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Some new class of special functions suggested by the confluent hypergeometric function

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Abstract In the present work, we introduce the function representing a rapidly convergent power series which extends the well-known confluent hypergeometric function ${}_1F_1[z]$ as well as the integral function $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!^n}$ considered by Sikkema (Differential operators and equations, P. Noordhoff N. V., Djakarta, 1953). We introduce the corresponding differential operators and obtain infinite order differential equations, for which these new special functions are the eigen functions. First we establish some properties, as the order zero of these entire (integral) functions, integral representations, differential equations involving a kind of hyper-Bessel type operators of infinite order. Then we emphasize on the special cases, especially the corresponding analogues of the exponential, circular and hyperbolic functions, called here as ℓ-H exponential function, ℓ-H circular and ℓ-H hyperbolic functions. At the end, the graphs of these functions are plotted using the Maple software.

Keywords Hypergeometric function · Entire functions · Differential equation · Eigen function

Mathematics Subject Classification 33B10 · 33C20 · 34A35

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1 Introduction

The hypergeometric function [4,11] is denoted and defined by

$$ {}_2F_1\left[\begin{matrix} a, b; \ z \\ c; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1} $$

where $c \neq 0, -1, -2, \ldots$ and $|z| < 1$.

When one of the numerator parameters is allowed to tend to *infinity*, then this function reduces to the lower order hypergeometric function which is termed as the confluent hypergeometric function. Thus, for each z interior to the disk $|z| = 1$,

$$ \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n\, n!} \frac{(b)_n}{b^n}\, z^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n\, n!}\, z^n = {}_1F_1\left[\begin{matrix} a; \ z \\ c; \end{matrix}\right]. \tag{2} $$

The function ${}_2F_1$ is analytic in the unit disk, but the confluent function ${}_1F_1$ has the series convergent for all finite values of $|z|$, and is an entire function.

In 1953, Sikkema [12, p. 6] considered the entire (integral) function

$$ f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!^n} \tag{3} $$

of order zero.

We introduce here a new class of special functions suggested by the power series representations (1), (2) and (3), as follows.

Definition 1 For $z \in \mathbf{C}$, define the ℓ-Hypergeometric function (ℓ-H function) as

$$ H\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\,(c)_n^{\ell n}} \frac{z^n}{n!}, \tag{4} $$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, $a, \ell \in \mathbf{C}$ with $\Re(\ell) \geq 0$, and $b, c \in \mathbf{C}/\{0, -1, -2, \ldots\}$.

Remark 1 We note that

$$ H\left[\begin{matrix} a; & & z \\ b; & (c:0); & \end{matrix}\right] = {}_1F_1\left[\begin{matrix} a; \ z \\ b; \end{matrix}\right]. $$

2 Main results

2.1 Convergence

The series in (4) is convergent for all $z \in \mathbf{C}$ which is proven in the following theorem.

Theorem 1 *If* $\Re(\ell) \geq 0$ *and* $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0$*, then the* ℓ*-H function is an entire function of* z*.*

Proof Put

$$\frac{(a)_n}{(b)_n\,(c)_n^{\ell n}\,n!} = \varphi_n,$$

then using the Cauchy–Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|\varphi_n|},$$

we have

$$\begin{aligned}\frac{1}{R} &= \lim_{n\to\infty} \sup \left|\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\,(c)_n^{\ell n}\,n!}\right|^{\frac{1}{n}}\\ &= \lim_{n\to\infty} \sup \left|\frac{\Gamma(b)}{\Gamma(a)}\right|^{\frac{1}{n}} \left|\frac{\Gamma(a+n)}{\Gamma(b+n)}\right|^{\frac{1}{n}} \times \left|\frac{\Gamma(c)}{\Gamma(c+n)}\right|^{\ell} \frac{1}{\Gamma^{\frac{1}{n}}(n+1)}.\end{aligned}$$

Now applying the Stirling's asymptotic formula [4] for large n, given by

$$\Gamma(\alpha+n) \sim \sqrt{2\pi}\, e^{-(\alpha+n)}\,(\alpha+n)^{(\alpha+n-1/2)}, \tag{5}$$

with $\alpha = a, b, c, 1$ in turn, we get

$$\begin{aligned}\frac{1}{R} &\sim \left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{\ell} \lim_{n\to\infty} \sup \left|\frac{\Gamma(b)}{\Gamma(a)}\right|^{\frac{1}{n}} \left|\frac{\sqrt{2\pi}\,e^{-(a+n)}\,(a+n)^{a+n-1/2}}{\sqrt{2\pi}\,e^{-(b+n)}\,(b+n)^{b+n-1/2}}\right|^{\frac{1}{n}}\\ &\quad\times \frac{1}{\left|e^{-(c+n)}\,(c+n)^{c+n-1/2}\right|^{\ell}\,\left|\sqrt{2\pi}\,e^{-(n+1)}\,(n+1)^{n+1-1/2}\right|^{\frac{1}{n}}}\\ &= \left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{\ell} \lim_{n\to\infty} \sup \left|\frac{1}{n^{c\ell-\frac{\ell}{2}+1}} \left(\frac{e}{n}\right)^{\ell n}\right|\\ &= 0,\end{aligned}$$

provided that $\Re(\ell) \geq 0$ and $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0$. □

Remark 2 1. The series $\sum \phi_n z^n$ thus converges uniformly, in any compact subset of **C**.
2. It may be observed that our new class of functions (4) preserves the entire function property of (2).

2.2 Order of ℓ-H function

Theorem 2 *If the conditions stated in Theorem 1 hold, then the ℓ-H function is an entire function of order zero.*

Proof We use the result which states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of f [2,10] is given by

$$\rho(f) = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln |\varphi_n|^{-1}}. \tag{6}$$

Now from the definition of the ℓ-H function,

$$|\varphi_n|^{-1} = \left| \frac{\Gamma(a)}{\Gamma(b)} \frac{\Gamma(b+n)\, \Gamma^{\ell n}(c+n)\, \Gamma(n+1)}{\Gamma(a+n)\, \Gamma^{\ell n}(c)} \right|.$$

Hence,

$$\begin{aligned} \ln |\varphi_n|^{-1} &= |\ln \Gamma(a) - \ln \Gamma(b) + \ln(\Gamma(b+n)) + \ell n \ \ln(\Gamma(c+n)) \\ &\quad + \ln(\Gamma(n+1)) - \ln(\Gamma(a+n)) - \ell n \ \ln(\Gamma(c)|. \end{aligned}$$

Since

$$\ln \Gamma(r) \sim \left(r - \frac{1}{2}\right) \ln r - r + \frac{1}{2} \ln \sqrt{2\pi}$$

for large r, we have

$$\begin{aligned} \ln |\varphi_n|^{-1} &\le |\ln \Gamma(a) - \ln \Gamma(b)| \\ &\quad + \left| \left(b + n - \frac{1}{2}\right) \ln(b+n) - (b+n) + \frac{1}{2} \ln \sqrt{2\pi} \right| \\ &\quad + \left| \ell n \left[\left(c + n - \frac{1}{2}\right) \ln(c+n) - (c+n) + \frac{1}{2} \ln \sqrt{2\pi} \right] \right| \\ &\quad + \left| \left(n + 1 - \frac{1}{2}\right) \ln(n+1) - (n+1) + \frac{1}{2} \ln \sqrt{2\pi} \right| + |\ell n (\ln \Gamma(c))| \\ &\quad + \left| \left(a + n - \frac{1}{2}\right) \ln(a+n) + (a+n) - \frac{1}{2} \ln \sqrt{2\pi} \right|. \end{aligned} \tag{7}$$

But since

$$\lim_{n\to\infty} \frac{\ln |\varphi_n|^{-1}}{n \ln n}$$

is unbounded as n increases without bound, hence it follows from (6) and (7) that

$$\rho\left(H \left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix} \right] \right) = \lim_{n\to\infty} \frac{n \ln n}{\ln |\varphi_n|^{-1}} = 0.$$

□

Remark 3 It is proved that [1, Theorem1.1] "If f is entire and $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros or it is a polynomial." Hence it follows that the ℓ-H function has infinitely many zeros.

2.3 Integral representation

The integral representation occurs by routine calculations; which is obtained below.

Theorem 3 *With* $\Re(b) > \Re(a) > 0, c, b \neq 0, -1, \ldots,$ *and* $\Re\left(c\ell - \frac{\ell}{2} + 1\right) > 0,$

$$H\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right] = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} \times H\left[\begin{matrix} -; & & zt \\ -; & (c:\ell); & \end{matrix}\right] dt. \tag{8}$$

The proof follows readily in view of the technique adopted in [11, Ch.4, p.47]. The integral in (8) can be viewed also as fractional order integral of order $b-a$ with $\Re(b-a) > 0$, and makes the result expectable from this point of view (see ideas in [5], also [9, Eq.(5.1), p. 245]).

However, the differential equation of the ℓ-H function is derived with the aid of the following operator.

Definition 2 Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbf{C}$, $p \in \mathbf{N} \cup \{0\}$ and $\alpha \in \mathbf{C}$. Define

$${}_p\Delta_\alpha^\Theta(f(z)) = \begin{cases} \sum_{n=1}^{\infty} a_n (\alpha)_{n-1}^p (\Theta + \alpha - 1)^{pn} z^n, & if\ p \in \mathbf{N} \\ f(z), & if\ p = 0 \end{cases}, \tag{9}$$

where Θ is the Euler differential operator $\theta = z\frac{d}{dz}$ and

$$(\Theta + \alpha)^r = \underbrace{(\Theta + \alpha)(\Theta + \alpha)\cdots(\Theta + \alpha)}_{r\ times}$$

is a special case of the hyper-Bessel differential operators (see e.g., [5–7]).

We then have

Theorem 4 *For* $\ell \in \mathbf{N} \cup \{0\}, a, z \in \mathbf{C}$, *and* $b, c \in \mathbf{C}/\{0, -1, -2, \ldots\}$, *the function* $w = H\left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix}\right]$ *satisfies the differential equation*

$$\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta + b - 1\}\theta - z(\theta + a)\right\} w = 0, \tag{10}$$

where the operator ${}_\ell\Delta_c^\theta$ *is as defined in* (9).

Since the operator defined by (9) involves the infinite series, its applicability is subject to the convergence of the series with the general term $a_n\ f_n(a,b,c,\ell;z),\ n\geq 0$. This is proved in the following lemma.

Lemma 1 *If* $\ell\in\mathbf{N}\cup\{0\},\ w=H\left[\begin{matrix}a; & & z\\ b; & (c:\ell); & \end{matrix}\right]=\sum_{n=0}^{\infty}a_n z^n$ *and* $\left(\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right)w=\sum_{n=0}^{\infty}f_n(a,b,c,\ell;z)$, *then* $\left(\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right)$ *is applicable to the* ℓ*-H function provided that*

$$\sum_{n=0}^{\infty}a_n\ f_n(a,b,c,\ell;z)$$

is convergent (cf. [12, Definition 11, p. 20]).

