}\ z^n}{(q;q)_{n-1}^{\ell n}\ (q;q)_{n-1}} \\
&= -\sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}\ z^n}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= -z\ w.
\end{aligned}$$

Note 5.24. *The case* $\ell = 0$ *in (5.56) yields the equation*

$$(\Delta_q + z)\ w = 0,$$

where $\Delta_q := {}_{(0,1)}\Lambda_{(1,0)}^{(1,1,\theta_q)} = \{{}_{\ell}\Delta_1^{\theta_q}\}\ \theta_q\ (-q)$ *and* $w = E_q(z)$.

In order to derive eigen function property for the q-ℓ-Ψ Exponential function, one more differential operator is required which is defined below.

Definition 5.25. Let $f(z;q) = \sum_{n=0}^{\infty} a_{n,q}\ z^n$, $|z| < R, R > 0$, $p \in \mathbb{N} \cup \{0\}$. Define an operator

$$ {}_p\mathbf{D}_M^q f(z;q) = \left(\left\{ {}_p\Omega_1^{\mathbb{D}_q^z} \right\} \ \theta_q \ q \right) f\left(\frac{z}{q}; q \right), \tag{5.57}$$

where ${}_p\Omega_1^{\mathbb{D}_q^z}$ is as defined in (5.37).

This operator enables one to prove

Theorem 5.26. *If $\ell \in \mathbb{N} \cup \{0\}$ then the q-ℓ-Ψ Exponential function is an eigen function of the operator ${}_\ell\mathbf{D}_M^q$. That is,*

$$ {}_\ell\mathbf{D}_M^q \left(E_\Psi^\ell(\lambda z;q) \right) = \lambda\ E_\Psi^\ell(\lambda z;q), \ \ \lambda \in \mathbb{C}. \tag{5.58}$$

Proof. The left hand side of (5.58) is

$$\begin{aligned}
&{}_\ell\mathbf{D}_M^q \left(E_\Psi^\ell(\lambda z;q) \right) \\
&= \left(\left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \ \theta_q \right) \left[\sum_{n=0}^{\infty} \frac{\lambda^n \ \ q^{n(n-1)/2}\ q}{(q;q)_n^{\ell n+1}} \ \left(\frac{z}{q} \right)^n \right] \\
&= \left(\left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \ \theta_q \right) \left[\sum_{n=0}^{\infty} \frac{\lambda^n \ \ q^{(n-1)(n-2)/2}\ z^n}{(q;q)_n^{\ell n+1}} \right] \\
&= \left\{ {}_\ell\Omega_1^{\mathbb{D}_q^z} \right\} \left[\sum_{n=1}^{\infty} \frac{\lambda^n \ \ q^{(n-1)(n-2)/2}\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} \right] \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n \ \ q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} (q;q)_{n-1}^\ell \ ((1-q)\mathbb{D}_q^z)^{\ell n} z^n.
\end{aligned}$$

In view of (5.42), this simplifies to

$$\begin{aligned}
{}_\ell\mathbf{D}_M^q \left(E_\Psi^\ell(\lambda z;q) \right) &= \sum_{n=1}^{\infty} \frac{\lambda^n \ \ q^{(n-1)(n-2)/2}}{(q;q)_n^{\ell n}\ (q;q)_{n-1}} (q;q)_{n-1}^\ell \ (1-q^n)^{\ell n} \ z^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n \ \ q^{(n-1)(n-2)/2}\ z^{n-1}}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{n+1} \ \ q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}} \\
&= \lambda\ E_\Psi^\ell(\lambda z;q).
\end{aligned}$$

□

5.3.5 The q-ℓ-Ψ Trigonometric functions

Just as the $\mathrm{Cos}_q x$ and $\mathrm{Sin}_q x$ are defined by means of the function $E_q(x)$, the q-ℓ-Ψ Exponential function also gives rise to such functions. This is shown below.
Consider