Proof Let

$$w=H\left[\begin{matrix}a; & & z\\ b; & (c:\ell); & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n\,(c)_n^{\ell n}}\frac{z^n}{n!}=\sum_{n=0}^{\infty}a_n z^n.$$

Now

$$\begin{aligned}\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right\}w&=\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\left(\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n\,(c)_n^{\ell n}}\frac{z^n}{n!}\right)\\&=\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\left(\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n\,(c)_n^{\ell n}}\frac{z^n}{(n-1)!}\right)\\&=\left\{{}_\ell\Delta_c^\theta\right\}\left(\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n\,(c)_n^{\ell n}}\frac{(n+b-1)\,z^n}{(n-1)!}\right)\\&=\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_{n-1}\,(c)_n^{\ell n}}\frac{(c)_{n-1}^{\ell}\,(\theta+c-1)^{\ell n}\,z^n}{(n-1)!}.\end{aligned}$$

Since,

$$(\theta+c-1)^{\ell n}\,z^n=(n+c-1)^{\ell n}\,z^n,\quad n\in\mathbf{N},$$

we have

$$\begin{aligned}\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{\theta+b-1\}\,\theta\right\}w&=\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_{n-1}\,(c)_n^{\ell n}}\frac{(c)_{n-1}^{\ell}\,(n+c-1)^{\ell n}\,z^n}{(n-1)!}\\&=\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n}}\frac{(c)_{n-1}^{\ell}\,z^n}{(n-1)!}\\&=\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}}\frac{z^n}{(n-1)!}\end{aligned}\qquad(11)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_n\,(c)_n^{\ell n}} \frac{z^{n+1}}{n!} \tag{12}$$

$$= \sum_{n=0}^{\infty} f_n(a, b, c, \ell; z).$$

To complete the proof of the lemma it suffices to show that

$$\sum_{n=0}^{\infty} a_n\; f_n(a, b, c, \ell; z) = \sum_{n=0}^{\infty} \frac{(a)_n (a)_{n+1}}{(b)_n^2\,(c)_n^{2\ell n}} \frac{z^{n+1}}{(n!)^2}$$

is convergent.

Put

$$\zeta_n = \frac{(a)_n^2\,(a+n)\,z^{n+1}}{(b)_n^2\,(c)_n^{2\ell n}\,(n!)^2} = \frac{\Gamma^2(b)}{\Gamma^2(a)} \frac{\Gamma^2(a+n)\,(a+n)\,\Gamma^{2\ell n}(c)\,z^{n+1}}{\Gamma^2(b+n)\,\Gamma^{2\ell n}(c+n)\,\Gamma^2(n+1)}.$$

Now applying the Stirling's asymptotic formula for large n given in (5), we have

$$|\zeta_n|^{\frac{1}{n}} \sim \left|\frac{\Gamma^2(b)}{\Gamma^2(a)}\right|^{\frac{1}{n}} \frac{\left|e^{-(a+n)}(a+n)^{a+n-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}{\left|e^{-(b+n)}(b+n)^{b+n-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}$$
$$\times \frac{|a+n|^{\frac{1}{n}}\,\Gamma^{2\ell}(c)\,|z|^{1+\frac{1}{n}}}{\left|e^{-(c+n)}(c+n)^{c+n-\frac{1}{2}}\sqrt{2\pi}\right|^{2\ell}\,\left|e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}\sqrt{2\pi}\right|^{\frac{2}{n}}}.$$

Hence,

$$\lim_{n\to\infty}\sup|\zeta_n|^{\frac{1}{n}} \sim \lim_{n\to\infty}\sup\left|\frac{\Gamma(c)}{\sqrt{2\pi}}\right|^{2\ell} |z| \frac{1}{|n^{2c\ell-\ell+2}|} \left|\frac{e}{n}\right|^{2n\ell}$$
$$= 0,$$

when $\Re\,(2c\ell - \ell + 2) > 0$. □

Proof (of Theorem 4) From (12),

$$\left\{\left\{{}_\ell\Delta_c^\theta\right\}\{(\theta+b-1)\}\,\theta\right\} w = \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_n\,(c)_n^{\ell n}} \frac{z^{n+1}}{n!}$$
$$= z\sum_{n=0}^{\infty} \frac{(a)_n(a+n)}{(b)_n\,(c)_n^{\ell n}} \frac{z^n}{n!}$$
$$= z(\theta+a)w$$

which gives (10). □

Remark 4 It is noteworthy here that when $\ell = 0$, the differential equation (10) reduces to the form [4, Ch. 4.2, Eq. (2)]

$$[(\theta + b - 1)\,\theta - z(\theta + a)]\,w = 0$$

which is the differential equation satisfied by $w = {}_1F_1\left[\begin{matrix} a; \; z \\ b; \end{matrix}\right]$.

To obtain the ℓ-H function as the eigen function, we need to define yet another operator as follows.

Definition 3 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n,\ |z| < R, z \neq 0, R > 0$ and $\Re(\alpha) > 0$. Define

$$_{\alpha}\mathcal{H}_{\beta}^{(\gamma:p)}(f(z)) = \left[I_{\alpha} \left(z^{-1} \left\{ {}_{p}\Delta_{\gamma}^{\theta} \right\} \{\theta + \beta - 1\}\,\theta \right) \right] (f(z)), \tag{13}$$

where the operator ${}_p\Delta_\gamma^\theta$ is as defined in (9) and

$$I_\alpha(f(z)) = z^{-\alpha} \int_0^z t^{\alpha-1} f(t) dt. \tag{14}$$

Remark 5 Note that (14) is nothing but the fractional integral of order α with $\Re(\alpha) > 0$.

Theorem 5 *If $\ell \in \mathbf{N} \cup \{0\}$ and $\Re(a) > 0$, then the ℓ-H function is the eigen function with respect to the operator ${}_a\mathcal{H}_b^{(c:\ell)}$ defined in (13). That is,*

$$_{a}\mathcal{H}_{b}^{(c:\ell)} \left(H \left[\begin{matrix} a; & & \lambda z \\ b; & (c:\ell); & \end{matrix} \right] \right) = \lambda\ H \left[\begin{matrix} a; & & z \\ b; & (c:\ell); & \end{matrix} \right], \quad \lambda \in \mathbf{C}. \tag{15}$$

Proof The applicability of this operator to the ℓ-H function follows from the Lemma 1.

We first note that for $z \neq 0$,

$$\begin{aligned} &{}_{a}\mathcal{H}_{b}^{(c:\ell)} \left(H \left[\begin{matrix} a; & & \lambda z \\ b; & (c:\ell); & \end{matrix} \right] \right) \\ &= \left[I_a \left\{ z^{-1} \left(\left\{ {}_{\ell}\Delta_c^{\theta} \right\} \{(\theta + b - 1)\}\,\theta \sum_{n=0}^{\infty} \frac{\lambda^n\ , (a)_n}{(b)_n\ (c)_n^{\ell n}\ n!} z^n \right) \right\} \right]. \end{aligned}$$

In view of (11), we have

$$\begin{aligned}
{}_a\mathcal{H}_b^{(c:\ell)}\left(H\left[\begin{matrix} a; & & \lambda z \\ b; & (c:\ell); & \end{matrix}\right]\right) &= I_a\left\{z^{-1}\left[\sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}}\frac{z^n}{(n-1)!}\right]\right\} \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!}z^{-a}\int_0^z t^{a-1}t^{n-1}dt \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_n\,z^{-a}}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!}\left[\frac{z^{a+n-1}}{a+n-1}\right] \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\,(a)_{n-1}\,z^{n-1}}{(b)_{n-1}\,(c)_{n-1}^{\ell n-\ell}\,(n-1)!} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}\,(a)_n\,z^n}{(b)_n\,(c)_n^{\ell n}\,n!}
\end{aligned}$$

which proves (15). □

Remark 6 If $z = 0$, then we must have $\lambda = 0$.

3 Special cases

When the parameter b is absent and $a = c = \ell = 1$ in (4), then

$$H\left[\begin{matrix} 1; & & z \\ -; & (1:1); & \end{matrix}\right] = 1 + \sum_{n=1}^{\infty}\frac{z^n}{n!^n}.$$

Thus, $H\left[\begin{matrix} 1; & & z \\ -; & (1:1); & \end{matrix}\right] - 1$ gives the Sikemma's function (3).

3.1 ℓ-H exponential function

From (4) if $a = b$, then we have

$$H\left[\begin{matrix} -; & & z \\ -; & (c:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(c)_n^{\ell n}\,n!}. \tag{16}$$

This leads to define the ℓ-H exponential function as follows:

Definition 4 The ℓ-H exponential function is denoted and defined by

$$e_H^{\ell}(z) = H\left[\begin{matrix} -; & & z \\ -; & (1:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\ell n+1}}, \tag{17}$$

for all $z \in \mathbf{C}$ and $\Re(\ell) \geq 0$.

Remark 7 Obviously, $e_H^0(z) = e^z$ *and* $e_H^\ell(0) = 1$,

Remark 8 By taking $\ell = 0$ in (10) we get the reduced differential equation

$$(\theta - z)\, w = 0,$$

where $w = e^z$.

In order to derive the eigen function property for the ℓ-H exponential function, we consider the particular eigen function operator of Definition 3 when the parameter $a = b$, as follows.

Definition 5 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, $R > 0$. Define the operator

$$_p\mathcal{D}_M\,(f(z)) = z^{-1}\,{}_p\Delta_1^\theta\,(\theta(f(z))), \tag{18}$$

where $z \neq 0$, $p \in \mathbf{N} \cup \{0\}$ and the operator $_p\Delta_1^\theta$ is as defined in (9).

Remark 9 When $a = b$,

$$_p\mathcal{D}_M = \mathcal{H}^{(1:p)}$$

where the operator $\mathcal{H}^{(1:p)}$ is as defined in Definition 3.