$$\begin{aligned}
&\frac{1}{2}\left[E^{\ell}_{\Psi}(iz;q)+E^{\ell}_{\Psi}(-iz;q)\right]\\
&\quad= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (iz)^n}{(q;q)_n^{\ell n+1}}+\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (-iz)^n}{(q;q)_n^{\ell n+1}}\right]\\
&\quad= \frac{1}{2}\Bigg[1+\frac{iz}{((q;q)_1)^{\ell+1}}+\frac{q^{2(2-1)/2}(iz)^2}{(q;q)_2^{2\ell+1}}+\dots\\
&\qquad +1+\frac{-iz}{(q;q)_1^{\ell+1}}+\frac{q^{2(2-1)/2}(iz)^2}{(q;q)_2^{2\ell+1}}+\dots\Bigg]\\
&\quad= \frac{1}{2}\left[2\left(1-\frac{q^{2(2-1)/2}\ z^2}{(q;q)_2^{2\ell+1}}+\frac{q^{4(4-1)/2}\ z^4}{(q;q)_4^{4\ell+1}}-\dots\right)\right]\\
&\quad= \sum_{n=0}^{\infty}\frac{(-1)^n\ q^{2n(2n-1)/2}\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}},
\end{aligned}\tag{5.59}$$

and,

$$\begin{aligned}
&\frac{1}{2i}\left[E^{\ell}_{\Psi}(iz;q)-E^{\ell}_{\Psi}(-iz;q)\right]\\
&\quad= \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (iz)^n}{(q;q)_n^{\ell n+1}}-\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (-iz)^n}{(q;q)_n^{\ell n+1}}\right]\\
&\quad= \frac{1}{2i}\Bigg[1+\frac{iz}{((q;q)_1)^{\ell+1}}+\frac{q^{2(2-1)/2}(iz)^2}{(q;q)_2^{2\ell+1}}+\dots\\
&\qquad -1-\frac{-iz}{(q;q)_1^{\ell+1}}-\frac{q^{2(2-1)/2}(iz)^2}{(q;q)_2^{2\ell+1}}-\dots\Bigg]\\
&\quad= \frac{1}{2i}\left[2i\left(\frac{z}{(q;q)_1^{\ell+1}}-\frac{q^{3(3-1)/2}\ z^3}{(q;q)_3^{3\ell+1}}+\dots\right)\right]\\
&\quad= \sum_{n=0}^{\infty}\frac{(-1)^n\ q^{2n(2n+1)/2}\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}.
\end{aligned}\tag{5.60}$$

Then the series on the right hand sides of (5.59) and (5.60) enable one to extend the q-Cosine and q-Sine functions [23, Ex. 1.14, p.23] respectively which are denoted here by $\mathrm{Cos}^{\ell}_{\Psi}(z;q)$ and $\mathrm{Sin}^{\ell}_{\Psi}(z;q)$. In fact, for any $z\in\mathbb{C}$,

$$\frac{1}{2}\left[E^{\ell}_{\Psi}(iz;q)+E^{\ell}_{\Psi}(-iz;q)\right]:=\mathrm{Cos}^{\ell}_{\Psi}(z;q),\tag{5.61}$$

and

$$\frac{1}{2i}\left[E_{\Psi}^{\ell}(iz;q)-E_{\Psi}^{\ell}(-iz;q)\right] := \mathrm{Sin}_{\Psi}^{\ell}(z;q), \tag{5.62}$$

which imply that

$$E_{\Psi}^{\ell}(iz;q)=\mathrm{Cos}_{\Psi}^{\ell}(z;q)+i\ \mathrm{Sin}_{\Psi}^{\ell}(z;q). \tag{5.63}$$

Remark 5.27. It is noteworthy that $\mathrm{Cos}_{\Psi}^{0}(z;q)=\mathrm{Cos}_{q}(z)$, and $\mathrm{Sin}_{\Psi}^{0}(z;q)=\mathrm{Sin}_{q}(z)$.