It can be easily verified that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $|z| < R$ then for $\alpha, \beta \in \mathbf{R}$

$$_p\mathcal{D}_M\,(\alpha f(z) + \beta g(z)) = \alpha\,{}_p\mathcal{D}_M(f(z)) + \beta\,{}_p\mathcal{D}_M(g(z)). \tag{19}$$

In view of Lemma 1, the operator $_\ell\mathcal{D}_M$ is applicable to the ℓ-H exponential function, we thus have

Theorem 6 *With* $\ell \in \mathbf{N} \cup \{0\}$, *the* ℓ*-H exponential function is the eigen function of the operator* $_\ell\mathcal{D}_M$ *as defined in* (18), *that is,*

$$_\ell\mathcal{D}_M\left(e_H^\ell(\lambda z)\right) = \lambda\, e_H^\ell(\lambda z), \quad \lambda \in \mathbf{C}. \tag{20}$$

Proof From (18),

$$\begin{aligned} _\ell\mathcal{D}_M\left(e_H^\ell(\lambda z)\right) &= z^{-1}\,{}_p\Delta_1^\theta\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^{\ell n+1}}(\theta z^n)\right) \\ &= z^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n!)^{\ell n}(n-1)!}\,{}_\ell\Delta_1^\theta(z^n) \\ &= z^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n!)^{\ell n}(n-1)!}\,((n-1)!)^\ell\,(\theta)^{\ell n}(z^n). \end{aligned} \tag{21}$$

Now

$$
\begin{aligned}
(\theta)^{\ell} z &= z (= (1^1)^{\ell} z),\\
(\theta)^{2\ell} z^2 &= \underbrace{\left(z\frac{d}{dz}z\frac{d}{dz}\cdots\frac{d}{dz}z\frac{d}{dz}\right)}_{2\ell\ derivatives} z^2\\
&= (2^2)^{\ell} z^2.
\end{aligned}
$$

In general,

$$
(\theta)^{\ell n} z^n = n^{\ell n} z^n, \quad n = 1, 2, \ldots. \tag{22}
$$

Using (22) in (21), we get

$$
\begin{aligned}
{}_{\ell}\mathcal{D}_M\left(e_H^{\ell}(\lambda z)\right) &= z^{-1}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n!)^{\ell n}((n-1)!)^{1-\ell}}\, n^{\ell n} z^n\\
&= \sum_{n=1}^{\infty}\frac{\lambda^n}{((n-1)!)^{\ell n+1-\ell}}\, z^{n-1}\\
&= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}\, z^n}{(n!)^{\ell n+1}}\\
&= \lambda\, e_H^{\ell}(\lambda z).
\end{aligned}
$$

As an interesting view of the definition (17), we further have

$$
\begin{aligned}
e_H^{\ell}(iz) &= \sum_{n=0}^{\infty}\frac{(iz)^n}{(n!)^{\ell n+1}}\\
&= \sum_{n=0}^{\infty}\frac{(i)^{2n}\,(z)^{2n}}{((2n)!)^{2\ell n+1}} + \sum_{n=0}^{\infty}\frac{(i)^{2n+1}\,(z)^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\,(z)^{2n}}{((2n)!)^{2\ell n+1}} + i\sum_{n=0}^{\infty}\frac{(-1)^n\,(z)^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}.
\end{aligned} \tag{23}
$$

These infinite series are resembling with those of cosine and sine series. They are further taken up in the Sect. 3.2.

3.2 ℓ-H circular functions

From the first and second series in r.h.s of (23), we define the extended cosine and sine functions respectively which are denoted here by $\cos_H^{\ell}(z)$ and $\sin_H^{\ell}(z)$. In fact, for any $z \in \mathbf{C}$,

$$\Re\left(e_H^\ell(iz)\right) = \Re\left(H\left[\begin{matrix} -; & & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \cos_H^\ell(z), \tag{24}$$

and

$$\Im\left(e_H^\ell(iz)\right) = \Im\left(H\left[\begin{matrix} -; & & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \sin_H^\ell(z), \tag{25}$$

whence we have

$$e_H^\ell(iz) = \cos_H^\ell(z) + i\,\sin_H^\ell(z). \tag{26}$$

Remark 10 It is noteworthy that $\cos_H^0(z) = \cos(z)$, and $\sin_H^0(z) = \sin(z)$.

Further,

$$\begin{aligned}
\frac{1}{2}\left[e_H^\ell(iz) + e_H^\ell(-iz)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{(n!)^{\ell n+1}} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{(n!)^{\ell n+1}}\right] \\
&= \frac{1}{2}\left[1 + \frac{iz}{(1!)^{\ell+1}} + \frac{(iz)^2}{(2!)^{2\ell+1}} + \cdots\right. \\
&\qquad \left. +1 + \frac{-iz}{(1!)^{\ell+1}} + \frac{(iz)^2}{(2!)^{2\ell+1}} + \cdots\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\;(z)^{2n}}{((2n)!)^{2\ell n+1}} \\
&= \cos_H^\ell(z)
\end{aligned}$$

and likewise,

$$\frac{1}{2i}\left[e_H^\ell(iz) - e_H^\ell(-iz)\right] = \sin_H^\ell(z).$$

We also find that

$$\begin{aligned}
\cos_H^\ell(0) &= \frac{1}{2}\left[e_H^\ell(0) + e_H^\ell(0)\right] = 1, \\
\sin_H^\ell(0) &= \frac{1}{2i}\left[e_H^\ell(0) - e_H^\ell(0)\right] = 0.
\end{aligned}$$

Remark 11 The operator (18) yields the identities:

1. ${}_\ell\mathcal{D}_M(\cos_H^\ell(z)) = -\sin_H^\ell(z)$,
2. ${}_\ell\mathcal{D}_M(\sin_H^\ell(z)) = \cos_H^\ell(z)$.

Just as the functions $\sin z$ and $\cos z$ are solutions of the equation $\frac{d^2y}{dz^2} + y = 0$, the ℓ-H sine and ℓ-H cosine functions are also solution of a differential equation. This is shown in

Theorem 7 *The ℓ-H cosine and ℓ-H sine functions are solutions of the differential equation*

$$ {}_{\ell}\mathcal{D}_M^2 \nu + \nu = 0. $$

Proof We first note that from (20),

$$ {}_{\ell}\mathcal{D}_M \left(e_H^{\ell}(iz) \right) = i \left(e_H^{\ell}(iz) \right). $$

Hence,

$$ {}_{\ell}\mathcal{D}_M^2 \left(e_H^{\ell}(iz) \right) = {}_{\ell}\mathcal{D}_M \left({}_{\ell}\mathcal{D}_M \left(e_H^{\ell}(iz) \right) \right) = {}_{\ell}\mathcal{D}_M \left(i \left(e_H^{\ell}(iz) \right) \right) = -e_H^{\ell}(iz). $$

Now using (26), this may be written as

$$ {}_{\ell}\mathcal{D}_M^2 \left(\cos_H^{\ell}(iz) + i \sin_H^{\ell}(iz) \right) = - \left(\cos_H^{\ell}(iz) + i \sin_H^{\ell}(iz) \right). $$

By making an appeal to the property (19) and comparing the real and imaginary parts, we find that

$$ {}_{\ell}\mathcal{D}_M^2 \left(\cos_H^{\ell}(iz) \right) + \cos_H^{\ell}(iz) = 0 \quad \textit{and} \quad {}_{\ell}\mathcal{D}_M^2 \left(\sin_H^{\ell}(iz) \right) + \sin_H^{\ell}(iz) = 0. $$

3.3 ℓ-H hyperbolic functions

Again from the definition of the ℓ-H exponential function (17), we observe that

$$ \begin{aligned} e_H^{\ell}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{((2n)!)^{2\ell n+1}} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}}. \end{aligned} \tag{27} $$

Let us denote the first series (with even powers of z) on r.h.s. by (cf. [3])

$$ \mathcal{E}(e_H^{\ell}(z)) = \mathcal{E}\left(H \left[\begin{matrix} -; \\ -; \ (1:\ell); \end{matrix} \ z \right] \right) = \cosh_H^{\ell}(z) \tag{28} $$

which we call the hyperbolic ℓ-H cosine function and the second series (with odd powers of z) on r.h.s. by (cf. [3])

$$ \mathcal{O}(e_H^{\ell}(z)) = \mathcal{O}\left(H \left[\begin{matrix} -; \\ -; \ (1:\ell); \end{matrix} \ z \right] \right) = \sinh_H^{\ell}(z) \tag{29} $$

which we call the hyperbolic ℓ-H sine function.

Hence from (27),

$$ e_H^{\ell}(z) = \cosh_H^{\ell}(z) + \sinh_H^{\ell}(z). \tag{30} $$

Remark 12 $\cosh_H^0(z) = \cosh(z)$, and $\sinh_H^0(z) = \sinh(z)$.

Also note that

$$
\begin{aligned}
\frac{1}{2}\left[e_H^{\ell}(z)+e_H^{\ell}(-z)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\ell n+1}}+\sum_{n=0}^{\infty}\frac{(-z)^n}{(n!)^{\ell n+1}}\right]\\
&= \frac{1}{2}\left[1+\frac{z}{(1!)^{\ell+1}}+\frac{z^2}{(2!)^{2\ell+1}}+\ldots\right.\\
&\qquad \left.+1+\frac{-z}{(1!)^{\ell+1}}+\frac{z^2}{(2!)^{2\ell+1}}+\ldots\right]\\
&= \sum_{n=0}^{\infty}\frac{z^{2n}}{((2n)!)^{2\ell n+1}}\\
&= \cosh_H^{\ell}(z). \qquad (31)
\end{aligned}
$$

Similarly,

$$
\frac{1}{2}\left[e_H^{\ell}(z)-e_H^{\ell}(-z)\right]=\sinh_H^{\ell}(z).
$$

In particular,

$$
\cosh_H^{\ell}(0)=\frac{1}{2}\left[e_H^{\ell}(0)+e_H^{\ell}(0)\right]=1,\quad \sinh_H^{\ell}(0)=\frac{1}{2}\left[e_H^{\ell}(0)-e_H^{\ell}(0)\right]=0.
$$

In parallel to Theorem 7, we have

Theorem 8 *The hyperbolic ℓ-H cosine and hyperbolic ℓ-H sine functions are solutions of the differential equation*

$$
{}_{\ell}\mathcal{D}_M^2 \nu - \nu = 0.
$$

Proof One can see that from (31), (19) and (20),

$$
\begin{aligned}
{}_{\ell}\mathcal{D}_M^2(\cosh_H^{\ell}(z))-\cosh_H^{\ell}(z) &= {}_{\ell}\mathcal{D}_M^2\left(\frac{e_H^{\ell}(z)+e_H^{\ell}(-z)}{2}\right)-\left(\frac{e_H^{\ell}(z)+e_H^{\ell}(-z)}{2}\right)\\
&= \frac{1}{2}\left[e_H^{\ell}(z)+e_H^{\ell}(-z)-e_H^{\ell}(z)-e_H^{\ell}(-z)\right]\\
&= 0.
\end{aligned}
$$

Likewise,

$$
{}_{\ell}\mathcal{D}_M^2\left(\sinh_H^{\ell}(z)\right)-\sinh_H^{\ell}(z)=0
$$

can be verified. □

Remark 13 The new functions (4) can evidently be considered as extensions of the generalized hypergeometric function ${}_1F_{1+q}$ (see [4, Ch.4]), reduced to the so-called hyper-Bessel functions ${}_0F_q$ if $a = b$, and being eigen functions of the hyper-Bessel operators already mentioned in this paper (details in [5,7]). But now q in the second index goes to infinity together with the summation index n in the power series. Then, it is interesting to mention an analogy of the ℓ-H circular and ℓ-H hyperbolic functions with the generalized cosine, sine and hyperbolic functions in the sense of [5], called also r-even functions in [8]. These are special cases of the hyper-Bessel functions in the same ways as the ℓ-H circular and ℓ-H hyperbolic functions come in the scheme of the new ℓ-Hypergeometric functions.

3.4 Graphs

Figures 1, 2 and 3 are the graphs of particular ℓ-H functions.