Further, from Remark 5.23,

$$\begin{aligned}\mathrm{Cos}_{\Psi}^{\ell}(0;q) &= \frac{1}{2}\left[E_{\Psi}^{\ell}(0;q)+E_{\Psi}^{\ell}(0;q)\right]=1,\\ \mathrm{Sin}_{\Psi}^{\ell}(0;q) &= \frac{1}{2i}\left[E_{\Psi}^{\ell}(0;q)-E_{\Psi}^{\ell}(0;q)\right]=0.\end{aligned}$$

Note 5.28. *If* $f(z;q)=\sum_{n=0}^{\infty}a_{n,q}\ z^{n}$ *and* $g(z;q)=\sum_{n=0}^{\infty}b_{n,q}\ z^{n}$, $|z|<R$ *then for* $\alpha,\beta\in\mathbb{R}$ *and* $p\in\mathbb{N}\cup\{0\}$,

$${}_{p}\mathbf{D}_{M}^{q}\left(\alpha\ f(z;q)+\beta\ g(z;q)\right)=\alpha\ {}_{p}\mathbf{D}_{M}^{q}(f(z;q))+\beta\ {}_{p}\mathbf{D}_{M}^{q}(g(z;q)). \tag{5.64}$$

Remark 5.29. The operator (5.57) when acts on (5.61) and (5.62), yields

1. ${}_{\ell}\mathbf{D}_{M}^{q}(\mathrm{Cos}_{\Psi}^{\ell}(z;q)) = -\mathrm{Sin}_{\Psi}^{\ell}(z;q)$,
2. ${}_{\ell}\mathbf{D}_{M}^{q}(\mathrm{Sin}_{\Psi}^{\ell}(z;q)) = \mathrm{Cos}_{\Psi}^{\ell}(z;q)$,

respectively.

Just as the functions $\sin z$ and $\cos z$ are solutions of equation $\frac{d^2y}{dz^2}+y=0$, these generalized functions are also solutions of a differential equation. This is shown in

Theorem 5.30. *The q-ℓ-Ψ Cosine and Sine functions are solutions of the differential equation*

$$\left({}_{\ell}\mathbf{D}_{M}^{q}\right)^{2}\nu+\nu=0.$$

Proof. From Theorem 5.26,

$${}_{\ell}\mathbf{D}_{M}^{q}(E_{\Psi}^{\ell}(iz;q))=i(E_{\Psi}^{\ell}(iz;q))).$$

Hence,

$$\begin{aligned}
\left({}_{\ell}\mathbf{D}_M^q\right)^2 \left(E_\Psi^\ell(iz;q)\right) &= {}_{\ell}\mathbf{D}_M^q(\ {}_{\ell}\mathbf{D}_M^q(E_\Psi^\ell(iz;q))) \\
&= \ {}_{\ell}\mathbf{D}_M^q(i(E_\Psi^\ell(iz;q))) \\
&= -E_\Psi^\ell(iz;q).
\end{aligned}$$

Now using (5.63), this may be written as

$$\left({}_{\ell}\mathbf{D}_M^q\right)^2 (\mathrm{Cos}_\Psi^\ell(z;q) + i\mathrm{Sin}_\Psi^\ell(z;q)) = -(\mathrm{Cos}_\Psi^\ell(z;q) + i\mathrm{Sin}_\Psi^\ell(z;q)).$$

By making an appeal to the property (5.64) and comparing real and imaginary parts, one finds that

$$\left({}_{\ell}\mathbf{D}_M^q\right)^2 \ \mathrm{Cos}_\Psi^\ell(z;q) + \mathrm{Cos}_\Psi^\ell(z;q) = 0,$$

and

$$\left({}_{\ell}\mathbf{D}_M^q\right)^2 \ \mathrm{Sin}_\Psi^\ell(z;q) + \mathrm{Sin}_\Psi^\ell(z;q) = 0.$$

Then $\nu = \mathrm{Cos}_\Psi^\ell(z;q)$ and with $\nu = \mathrm{Sin}_\Psi^\ell(z;q)$ are solutions. □

5.3.6 The q-ℓ-Ψ Hyperbolic functions

From the Definition 5.22,

$$\begin{aligned}
&\frac{1}{2}\left[E_\Psi^\ell(z;q) + E_\Psi^\ell(-z;q)\right] \\
&\quad= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}} + \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\ (-z)^n}{(q;q)_n^{\ell n+1}}\right] \\
&\quad= \frac{1}{2}\left[1 + \frac{z}{(q;q)_1^{\ell+1}} + \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} + \ldots - 1 - \frac{z}{(q;q)_1^{\ell+1}} - \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} - \ldots\right] \\
&\quad= \frac{1}{2}\left[2\left(1 + \frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}} + \frac{q^{4(4-1)}\ z^4}{(q;q)_4^{4\ell+1}} + \ldots\right)\right] \\
&\quad= \sum_{n=0}^{\infty} \frac{q^{2n(2n-1)}\ z^{2n}}{(q;q)_{2n}^{2\ell n+1}},
\end{aligned} \tag{5.65}$$

and

$$\frac{1}{2}\left[E_\Psi^\ell(z;q) - E_\Psi^\ell(-z;q)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ z^n}{(q;q)_n^{\ell n+1}} - \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\ (-z)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2}\left[1+\frac{z}{(q;q)_1^{\ell+1}}+\frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}}+\ldots-1-\frac{-z}{(q;q)_1^{\ell+1}}-\frac{q^{2(2-1)}\ z^2}{(q;q)_2^{2\ell+1}}-\ldots\right] \\
&= \frac{1}{2}\left[2\left(\frac{z}{(q;q)_1^{\ell+1}}+\frac{q^{3(3-1)}\ z^3}{(q;q)_3^{3\ell+1}}+\frac{q^{5(5-1)}\ z^5}{(q;q)_5^{5\ell+1}}+\ldots\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{q^{2n(2n+1)}\ z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}.
\end{aligned}
\tag{5.66}
$$

From first and second series on the right hand sides of (5.65) and (5.66), the hyperbolic q-Cosine and q-Sine functions admit extensions which are denoted here by $\mathrm{Cosh}_{\Psi}^{\ell}(z;q)$ and $\mathrm{Sinh}_{\Psi}^{\ell}(z;q)$ respectively. In fact, for any $z \in \mathbb{C}$,

$$\frac{1}{2}\left[E_{\Psi}^{\ell}(z;q)+E_{\Psi}^{\ell}(-z;q)\right] := \mathrm{Cosh}_{\Psi}^{\ell}(z;q), \tag{5.67}$$

and

$$\frac{1}{2}\left[E_{\Psi}^{\ell}(z;q)-E_{\Psi}^{\ell}(-z;q)\right] := \mathrm{Sinh}_{\Psi}^{\ell}(z;q), \tag{5.68}$$

which together imply that

$$E_{\Psi}^{\ell}(z;q) = \mathrm{Cosh}_{\Psi}^{\ell}(z;q)+\mathrm{Sinh}_{\Psi}^{\ell}(z;q). \tag{5.69}$$

Remark 5.31. $\mathrm{Cosh}_{\Psi}^{0}(z;q) = \mathrm{Cosh}_{q}(z)$, and $\mathrm{Sinh}_{\Psi}^{0}(z;q) = \mathrm{Sinh}_{q}(z)$.

In particular, from Remark 5.23,

$$
\begin{aligned}
\mathrm{Cosh}_{\Psi}^{\ell}(0;q) &= \frac{1}{2}\left[E_{\Psi}^{\ell}(0;q)+E_{\Psi}^{\ell}(0;q)\right] = 1, \\
\mathrm{Sinh}_{\Psi}^{\ell}(0;q) &= \frac{1}{2}\left[E_{\Psi}^{\ell}(0;q)-E_{\Psi}^{\ell}(0;q)\right] = 0.
\end{aligned}
$$

The difference equation satisfied by these functions is given in

Theorem 5.32. *The hyperbolic q-ℓ-Ψ Cosine and Sine functions are solutions of the differential equation*

$$\left({}_{\ell}\mathbf{D}_{M}^{q}\right)^{2}\nu-\nu=0.$$

Proof. It is seen from (5.67), (5.64) and (5.58) that

$$\left({}_{\ell}\mathbf{D}_{M}^{q}\right)^{2}\mathrm{Cosh}_{\Psi}^{\ell}(z;q)-\mathrm{Cosh}_{\Psi}^{\ell}(z;q)$$

$$
\begin{aligned}
&= ({}_\ell\mathbf{D}_M^q)^2 \left(\frac{E_\Psi^\ell(z;q)+E_\Psi^\ell(-z;q)}{2}\right) - \left(\frac{E_\Psi^\ell(z;q)+E_\Psi^\ell(-z;q)}{2}\right)\\
&= \frac{1}{2}\left[E_\Psi^\ell(z;q)+E_\Psi^\ell(-z;q)-E_\Psi^\ell(z;q)-E_\Psi^\ell(-z;q)\right]\\
&= 0.
\end{aligned}
$$