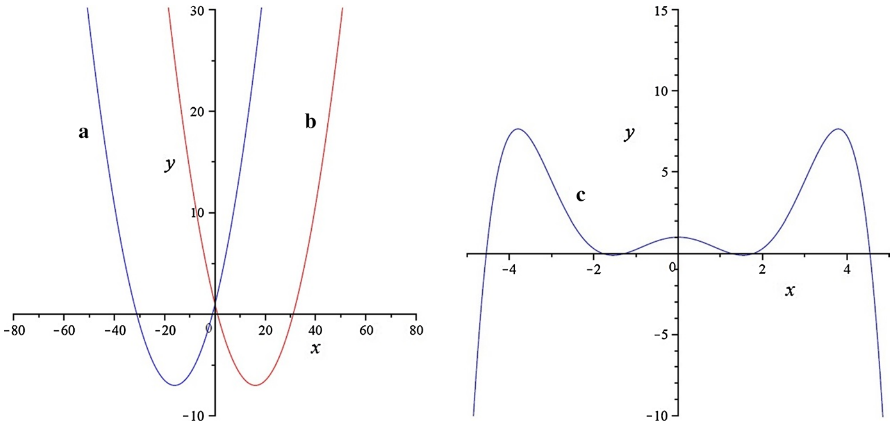

Fig. 1 **a** Graph of $e_H^2(x)$; **b** graph of $e_H^2(-x)$ and **c** graph of $e_H^{\frac{1}{2}}(-x^2)$

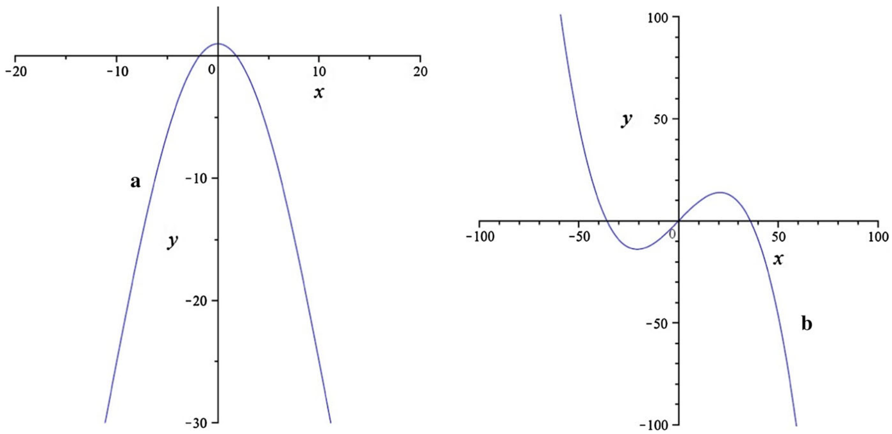

Fig. 2 **a** Graph of $\cos_H^{\frac{5}{14}}(x)$ and **b** Graph of $\sin_H^1(x)$

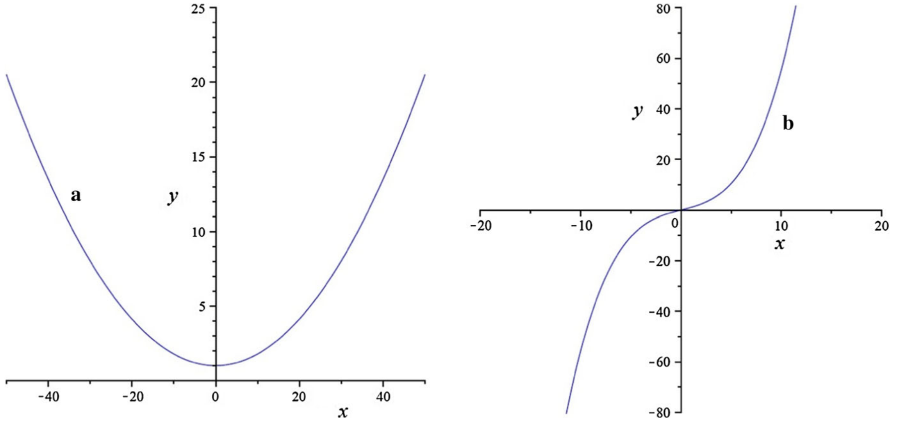

Fig. 3 **a** Graph of $\cosh_H^3(x)$ and **b** Graph of $\sinh_H^{\frac{1}{4}}(x)$

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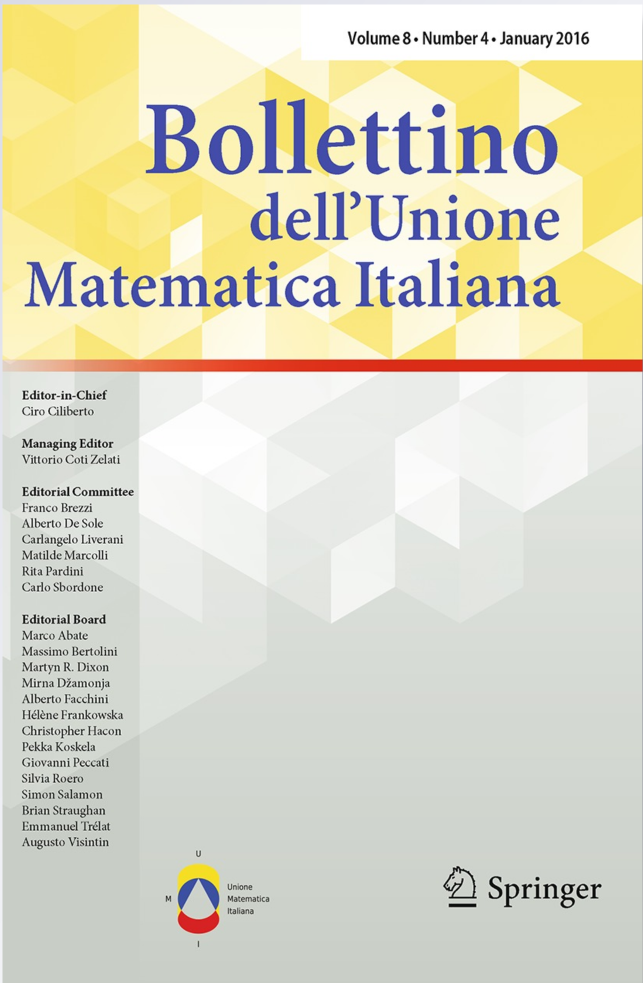

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A new generalization of q-hypergeometric function

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Abstract In the present work, we introduce a function representing a rapidly convergent q-power series which extends the well-known basic hypergeometric function ${}_2\phi_1[z]$. We introduce infinite order q-difference operators to obtain its difference equation. Besides this, we also derive the eigen function property and contiguous functions relations for this function. The work characterizes the q-exponential function which we refer to as q-ℓ-Ψ exponential function and derive its eigen function property by introducing an infinite order q-hyper-Bessel type operator. The definitions of q-ℓ-Ψ circular and q-ℓ-Ψ hyperbolic functions follow immediately. It is also shown that the product of two q-ℓ-Ψ exponential functions generates q-ℓ-Ψ Bessel function.

Keywords Basic hypergeometric function · q-Contiguous function · q-Integral · q-Derivative · Eigen function

Mathematics Subject Classification 33D05 · 33D15 · 33D99 · 34A35

1 Introduction

Let $0 < q < 1$. A q-analogue of factorial function

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1)$$

is defined by [4, Eq. (1.2.15) and (1.2.30), p. 3, 6]

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$$(a;q)_n = \begin{cases} 1, & \text{if } \ n=0 \\ (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), & \text{if } \ n \in \mathbf{Z}_{>0} \\ \left[\left(1-aq^{-1}\right)\left(1-aq^{-2}\right)\cdots\left(1-aq^{-n}\right)\right]^{-1}, & \text{if } \ n \in \mathbf{Z}_{<0} \\ \frac{(a;q)_\infty}{(aq^n;q)_\infty} & \text{if } \ n \in \mathbf{C}, \end{cases} \tag{1}$$

where $a \in \mathbf{C}$ in general, and $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$.

For $a \equiv q^a = q$,

$$(q;q)_n = (1-q)\left(1-q^2\right)\cdots\left(1-q^n\right)$$

is q-analogue of $n!$.

In these notations, the q-hypergeometric function is defined as [4, Eq. (1.2.13), p. 3]

$${}_2\phi_1\left[\begin{matrix} a; \ b; \ z \\ c; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(c;q)_n}\frac{z^n}{(q;q)_n} \tag{2}$$

which is an analytic function for $|z| < 1$ and $a, b, c \in \mathbf{C}$ with $c \neq 0, -1, -2, \ldots$.

Let us consider the series

$$\sum_{n=1}^{\infty} \frac{z^n}{(q;q)_n^n}. \tag{3}$$

as a q-form of the integral function

$$\sum_{n=1}^{\infty} \frac{z^n}{n!^n}$$

due to Sikkema [8, p. 6]. The series in (3) leads us to extend the q-Hypergeometric series in (2) which we define as follows.

Definition 1 For $\Re(\ell) \geq 0$, $c, d \in \mathbf{C}/\{0, -1, -2, \ldots\}$ and $a, b, z \in \mathbf{C}$, define the function

$$\Psi\left[\begin{matrix} a; & b; & q; \ z \\ c; & (d:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(d;q)_n^{\ell n}}\frac{z^n}{(q;q)_n}. \tag{4}$$

We shall call this function to be q-ℓ-Ψ hypergeometric function and in brief, the q-ℓ-Ψ function.

The q-series in (3) is an evident special case of (4) when $a = c, \ b = d = q$ and $\ell = 1$.

2 Main results

In this section, we derive difference equation, eigen function property and contiguous function relations of the series (4); but first we investigate the convergence behavior.

2.1 Convergence

Theorem 1 *If* $0 < q < 1, \Re(\ell) > 0$ *then* q-ℓ-Ψ *function is an analytic function of* z.

Proof Put

$$\frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(d;q)_n^{\ell n}\,(q;q)_n} = \xi_n$$

then in view of the formula

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}\,(1-q)^n,$$

we have

$$\begin{aligned}|\xi_n|^{\frac{1}{n}} &= \left|\frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(d;q)_n^{\ell n}\,(q;q)_n}\right|^{\frac{1}{n}}\\ &= \left|\frac{\Gamma_q(c)}{\Gamma_q(a)\,\Gamma_q(b)}\right|^{\frac{1}{n}}\left|\frac{\Gamma_q(a+n)\,\Gamma_q(b+n)}{\Gamma_q(c+n)\,\Gamma_q(n+1)}\right|^{\frac{1}{n}}\left|\frac{\Gamma_q(d)}{\Gamma_q(d+n)\,(1-q)^n}\right|^{\ell}.\end{aligned}$$