Also from (5.68), (5.64) and (5.58),

$$
\begin{aligned}
&({}_\ell\mathbf{D}_M^q)^2 \operatorname{Sinh}_\Psi^\ell(z;q) - \operatorname{Sinh}_\Psi^\ell(z;q)\\
&= ({}_\ell\mathbf{D}_M^q)^2 \left(\frac{E_\Psi^\ell(-z;q)-E_\Psi^\ell(z;q)}{2}\right) - \left(\frac{E_\Psi^\ell(-z;q)-E_\Psi^\ell(z;q)}{2}\right)\\
&= \frac{1}{2}\left[E_\Psi^\ell(-z;q)-E_\Psi^\ell(z;q)-E_\Psi^\ell(-z;q)+E_\Psi^\ell(z;q)\right]\\
&= 0.
\end{aligned}
$$

□

5.3.7 Series transformation

In this subsection, a series transformation formula is derived involving a particular case: $r = 1, s = 0$ and $p = 1$ of the generalized q-ℓ- function (5.1). By taking $a_1 = a$ and $c_1 = c$, this reduced function takes the form:

$$
{}_1\Psi_0^1\left[\begin{matrix} a; & & q; & z \\ -; & (c:\ell); & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n\ z^n}{(c;q)_n^{\ell n}\ (q;q)_n} := \Psi_a^\ell(c;z). \tag{5.70}
$$

For this function, the series transformation is established here by means of the following lemma.

Lemma 5.33. *If* $0 < q < 1, a \in \mathbb{C}, c \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$ *and* $\ell \in \mathbb{N}\cup\{0\}$ *then there holds the series transformation formula:*

$$
\Psi_a^\ell(c;z) = (1-a)\Psi_{aq}^\ell(c;z) + a\Psi_a^\ell(c;zq). \tag{5.71}
$$

Proof. From (5.70),

$$
\begin{aligned}
&\Psi_a^\ell(c;z) - \Psi_{aq}^\ell(c;z)\\
&= \sum_{n=0}^{\infty} \frac{[(a;q)_n - (aq;q)_n]\ z^n}{(c:q)_n^{\ell n}\ (q;q)_n}\\
&= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(c:q)_n^{\ell n}\ (q;q)_n}\ (1-a-1+aq^n)\ z^n
\end{aligned}
$$

$$
\begin{aligned}
&= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}\ z^n}{(c:q)_n^{\ell n}\ (q;q)_{n-1}} \\
&= -a\ z \sum_{n=0}^{\infty} \frac{(aq;q)_n\ z^n}{(c:q)_{n+1}^{\ell n+\ell}\ (q;q)_n} \\
&= -a\ z \sum_{n=0}^{\infty} \frac{(aq;q)_n\ z^n}{[(1-c)\ (c:q)_n]^{\ell n+\ell}\ (q;q)_n} \\
&= \frac{-a\ z}{(1-c)^\ell} \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(cq:q)_n^\ell\ (cq:q)_n^{\ell n}\ (q;q)_n} \frac{z^n}{(1-c)^{\ell n}} \\
&= \frac{-a\ z}{(1-c)^\ell}\ {}_2\Psi_{\ell+1}^1 \left[\begin{matrix} aq, 0; & q; & \frac{z}{(1-c)^\ell} \\ cq, cq, \ldots, cq, 0; & (cq:\ell); & \end{matrix} \right]. \qquad (5.72)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\Psi_a^\ell(c;z) - \Psi_a^\ell(c;zq) \\
&= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(c;q)_n^{\ell n}\ (q;q)_n}(z^n - z^n q^n) \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}\ z^{n+1}}{(c;q)_{n+1}^{\ell n+\ell}\ (q;q)_n} \\
&= \frac{(1-a)\ z}{(1-c)^\ell} \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(cq:q)_n^\ell\ (cq:q)_n^{\ell n}\ (q;q)_n} \frac{z^n}{(1-c)^{\ell n}} \\
&= \frac{(1-a)\ z}{(1-c)^\ell}\ {}_2\Psi_{\ell+1}^1 \left[\begin{matrix} aq, 0; & q; & \frac{z}{(1-c)^\ell} \\ cq, cq, \ldots, cq, 0; & (cq:\ell); & \end{matrix} \right]. \qquad (5.73)
\end{aligned}
$$