Here using Stirling's formula of q-Gamma function [6, Eq. (2.25), p. 482]:

$$\Gamma_q(z) \sim (1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-z}\,e^{\frac{\theta q^z}{1-q-q^z}}, \tag{5}$$

for large $|z|$ and $0<\theta<1$, we further have

$$\begin{aligned}|\xi_n|^{\frac{1}{n}} &\sim \left|\frac{\Gamma_q(c)}{\Gamma_q(a)\,\Gamma_q(b)}\right|^{\frac{1}{n}}\left|\frac{(1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(a+n)}\,e^{\frac{\theta\,q^{a+n}}{1-q-q^{a+n}}}}{(1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(c+n)}\,e^{\frac{\theta\,q^{c+n}}{1-q-q^{c+n}}}}\right|^{\frac{1}{n}}\\ &\quad\times\left|\frac{(1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(b+n)}\,e^{\frac{\theta\,q^{b+n}}{1-q-q^{b+n}}}}{(1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(n+1)}\,e^{\frac{\theta\,q^{n+1}}{1-q-q^{n+1}}}}\right|^{\frac{1}{n}}\\ &\quad\times\frac{\left|\Gamma_q^{\ell}(d)\,(1-q)^{-\ell n}\right|}{\left|(1+q)^{\frac{1}{2}}\,\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-(d+n)}\,e^{\frac{\theta\,q^{d+n}}{1-q-q^{d+n}}}\right|^{\ell}}\\ &= \left|\frac{(1-q)^{1+c-a-b}\,\Gamma_q(c)}{\Gamma_q(a)\,\Gamma_q(b)}\right|^{\frac{1}{n}}\left|e^{\frac{\theta\,q^{a+n}}{1-q-q^{a+n}}}\right|^{\frac{1}{n}}\left|e^{\frac{\theta\,q^{c+n}}{1-q-q^{c+n}}}\right|^{-\frac{1}{n}}\\ &\quad\times\left|e^{\frac{\theta\,q^{b+n}}{1-q-q^{b+n}}}\right|^{\frac{1}{n}}\left|e^{\frac{\theta\,q^{n+1}}{1-q-q^{n+1}}}\right|^{-\frac{1}{n}}\frac{\left|\Gamma_q^{\ell}(d)\,(1-q)^{d\ell}\right|}{\left|\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q^2)^{\frac{1}{2}}\,e^{\frac{\theta\,q^{d+n}}{1-q-q^{d+n}}}\right|^{\ell}}\end{aligned}$$

for large n. Now from the Cauchy Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty}\sup\sqrt[n]{|\xi_n|},$$

it follows that the radius of convergence

$$R = \left|\frac{\left(1-q^2\right)^{\frac{1}{2}}\;\Gamma_{q^2}\left(\frac{1}{2}\right)}{\Gamma_q(d)\,(1-q)^d}\right|^{\ell}. \tag{6}$$

Remark 1 From (6), it is evident that if $\ell=0$ then q-ℓ-Ψ function reduces to the basic hypergeometric series representing ${}_2\phi_1(a,b;c;q,z)$ with radius of convergence unity.

2.2 q-Difference equation

The differential equation of q-ℓ-Ψ function occurs for $\ell \in \mathbf{N} \cup \{0\}$, which is obtained by means of the following operator.

Definition 2 Let $f(z) = \sum_{n=1}^{\infty} a_{n,q}\, z^n, 0 \neq z \in \mathbf{C},\ p \in \mathbf{N} \cup \{0\},\ \alpha \in \mathbf{C}$ and the difference operator θ_q be defined by

$$\theta_q f(z) = f(z) - f(zq). \tag{7}$$

Define

$$ {}_p\Delta_{\alpha}^{\theta_q} f(z) = \begin{cases} \sum_{n=1}^{\infty} a_{n,q}\ (\alpha; q)_{n-1}^{p}\ (q^{\alpha-1}\,\theta_q - q^{\alpha-1} + 1)^{pn}\, z^n, & \text{if } p \in \mathbf{N} \\ f(z), & \text{if } p = 0. \end{cases} \tag{8}$$

Then we have

Theorem 2 *For $\ell \in \mathbf{N} \cup \{0\}$, $a, z \in \mathbf{C}$, and $c, d \in \mathbf{C}/\{0, -1, -2, \ldots\}$, the function $w = \Psi\left[\begin{matrix} a; & b; & q;\ z \\ c; & (d:\ell); & \end{matrix}\right]$ satisfies the difference equation*

$$\left[{}_\ell\Delta_d^{\theta_q}\left\{q^{c-1}\theta_q - q^{c-1} + 1\right\}\theta_q - z\left(q^a\theta_q - q^a + 1\right)\left(q^b\theta_q - q^b + 1\right)\right] w = 0, \tag{9}$$

where the operator ${}_\ell\Delta_d^{\theta_q}$ is as defined in (8).

Note 1 The operators ${}_\ell\Delta_d^{\theta_q}$ and θ_q *do not* commute.

The proof needs the following lemma which actually permits us to apply the operator $\left\{{}_\ell\Delta_d^{\theta_q}\right\}\left\{q^{c-1}\theta_q - q^{c-1} + 1\right\}\theta_q$ on the operand w.

For the sake of brevity, we put

$$\left\{{}_\ell\Delta_d^{\theta_q}\right\}\left\{q^{c-1}\theta_q - q^{c-1} + 1\right\}\theta_q = {}_c\Lambda_{(d,\ell)}^{\theta_q} \tag{10}$$

and as mentioned earlier, $\alpha \equiv q^{\alpha}$. In this notation, we have

Lemma 1 *If $\ell \in \mathbf{N} \cup \{0\}$, $a, b, z \in \mathbf{C}$, $c, d \in \mathbf{C}/\{0, -1, -2, \ldots\}$,*

$$w = \Psi\left[\begin{matrix} a; & b; & q;\ z \\ c; & (d:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty} \xi_n\, z^n$$

and ${}_c\Lambda_{\{d,\ell\}}^{\theta_q} w = \sum_{n=0}^{\infty} f_{n,q}(a, b; c, (d:\ell); z)$ then the operator ${}_c\Lambda_{\{d,\ell\}}^{\theta_q}$ is applicable to the q-ℓ-Ψ function provided that the series

$$\sum_{n=0}^{\infty} \xi_n\ f_{n,q}(a, b; c, (d:\ell); z)$$

converges (cf. [8, Definition 11, p. 20]).

Proof We begin with

$$
\begin{aligned}
&{}_c\Lambda^{\theta_q}_{(d,\ell)} w \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_d\right\}\left\{q^{c-1}\theta_q - q^{c-1}+1\right\}\right]\left(\sum_{n=0}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}}\frac{z^n - z^n q^n}{(q;q)_n}\right) \\
&= \left[\left\{{}_\ell\Delta^{\theta_q}_d\right\}\left\{q^{c-1}\theta_q - q^{c-1}+1\right\}\right]\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}}\frac{z^n}{(q;q)_{n-1}}\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_d\right\}\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}\left(q^{c-1}(z^n - z^n q^n) - z^n\left(q^{c-1}-1\right)\right)\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_d\right\}\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_{n-1}} z^n\ \left(1-q^{c+n-1}\right)\right) \\
&= \left\{{}_\ell\Delta^{\theta_q}_d\right\}\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_n^{\ell n}\ (q;q)_{n-1}} z^n\right) \\
&= \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}(d;q)_{n-1}^{\ell}\ \left(q^{d-1}\theta_q - q^{d-1}+1\right)^{\ell n} z^n.
\end{aligned}
$$

Here a little computations show that

$$
\left(q^{d-1}\theta_q - q^{d-1}+1\right)^{\ell n}\ z^n = \left(1-q^{n+d-1}\right)^{\ell n} z^n.
$$

We thus have

$$
\begin{aligned}
{}_c\Lambda^{\theta_q}_{(d,\ell)} w &= \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n\ (d;q)_{n-1}^{\ell}}{(c;q)_{n-1}\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}(1-q^{n+d-1})^{\ell n} z^n \\
&= \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n\ z^n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \qquad (11)\\
&= \sum_{n=0}^{\infty}\frac{(a;q)_{n+1}\ (b;q)_{n+1}\ z^{n+1}}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_n} \\
&= \sum_{n=0}^{\infty} f_n(a,b;c,(d:\ell);z). \qquad (12)
\end{aligned}
$$

To complete the proof of lemma, it suffices to show that

$$
\sum_{n=0}^{\infty}\xi_n\ f_{n,q}(a,b,c,(d:\ell);z) = \sum_{n=0}^{\infty}\frac{(a;q)_n^2\ (b;q)_n^2\ (1-aq^n)\ (1-bq^n)\ z^{n+1}}{(c;q)_n^2\ (d;q)_n^{2\ell n}\ (q;q)_n^2}
$$

is convergent. For that, take

$$\begin{aligned}
|\mu_n| &= \left|\xi_n\ f_{n,q}(a, b; c, (d:\ell); z)\right| \\
&= \left|\frac{(a;q)_n^2\ (b;q)_n^2\ \left(1-aq^n\right)\ \left(1-bq^n\right)\ z^{n+1}}{(c;q)_n^2\ (d;q)_n^{2\ell n}\ (q;q)_n^2}\right| \\
&= \left|\frac{\Gamma_q(a+n)\ \Gamma_q(b+n)\ \Gamma_q(c)}{\Gamma_q(a)\ \Gamma_q(b)\ \Gamma_q(c+n)\ \Gamma_q(n+1)}\ \left(\frac{\Gamma_q(d)}{\Gamma_q(d+n)\ (1-q)^n}\right)^{\ell n}\right|^2 \\
&\quad \times \left|\left(1-aq^n\right)\ \left(1-bq^n\right)\right|\ |z|^{n+1}.
\end{aligned}$$

Once again applying the q-analogue of Stirling's formula:

$$\Gamma_q(\alpha+n) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{\frac{1}{2}-\alpha-n}\ e^{\theta\frac{q^{\alpha+n}}{1-q-q^{\alpha+n}}}, \qquad (0<\theta<1)$$

with α replaced by a, b, c and d in turn, then similar computations as in Theorem 1 yields

$$\lim_{n\to\infty}\sup|\mu_n|^{\frac{1}{n}} = \left|\frac{\Gamma_q^2(d)\ (1-q)^{2d}}{\left(1-q^2\right)\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)}\right|^{\ell}\ |z|.$$

Hence lemma follows under the condition

$$|z| < \left|\frac{\left(1-q^2\right)\ \Gamma_{q^2}^2\left(\frac{1}{2}\right)}{\Gamma_q^2(d)\ (1-q)^{2d}}\right|^{\ell}.$$

We now prove the theorem.

Proof of Theorem 2 From (12),

$$\begin{aligned}
&{}_b\Lambda_{(c,\ell)}^{\theta_q} w \\
&= \sum_{n=0}^{\infty}\frac{(a;q)_{n+1}\ (b;q)_{n+1}\ z^{n+1}}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_n} \\
&= z\sum_{n=0}^{\infty}\frac{(a;q)_n(q^a z^n - q^a z^n q^n - q^a z^n + z^n)\ (b;q)_{n+1}}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ \left(q^a\theta_q - q^a + 1\right)\quad \sum_{n=0}^{\infty}\frac{(a;q)_n\ (b;q)_{n+1}\ z^n}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ \left(q^a\theta_q - q^a + 1\right)\quad \sum_{n=0}^{\infty}\frac{(a;q)_n\ (b;q)_n\ \left(q^b z^n - q^b z^n q^n - q^b z^n + z^n\right)}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_n} \\
&= z\ \left(q^a\theta_q - q^a + 1\right)\ \left(q^b\theta_q - q^b + 1\right)\ w.
\end{aligned}$$

Hence (9) holds.