Hence from (5.72) and (5.73),

$$
(1-a)\left[\Psi_a^\ell(c;z) - \Psi_{aq}^\ell(c;z)\right] = -a\left[\Psi_a^\ell(c;z) - \Psi_a^\ell(c;zq)\right],
$$

which simplifies to

$$
\Psi_a^\ell(c;z) = (1-a)\ \Psi_{aq}^\ell(c;z) + a\ \Psi_a^\ell(c;zq).
$$

□

By iterating the identity (5.71) and then taking limit as $n \to \infty$, the desired series transformation is obtained. This is given as

Theorem 5.34. *If $|a| < 1$ and (5.71) holds then*

$$\sum_{n=0}^{\infty} \Psi^{\ell}_{aq}(c; zq^n)\ a^n = (1-a)^{-1}\ \Psi^{\ell}_{a}(c; z). \tag{5.74}$$

Proof. Upon iterating (5.71) by replacing z by zq one finds that

$$\Psi^{\ell}_{a}(c; zq) = (1-a)\ \Psi^{\ell}_{aq}(c; zq) + a\ \Psi^{\ell}_{a}(c; zq^2). \tag{5.75}$$

Substituting (5.75) in (5.71), gives

$$\Psi^{\ell}_{a}(c; z) = (1-a)\ \Psi^{\ell}_{aq}(c; z) + a(1-a)\ \Psi^{\ell}_{aq}(c; zq) + a^2\ \Psi^{\ell}_{a}(c; zq^2). \tag{5.76}$$

Again iterating (5.75) by replacing z by zq, one gets

$$\Psi^{\ell}_{a}(c; zq^2) = (1-a)\ \Psi^{\ell}_{aq}(c; zq^2) + a\ \Psi^{\ell}_{a}(c; zq^3). \tag{5.77}$$

Substituting (5.77) in (5.76), yields

$$\begin{aligned}\Psi^{\ell}_{a}(c; z) &= (1-a)\ \Psi^{\ell}_{aq}(c; z) + a(1-a)\ \Psi^{\ell}_{aq}(c; zq) + a^2\ (1-a)\ \Psi^{\ell}_{aq}(c; zq^2)\\ &\quad + a^3\ \Psi^{\ell}_{a}(c; zq^3).\end{aligned}$$

In this manner iterating (5.71) total n times by replacing z by zq one finally arrives at

$$\Psi^{\ell}_{a}(c; z) = (1-a)\sum_{k=0}^{n} a^k\ \Psi^{\ell}_{aq}(c; zq^k) + a^{n+1}\ \Psi^{\ell}_{a}(c; zq^{n+1}).$$

In this letting $n \to \infty$, it takes the form:

$$\begin{aligned}\Psi^{\ell}_{a}(c; z) &= (1-a)\sum_{k=0}^{\infty} a^k\ \Psi^{\ell}_{aq}(c; zq^k) + \lim_{n\to\infty} a^{n+1}\ \Psi^{\ell}_{a}(c; zq^{n+1})\\ &= (1-a)\sum_{k=0}^{\infty} a^k\ \Psi^{\ell}_{aq}(c; zq^k)\end{aligned}$$

as $\Psi^{\ell}_{a}(c; 0) = 1$ and $|a| < 1$. □

Remark 5.35. If $\ell = 0$ then (5.74) reduces to the transformation formula:

$$\sum_{n=0}^{\infty} {}_1\phi_0\left[\begin{matrix} aq; & q; & zq^n \\ -; & & \end{matrix}\right] a^n = (1-a)^{-1}\ {}_1\phi_0\left[\begin{matrix} a; & q; & z \\ -; & & \end{matrix}\right],\ |z| < 1,\ |a| < 1.$$

In this, using the formula [23, Eq. (1.3.12), p.8]:

$$
{}_1\phi_0 \left[\begin{array}{ccc} \alpha; & q; & z \\ -; & & \end{array} \right] = \frac{(\alpha z; q)_\infty}{(z; q)_\infty},
$$

the following particular q-sum is obtained. In fact,

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(azq^{n+1}; q)_\infty}{(zq^n; q)_\infty} \, a^n = (1-a)^{-1} \, \frac{(az; q)_\infty}{(z; q)_\infty} \\
\Rightarrow \quad & \sum_{n=0}^{\infty} \frac{(azq^{n+1}; q)_\infty}{(az; q)_\infty} \, \frac{(z; q)_\infty}{(zq^n; q)_\infty} \, a^n = \frac{1}{1-a} \\
\Rightarrow \quad & \sum_{n=0}^{\infty} \frac{(z; q)_n}{(az; q)_{n+1}} \, a^n = \frac{1}{1-a} \\
\Rightarrow \quad & \frac{1}{1-az} \sum_{n=0}^{\infty} \frac{(z; q)_n}{(azq; q)_n} \, a^n = \frac{1}{1-a} \\
\Rightarrow \quad & {}_2\phi_1 \left[\begin{array}{cccc} z, & q; & q; & a \\ & azq; & & \end{array} \right] = \frac{1-az}{1-a}.
\end{aligned}
$$

This can also be verified from q-Gauss summation formula (1.14) by substituting $a = z, \ b = q, \ c = azq$, and $z = a$.

5.3.8 q-ℓ-analogue of Maclaurin's theorem

In [31], F. H. Jackson introduced the following q-counterpart of Taylor series:

$$
f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q; q)_n} \, D_q^n f(a)[x-a]_n, \tag{5.78}
$$

where D_q is the usual q-differential operator as defined in (5.38) and for $n \geq 1$,

$$
[x-a]_n = (x-a)(x-aq)\cdots(x-aq^{n-1})
$$

and $[x-a]_0 = 1$.
The q-Taylor series (5.78) leads to

Theorem 5.36. *For $z \in \mathbb{C}$,*

$$
{}_r\Psi_s^p(\ell; q; z) = \sum_{m=0}^{\infty} \left[D_q^m \left({}_r\Psi_s^p(\ell; q; z) \right) \right]_{z=0} \frac{z^m \, (1-q)^m}{(q:q)_m}. \tag{5.79}
$$

Proof. From the definition the generalized q-ℓ-Ψ function (5.1) and from (5.38),

$$
\begin{aligned}
&D_q^m\left({}_r\Psi_s^p(\ell;q;z)\right)\\
&= D_q^m\left(\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\,\frac{\left[(-1)^n\,q^{\binom{n}{2}}\right]^{\kappa}\,z^n}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}\,(q;q)_n}\right),\\
&= \sum_{n=m}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\,\frac{\left[(-1)^n\,q^{\binom{n}{2}}\right]^{\kappa}\,z^{n-m}}{(c_1,c_2,\ldots,c_p;q)_n^{\ell n}\,(q;q)_{n-m}\,(1-q)^m},
\end{aligned}
$$

so that

$$
\left[D_q^m\left({}_r\Psi_s^p(\ell;q;z)\right)\right]_{z=0} = \frac{(a_1,a_2,\ldots,a_r;q)_m}{(b_1,b_2,\ldots,b_s;q)_m}\,\frac{\left[(-1)^m\,q^{\binom{m}{2}}\right]^{\kappa}}{(c_1,c_2,\ldots,c_p;q)_m^{\ell m}\,(1-q)^m}.
$$

Hence,

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\left[D_q^m\left({}_r\Psi_s^p(\ell;q;z)\right)\right]_{z=0}\frac{z^m\,(1-q)^m}{(q:q)_m}\\
&= \sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_s;q)_k}\,\frac{\left[(-1)^k\,q^{\binom{k}{2}}\right]^{\kappa}\,z^k}{(c_1,c_2,\ldots,c_p;q)_k^{\ell k}\,(q;q)_k}\\
&= {}_r\Psi_s^p(\ell;q;z)
\end{aligned}
$$

□

Remark 5.37. For $\ell=0$, (5.79) reduces to

$$
{}_r\phi_s[z] = \sum_{m=0}^{\infty}\left[D_q^m\left({}_r\phi_s[z]\right)\right]_{z=0}\frac{z^m\,(1-q)^m}{(q:q)_m}.
$$