2.3 Eigen function property

We recall the q-integral formula of an integrable function $f(z)$ which is given by [4, Eq. (1.11.1), p. 19]

$$I_q f(z) = \int_0^z f(t) d_q t = z\,(1-q) \sum_{n=0}^{\infty} f\left(zq^n\right)\, q^n. \tag{13}$$

In order to obtain q-ℓ-Ψ function as an eigen function, we need to define two more operators using following definitions.

Definition 3 Let $f(z) = \sum_{n=1}^{\infty} a_{n,q}\, z^n,\ 0 \neq |z| < R, R > 0$ and $\alpha \in \mathbf{C}$ with $\Re(\alpha) \geq 1$. Define

$$I_q^{\alpha} f(z) = \frac{z^{-\alpha}}{1-q}\, I_q\left(z^{\alpha-1} f(z)\right). \tag{14}$$

Definition 4 Let $f(z) = \sum_{n=0}^{\infty} a_{n,q}\, z^n,\ 0 \neq |z| < R, R > 0,\ p \in \mathbf{N} \cup \{0\}$ and $\alpha, \beta \in \mathbf{C}$ with $\Re(\alpha, \beta) \geq 1$. Define

$${}_{\alpha}^{\delta}\mathcal{E}_{\beta}^{(\gamma:p)} f(z) = \left[I_q^{\beta} I_q^{\alpha}\, z^{-1}\, {}_{\delta}\Lambda_{(\gamma,p)}^{\theta_q}\right] f(z), \tag{15}$$

where ${}_{\delta}\Lambda_{(\gamma,p)}^{\theta_q}$ and I_q^{α} are as defined in (10) and (14) respectively.

In these notations, we prove

Theorem 3 *If $\ell \in \mathbf{N} \cup \{0\}$ and $\Re(a,b) \geq 1$ then the q-ℓ-Ψ function is an eigen function with respect to the operator ${}_{a}^{c}\mathcal{E}_{b}^{(d:\ell)}$ defined in (15). That is,*

$${}_{a}^{c}\mathcal{E}_{b}^{(d:\ell)}\left(\Psi\begin{bmatrix} a; & b; & q;\ \lambda z \\ c; & (d:\ell); & \end{bmatrix}\right) = \lambda\, \Psi\begin{bmatrix} a; & b; & q;\ \lambda z \\ c; & (d:\ell); & \end{bmatrix},\quad \lambda \in \mathbf{C}. \tag{16}$$

Proof The applicability of this operator to the q-ℓ-Ψ function follows from Lemma 1.

Now for $z \neq 0$,

$$\begin{aligned} &{}_{a}^{c}\mathcal{E}_{b}^{(d:\ell)}\left(\Psi\begin{bmatrix} a; & b; & q;\ \lambda z \\ c; & (d:\ell); & \end{bmatrix}\right) \\ &= \left[I_q^{b} I_q^{a}\, z^{-1}\, {}_{c}\Lambda_{(d,\ell)}^{\theta_q}\left(\sum_{n=0}^{\infty} \frac{\lambda^n\,(a;q)_n\,(b;q)_n\, z^n}{(c;q)_n\,(d;q)_n^{\ell n}\,(q;q)_n}\right)\right]. \end{aligned}$$

In view of (11),

$$
\begin{aligned}
&{}_{a}^{c}\mathcal{E}_{b}^{(d:\ell)}\left(\Psi\begin{bmatrix} a; & b; & q;\ \lambda z \\ c; & (d:\ell); & \end{bmatrix}\right)\\
&= I_q^b\, I_q^a \left[\sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}} \frac{z^{n-1}}{(q;q)_{n-1}}\right]\\
&= I_q^b \left\{\frac{z^{-a}}{1-q}\ I_q \left[\sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}} \frac{z^{a+n-2}}{(q;q)_{n-1}}\right]\right\}\\
&= I_q^b \left\{\frac{z^{-a}}{1-q} \quad \sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\ z(1-q) \sum_{k=0}^{\infty} \left(zq^k\right)^{a+n-2} \quad q^k\right\}\\
&= I_q^b \left\{\frac{z^{-a}}{1-q} \quad \sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\ z(1-q)\ z^{a+n-2} \sum_{k=0}^{\infty} q^{k(a+n-1)}\right\}\\
&= I_q^b \sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \frac{z^{n-1}}{1-q^{a+n-1}}\\
&= I_q^b \sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_{n-1}\ (b;q)_n\ z^{n-1}}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}.
\end{aligned}
$$

In the similar manner, by applying operator I_q^b we finally arrive at

$$
\begin{aligned}
{}_{a}^{c}\mathcal{E}_{b}^{(d:\ell)}\left(\Psi\begin{bmatrix} a; & b; & q;\ \lambda z \\ c; & (d:\ell); & \end{bmatrix}\right) &= \sum_{n=1}^{\infty} \frac{\lambda^n\ (a;q)_{n-1}\ (b;q)_{n-1}\ z^{n-1}}{(c;q)_{n-1}\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}\\
&= \lambda \sum_{n=0}^{\infty} \frac{\lambda^n\ (a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}} \frac{z^n}{(q;q)_n}.
\end{aligned}
$$

Thus the theorem.

2.4 Contiguous function relations

The contiguous function relations for basic hypergeometric series have been derived by Swarttouw [9]. Our attempt made in this direction led us to the following identities:

$$(cq^{-1}-a)\Psi = cq^{-1}(1-a)\Psi(a+) - a(1-cq^{-1})\Psi(c-), \tag{17}$$

$$(1-c)\ \Psi = (1-a)\ \Psi(a+,c+) - (c-a)\ \Psi(c+,zq), \tag{18}$$

$$(cq^{-1}-b)\Psi = cq^{-1}(1-b)\Psi(b+) - b(1-cq^{-1})\Psi(c-), \tag{19}$$

and

$$(1-c)\ \Psi = (1-b)\ \Psi(b+,c+) - (c-b)\ \Psi(c+,zq), \tag{20}$$

in which the function notations carry the following meaning.

We take

$$\Psi = \Psi\begin{bmatrix} a; & b; & q;\ z \\ c; & (d:\ell); & \end{bmatrix},$$

then

$$\begin{aligned}\Psi(a+) &:= \Psi\left[\begin{matrix} aq; & b; & q;\, z \\ c; & (d:\ell); & \end{matrix}\right],\\ \Psi(a-) &:= \Psi\left[\begin{matrix} aq^{-1}; & b; & q;\, z \\ c; & (d:\ell); & \end{matrix}\right],\\ \Psi(a+, c+) &:= \Psi\left[\begin{matrix} aq; & b; & q;\, z \\ cq; & (d:\ell); & \end{matrix}\right],\\ \Psi(c+, zq) &:= \Psi\left[\begin{matrix} a; & b; & q;\, zq \\ cq; & (d:\ell); & \end{matrix}\right].\end{aligned}$$

The functions $\Psi(b+), \Psi(b-), \Psi(c+), \Psi(c-)$ are defined similarly.

Now with

$$\xi_n = \frac{(a;q)_n\,(b;q)_n}{(c;q)_n\,(d;q)_n^{\ell n}\,(q;q)_n},$$

we have

$$\Psi = \sum_{n=0}^{\infty} \xi_n z^n$$

whence the following series representations are evident.

$$\left.\begin{aligned}\Psi(a+) &= \sum_{n=0}^{\infty} \tfrac{1-aq^n}{1-a}\,\xi_n\, z^n, \quad \Psi(a-) = \sum_{n=0}^{\infty} \tfrac{1-aq^{-1}}{1-aq^{n-1}}\,\xi_n\, z^n,\\ \Psi(b+) &= \sum_{n=0}^{\infty} \tfrac{1-bq^n}{1-b}\,\xi_n\, z^n, \quad \Psi(b-) = \sum_{n=0}^{\infty} \tfrac{1-bq^{-1}}{1-bq^{n-1}}\,\xi_n\, z^n,\\ \Psi(c+) &= \sum_{n=0}^{\infty} \tfrac{1-c}{1-cq^n}\,\xi_n\, z^n, \quad \Psi(c-) = \sum_{n=0}^{\infty} \tfrac{1-cq^{n-1}}{1-cq^{-1}}\,\xi_n\, z^n,\\ \Psi(d+) &= \sum_{n=0}^{\infty} \tfrac{(1-d)^{\ell n}}{(1-dq^n)^{\ell n}}\,\xi_n\, z^n, \ \Psi(d-) = \sum_{n=0}^{\infty} \tfrac{(1-dq^{n-1})^{\ell n}}{(1-dq^{-1})^{\ell n}}\,\xi_n\, z^n.\end{aligned}\right\} \tag{21}$$

Now, the q-derivative operator D_q is defined by [4, Ex. (1.12), p. 22]

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}. \tag{22}$$

If $zD_q = \Theta_q$ then

$$\Theta_q \Psi = \sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\,\xi_n\, z^n \tag{23}$$

whence we have

$$\begin{aligned}\left(a\Theta_q + \frac{1-a}{1-q}\right)\Psi &= a\sum_{n=0}^{\infty} \frac{(1-q^n)}{(1-q)}\,\xi_n\, z^n + \frac{1-a}{1-q}\sum_{n=0}^{\infty} \xi_n\, z^n\\ &= \sum_{n=0}^{\infty} \xi_n\, \frac{z^n}{1-q}\,[a - aq^n + 1 - a]\\ &= \frac{1-a}{1-q}\,\Psi(a+).\end{aligned} \tag{24}$$

Also,

$$\left(cq^{-1}\Theta_q + \frac{1-cq^{-1}}{1-q}\right)\Psi = cq^{-1}\sum_{n=0}^{\infty}\frac{(1-q^n)}{(1-q)}\,\xi_n\,z^n + \frac{1-cq^{-1}}{1-q}\quad\sum_{n=0}^{\infty}\xi_n\,z^n$$
$$= \sum_{n=0}^{\infty}\xi_n\,\frac{z^n}{1-q}\,\left[cq^{-1} - cq^{n-1} + 1 - cq^{-1}\right]$$
$$= \frac{1-cq^{-1}}{1-q}\,\Psi(b-). \tag{25}$$

Eliminating Θ_q from (24) and (25), yields (17).

Next, for $z \neq 0$, using the technique adopted in the proof of Lemma 1 and Theorem 3, we have

$$\left\{I_q^b\ z^{-1}\ {}_{\ell}\Delta_d^{\theta_q}\right\}(\theta_q\Psi)$$
$$= \left\{{}_{\ell}\Delta_d^{\theta_q}\right\}\left(\sum_{n=0}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}}\,\frac{z^n - z^n q^n}{(q;q)_n}\right)$$
$$= \left\{I_q^b\ z^{-1}\ {}_{\ell}\Delta_d^{\theta_q}\right\}\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}}\,\frac{z^n}{(q;q)_{n-1}}\right)$$
$$= \left\{I_q^b\ z^{-1}\ {}_{\ell}\Delta_d^{\theta_q}\right\}\left(\sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}\,z^n\right)$$
$$= I_q^b\ z^{-1}\ \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}(d;q)_{n-1}\ \left(q^{d-1}\theta_q - q^{d-1} + 1\right)^{\ell n}\ z^n$$
$$= I_q^b\ z^{-1}\ \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n\ (d;q)_{n-1}^{\ell}}{(c;q)_n\ (d;q)_n^{\ell n}\ (q;q)_{n-1}}\ \left(1 - q^{n+d-1}\right)^{n\ell}\ z^n$$
$$= I_q^b\ z^{-1}\ \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_n\ z^n}{(c;q)_n\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}$$
$$= \sum_{n=1}^{\infty}\frac{(a;q)_n\ (b;q)_{n-1}\ z^{n-1}}{(c;q)_n\ (d;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}}$$
$$= \sum_{n=0}^{\infty}\frac{(a;q)_{n+1}\ (b;q)_n\ z^n}{(c;q)_{n+1}\ (d;q)_n^{\ell n}\ (q;q)_n}$$
$$= \sum_{n=0}^{\infty}\frac{1-aq^n}{1-cq^n}\ \xi_n\ z^n \tag{26}$$
$$= \frac{1-a}{1-c}\ \Psi(a+, c+). \tag{27}$$

Now in (26), putting

$$\frac{1-aq^n}{1-cq^n} = 1 + \frac{cq^n - aq^n}{1-cq^n}$$

we find that

$$\begin{aligned}
&\left\{I_q^b\, z^{-1}\, {}_\ell\Delta_d^{\theta_q}\right\}(\theta_q \Psi) \\
&= \sum_{n=0}^{\infty}\left(1+\frac{cq^n-aq^n}{1-cq^n}\right)\xi_n z^n \\
&= \sum_{n=0}^{\infty}\xi_n z^n + \sum_{n=0}^{\infty}\frac{c-a}{1-c}\frac{1-c}{1-cq^n}\,\xi_n\,(zq)^n \\
&= \Psi + \frac{c-a}{1-c}\Psi(c+, zq).
\end{aligned} \tag{28}$$

The relation (18) now follows from (26) and (28).

Since the q-ℓ-Ψ function (4) is symmetric in its numerator parameters a and b, the identities (19) and (20) follow immediately from (17) and (18) respectively.

3 Particular q-ℓ-Ψ functions

3.1 q-ℓ-Ψ Exponential function

In (4), if the parameters a, b and c are absent then

$$\Psi\left[\begin{matrix} -; & q; & z \\ -; & (d:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(d;q)_n^{\ell n}\,(q;q)_n}, \quad (\Re(\ell)\geq 0). \tag{29}$$

This function characterizes the q-exponential function. In fact for $d=1$, the above series would reduce to

$$\Psi\left[\begin{matrix} -; & q; & z \\ -; & (1:\ell); & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n^{\ell n+1}}.$$

This enables us to define q-ℓ-Ψ exponential function as follows:

Definition 5 The q-ℓ-Ψ exponential function is denoted and defined by

$$e_\Psi^\ell(z;q) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n^{\ell n+1}}, \tag{30}$$

for all $z \in \mathbf{C}$ and $\Re(\ell)\geq 0$.

Remark 2 Obviously, $e_\Psi^0(z;q) = e_q(z),\ |z|<1$ and $e_\Psi^\ell(0;q)=1$.

Eventually, when the parameters a and b are absent, the difference equation obtained in Theorem 2 would reduce to the form

$$\left(\left\{{}_\ell\Delta_1^{\theta_q}\right\}\theta_q - z\right)w = 0 \tag{31}$$

which is satisfied by $w = e_{\Psi}^{\ell}(z; q)$. This can be verified as follows.

$$\begin{aligned}
\left[\left\{{}_{\ell}\Delta_1^{\theta_q}\right\} \theta_q\right] w &= \left\{{}_{\ell}\Delta_1^{\theta_q}\right\} \sum_{n=0}^{\infty} \frac{1}{(q;q)_n^{\ell n+1}} \left(z^n - z^n q^n\right) \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell} \, (\theta_q)^{\ell n} \, z^n}{(q;q)_n^{\ell n} \, (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell} \, (1-q^n)^{\ell n} \, z^n}{(q;q)_n^{\ell n} \, (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}^{\ell} \, z^n}{(q;q)_{n-1}^{\ell n} \, (q;q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{z^n}{(q;q)_{n-1}^{\ell n-\ell} \, (q;q)_{n-1}} \\
&= z \, w.
\end{aligned}$$

Note 2 The case $\ell = 0$ in (31) yields the equation

$$(\theta_q - z) \, w = 0,$$

where

$$w = e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \quad |z| < 1$$

is a q-exponential function.

In order to derive eigen function property for q-ℓ-Ψ exponential function, we introduce a differential operator as follows.

Definition 6 Let $f(z;q) = \sum_{n=0}^{\infty} a_{n,q} z^n$, $|z| < R$, $R > 0$, $p \in \mathbf{N} \cup \{0\}$. Define an operator

$$ {}_p\Omega_{\alpha}^{\mathbf{D}_q^z} = \begin{cases} \sum_{n=1}^{\infty} a_{n,q} \, (\alpha;q)_{n-1}^{p} \, \left((1-q)\,\mathbf{D}_q^z\right)^{pn} \, z^n, & \text{if } \ p \in \mathbf{N} \\ f(z;q), & \text{if } \ p = 0 \end{cases}, \tag{32}$$

where $\mathbf{D}_q^z$ is the q-hyper-Bessel type operator given by (cf. [5])

$$(\mathbf{D}_q^z)^n = \underbrace{D_q z D_q \ldots D_q z D_q}_{n \text{ derivatives}}$$

in which D_q is q-differential operator defined in (22).

With this, we have

Theorem 4 *The q-ℓ-Ψ exponential function is an eigen function of the operator*

$${}_{\ell}\mathcal{D}_M^q = \left(\left\{{}_{\ell}\Omega_1^{\mathbf{D}_q^z}\right\} \theta_q\right), \tag{33}$$

where ${}_{\ell}\Omega_1^{\mathbf{D}_q^z}$ is as defined in (32). That is,

$${}_{\ell}\mathcal{D}_M^q \left(e_{\Psi}^{\ell}(q;\lambda z)\right) = \lambda \, e_{\Psi}^{\ell}(q;\lambda z), \ \lambda \in \mathbf{C}. \tag{34}$$

Proof We begin with

$$
\begin{aligned}
{}_{\ell}\mathcal{D}_M^q\left(e_\Psi^\ell(q;\lambda z)\right) &= \left(\left\{{}_{\ell}\Omega_1^{\mathbf{D}_q^z}\right\}\ \theta_q\right)\left[\sum_{n=0}^{\infty}\frac{\lambda^n\ z^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \left(\left\{{}_{\ell}\Omega_1^{\mathbf{D}_q^z}\right\}\right)\left[\sum_{n=1}^{\infty}\frac{\lambda^n\ z^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}}\right] \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}}(q;q)_{n-1}^{\ell}\ \left((1-q)\mathbf{D}_q^z\right)^{\ell n}\ z^n.
\end{aligned}
$$

Now it can be easily verified that for $n=1$,

$$(1-q)^\ell\left(\mathbf{D}_q^z\right)^\ell\ z=(1-q)^\ell\,.$$

For $n=2$,

$$(1-q)^{2\ell}\left(\mathbf{D}_q^z\right)^{2\ell}\ z^2=\left(1-q^2\right)^{2\ell}\ z.$$

In general for $n\in\mathbf{N}$,

$$(1-q)^{n\ell}\left(\mathbf{D}_q^z\right)^{n\ell}\ z^n=\left(1-q^n\right)^{\ell n}\ z^{n-1}.$$

Hence

$$
\begin{aligned}
{}_{\ell}\mathcal{D}_M^q\left(e_\Psi^\ell(q;\lambda z)\right) &= \sum_{n=1}^{\infty}\frac{\lambda^n}{(q;q)_n^{\ell n}\ (q;q)_{n-1}}(q;q)_{n-1}^{\ell}\ \left(1-q^n\right)^{\ell n}\ z^{n-1} \\
&= \sum_{n=1}^{\infty}\frac{\lambda^n\ z^{n-1}}{(q;q)_{n-1}^{\ell n-\ell}\ (q;q)_{n-1}} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}\ z^n}{(q;q)_n^{\ell n+1}} \\
&= \lambda\ e_\Psi^\ell(q;\lambda z).
\end{aligned}
$$

Note 3 If $f(z;q)=\sum_{n=0}^{\infty}a_{n,q}\ z^n$ and $g(z;q)=\sum_{n=0}^{\infty}b_{n,q}\ z^n$, $|z|<R$ then for $\alpha,\beta\in\mathbf{R}$ and $p\in\mathbf{N}\cup\{0\}$,

$$ {}_{p}\mathcal{D}_M^q\left(\alpha\ f(z;q)+\beta\ g(z;q)\right)=\alpha\ {}_{p}\mathcal{D}_M^q\left(f(z;q)\right)+\beta\ {}_{p}\mathcal{D}_M^q(g(z;q)). \tag{35}$$

Interestingly, in view of definition (30),

$$
\begin{aligned}
e_\Psi^\ell(q;iz) &= \sum_{n=0}^{\infty}\frac{(iz)^n}{((q;q)_n)^{\ell n+1}} \\
&= \sum_{n=0}^{\infty}\frac{(i)^{2n}\ (z)^{2n}}{(q;q)_{2n}^{2\ell n+1}}\quad+\sum_{n=0}^{\infty}\frac{(i)^{2n+1}\ (z)^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\ (z)^{2n}}{(q;q)_{2n}^{2\ell n+1}}\quad+i\sum_{n=0}^{\infty}\frac{(-1)^n\ (z)^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}.
\end{aligned}\tag{36}
$$

These infinite series are resembling with those of q-cosine and q-sine series. They are further taken up in Sect. 3.2.

3.2 q-ℓ-Ψ Circular functions

The first and second series in r.h.s of (36), extend the q-cosine and q-sine functions [4, Ex. 1.14, p. 23] respectively which are denoted here by $\cos^{\ell}_{\Psi}(z; q)$ and $\sin^{\ell}_{\Psi}(z; q)$. In fact for any $z \in \mathbf{C}$,

$$\Re(e^{\ell}_{\Psi}(q; iz))) = \Re\left(\Psi\left[\begin{matrix} -; & q; & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \cos^{\ell}_{\Psi}(z; q) \tag{37}$$

and

$$\Im(e^{\ell}_{\Psi}(q; iz))) = \Im\left(\Psi\left[\begin{matrix} -; & q; & iz \\ -; & (1:\ell); & \end{matrix}\right]\right) := \sin^{\ell}_{\Psi}(z; q) \tag{38}$$

which implies that

$$e^{\ell}_{\Psi}(q; iz) = \cos^{\ell}_{\Psi}(z; q) + i\,\sin^{\ell}_{\Psi}(z; q). \tag{39}$$

Remark 3 It is noteworthy that $\cos^{0}_{\Psi}(z; q) = \cos_q(z)$, and $\sin^{0}_{\Psi}(z; q) = \sin_q(z)$.

Further,

$$\begin{aligned}
\frac{1}{2}\left[e^{\ell}_{\Psi}(q; iz) + e^{\ell}_{\Psi}(q; -iz)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{(q;q)_n^{\ell n+1}} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{(q;q)_n^{\ell n+1}}\right] \\
&= \frac{1}{2}\Bigg[1 + \frac{iz}{((q;q)_1)^{\ell+1}} + \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \cdots \\
&\qquad +1 + \frac{-iz}{(q;q)_1^{\ell+1}} + \frac{(iz)^2}{(q;q)_2^{2\ell+1}} + \cdots\Bigg] \\
&= \frac{1}{2}\left[2\left(1 - \frac{z^2}{(q;q)_2^{2\ell+1}} + \frac{z^4}{(q;q)_4^{4\ell+1}} - \cdots\right)\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n}}{(q;q)_{2n}^{2\ell n+1}} \\
&= \cos^{\ell}_{\Psi}(z; q)
\end{aligned}$$

and likewise,

$$\frac{1}{2i}\left[e^{\ell}_{\Psi}(q; iz) - e^{\ell}_{\Psi}(q; -iz)\right] = \sin^{\ell}_{\Psi}(z; q).$$

Also, simple calculations show that

$$\begin{aligned}
\cos^{\ell}_{\Psi}(0; q) &= \frac{1}{2}\left[e^{\ell}_{\Psi}(0; q) + e^{\ell}_{\Psi}(0; q)\right] = 1, \\
\sin^{\ell}_{\Psi}(0; q) &= \frac{1}{2i}\left[e^{\ell}_{\Psi}(0; q) - e^{\ell}_{\Psi}(0; q)\right] = 0.
\end{aligned}$$

Remark 4 The operator (33) yields

1. ${}_{\ell}\mathcal{D}^{q}_{M}(\cos^{\ell}_{\Psi}(z; q)) = -\sin^{\ell}_{\Psi}(z; q)$,
2. ${}_{\ell}\mathcal{D}^{q}_{M}(\sin^{\ell}_{\Psi}(z; q)) = \cos^{\ell}_{\Psi}(z; q)$.

Just as the functions $\sin z$ and $\cos z$ are solutions of the equation $\frac{d^2y}{dz^2} + y = 0$, these generalized functions are also the solutions of a differential equation. This is shown in

Theorem 5 *The q-ℓ-Ψ cosine and sine functions are solutions of the differential equation*

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2 \nu + \nu = 0.$$

Proof We first note that from (34),

$${}_{\ell}\mathcal{D}_M^q\left(e_\Psi^\ell(q;iz)\right) = i\left(e_\Psi^\ell\ (q;iz))\right).$$

Hence,

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2\left(e_\Psi^\ell(q;iz)\right) = {}_{\ell}\mathcal{D}_M^q\left({}_{\ell}\mathcal{D}_M^q\left(e_\Psi^\ell(q;iz)\right)\right) = {}_{\ell}\mathcal{D}_M^q\left(i\left(e_\Psi^\ell(q;iz)\right)\right) = -e_\Psi^\ell(q;iz).$$

Now using (39), this may be written as

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2\left(\cos_\Psi^\ell(q;iz) + i\sin_\Psi^\ell(q;iz)\right) = -\left(\cos_\Psi^\ell(q;iz) + i\sin_\Psi^\ell(q;iz)\right).$$

By making an appeal to the property (35) and comparing real and imaginary parts, we find that

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2\left(\cos_\Psi^\ell(q;iz)\right) + \cos_\Psi^\ell(q;iz) = 0,$$

and

$$\left({}_{\ell}\mathcal{D}_M^q\right)^2(\sin_\Psi^\ell(q;iz)) + \sin_\Psi^\ell(q;iz) = 0.$$

3.3 q-ℓ-Ψ Hyperbolic functions

Again from the definition of q-ℓ-Ψ exponential function (30), we observe that

$$\begin{aligned} e_\Psi^\ell(z;q) &= \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n^{\ell n+1}} \\ &= \sum_{n=0}^{\infty}\frac{z^{2n}}{(q;q)_{2n}^{2\ell n+1}} + \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(q;q)_{2n+1}^{2\ell n+\ell+1}}. \end{aligned} \tag{40}$$

Let us denote the first series (with even powers of z) on r.h.s. by (cf. [1, Eq. (3.15), p. 487])

$$\mathcal{E}(e_\Psi^\ell(z;q)) = \mathcal{E}\left(\Psi\left[\begin{matrix} -; & q; & z \\ -; & (1:\ell); & \end{matrix}\right]\right) = \cosh_\Psi^\ell(z;q) \tag{41}$$

which we call hyperbolic q-ℓ-Ψ cosine function and the second series (with odd powers of z) on r.h.s. by (cf. [1, Eq. (3.16), p. 487])

$$\mathcal{O}(e_\Psi^\ell(z;q)) = \mathcal{O}\left(\Psi\left[\begin{matrix} -; & q; & z \\ -; & (1:\ell); & \end{matrix}\right]\right) = \sinh_\Psi^\ell(z;q) \tag{42}$$

which we call hyperbolic q-ℓ-Ψ sine function.

Hence from (40),

$$e_\Psi^\ell(z;q) = \cosh_\Psi^\ell(z;q) + \sinh_\Psi^\ell(z;q). \tag{43}$$

Remark 5 $\cosh^0_\Psi(z;q) = \cosh_q(z)$, and $\sinh^0_\Psi(z;q) = \sinh_q(z)$.

Also note that

$$\begin{aligned}
\frac{1}{2}\left[e^\ell_\Psi(z;q) + e^\ell_\Psi(-z;q)\right] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{(q;q)^{\ell n+1}} + \sum_{n=0}^{\infty}\frac{(-z)^n}{(q;q)^{\ell n+1}}\right] \\
&= \frac{1}{2}\Bigg[1 + \frac{z}{(q;q)_1^{\ell+1}} + \frac{z^2}{(q;q)_2^{2\ell+1}} + \cdots \\
&\quad +1 + \frac{-z}{(q;q)_1^{\ell+1}} + \frac{z^2}{(q;q)_2^{2\ell+1}} + \cdots\Bigg] \\
&= \frac{1}{2}\left[2\left(1 + \frac{z^2}{(q;q)_2^{2\ell+1}} + \frac{z^4}{(q;q)_4^{4\ell+1}} + \cdots\right)\right] \\
&= \sum_{n=0}^{\infty}\frac{z^{2n}}{(q;q)_{2n}^{2\ell n+1}} \\
&= \cosh^\ell_\Psi(z;q).
\end{aligned} \tag{44}$$

Similarly,

$$\frac{1}{2}\left[e^\ell_\Psi(z;q) - e^\ell_\Psi(-z;q)\right] = \sinh^\ell_\Psi(z;q).$$

In particular,

$$\begin{aligned}
\cosh^\ell_\Psi(0;q) &= \frac{1}{2}\left[e^\ell_\Psi(0;q) + e^\ell_\Psi(0;q)\right] = 1, \\
\sinh^\ell_\Psi(0;q) &= \frac{1}{2}\left[e^\ell_\Psi(0;q) - e^\ell_\Psi(0;q)\right] = 0.
\end{aligned}$$

In parallel to Theorem 5, we have

Theorem 6 *The hyperbolic q-ℓ-Ψ cosine and sine functions are solutions of the differential equation*

$$({}_\ell\mathcal{D}_M)^2\, \nu - \nu = 0.$$

Proof One can see that from (44), (35) and (34),

$$\begin{aligned}
&({}_\ell\mathcal{D}_M)^2\left(\cosh^\ell_\Psi(z;q)\right) - \cosh^\ell_\Psi(z;q) \\
&= ({}_\ell\mathcal{D}_M)^2\left(\frac{e^\ell_\Psi(z;q) + e^\ell_\Psi(-z;q)}{2}\right) - \left(\frac{e^\ell_\Psi(z;q) + e^\ell_\Psi(-z;q)}{2}\right) \\
&= \frac{1}{2}\left[e^\ell_\Psi(z;q) + e^\ell_\Psi(-z;q) - e^\ell_\Psi(z;q) - e^\ell_\Psi(-z;q)\right] \\
&= 0.
\end{aligned}$$

Likewise,

$$({}_\ell\mathcal{D}_M)^2 \sinh^\ell_\Psi(z;q) - \sinh^\ell_\Psi(z;q) = 0$$

follows.

4 q-ℓ-Ψ Bessel function

Having motivated by the classical theory of Bessel function [7, Ch. 6], in particular the generating function relation [7, Theorem 39, p. 113]:

$$e^{\frac{zt}{2}}\, e^{\frac{-z}{2t}} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$$

and its q-extension

$$E_q\left(\frac{zt}{2}\right)\, E_q\left(\frac{-z}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n(z:q)\, t^n$$

due to Dattoli and Torre [2,3], we consider the product of two q-ℓ-Ψ exponential functions and proceed as follows.

Let

$$\begin{aligned}\mathcal{M}^{\ell}(z,t;q) &= e^{\ell}_{\Psi}\left(\frac{zt}{2}\right)\, e^{\ell}_{\Psi}\left(\frac{-z}{2t}\right)\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\, z^{r+s}\, t^{r-s}}{2^{r+s}\,(q;q)_r^{\ell r+1}\,(q;q)_s^{\ell s+1}}.\end{aligned}$$

Taking $s=r-n$ and putting $(q;q)_{r-n}=(q;q)_{-n}\,(q^{-n+1};q)_r$ for $n\le 0$, this gives

$$\begin{aligned}\mathcal{M}^{\ell}(z,t;q) &= \sum_{n=-\infty}^{\infty}\sum_{r=n}^{\infty}\frac{(-1)^{r-n} z^{r+r-n}\ \ t^{r-r+n}}{2^{r+r-n}\,(q;q)_r^{\ell r+1}\,(q;q)_{r-n}^{\ell r-\ell n+1}}\\ &= \sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r+n}}{2^{2r+n}\,(q;q)_r^{\ell r+1}\,(q;q)_{r+n}^{\ell r+\ell n+1}}\, t^n.\end{aligned}\tag{45}$$

It is worth mentioning here that when $\ell=0$, the inner infinite series on the r. h. s. reduces to the series for q-Bessel function $J_{\nu}^{(1)}(z;q)$ (cf. [4, Ex. 1.24, p. 25]). Hence for $\ell>0$, the inner series provides an extension to $J_{\nu}^{(1)}(z;q)$. We denote it by $J^{\ell}_{n,\Psi}(z;q)$ and call it q-ℓ-Ψ *Bessel function*. In fact, we have

Definition 7 For $\ell\in\mathbf{N}\cup\{0\}$, $z\in\mathbf{C}$, the q-ℓ-Ψ Bessel function $J^{\ell}_{n,\Psi}(z;q)$ is defined as

$$J^{\ell}_{n,\Psi}(z;q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(q;q)_{k+n}^{\ell k+\ell n+1}\,(q;q)_k^{\ell k+1}}\left(\frac{z}{2}\right)^{n+2k}.\tag{46}$$

The series relation in (45), thus provides us the *generating function relation* of q-ℓ-Ψ Bessel function in the form:

$$e^{\ell}_{\Psi}\left(\frac{zt}{2}\right)\, e^{\ell}_{\Psi}\left(\frac{-z}{2t}\right) = \sum_{n=-\infty}^{\infty} J^{\ell}_{n,\Psi}(z;q)\, t^n.$$

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