Chapter 2

Theoretical Models

2.1 Optical Model

The interaction between nucleons and a nucleus or between two nucleons, is a very complicated mechanism to be considered for the analysis of elastic scattering. In the optical model, these complicated interactions are approximated by the simple two body potential between the incident particle and the target nucleus. Historically, the Optical Model was proposed by H. Feshbach, C.E. Porter and V.F. Weisskopf in 1954. It was suggested that the incident single particle, in its encounter with the target nucleus faces an average complex potential[2, 43]. The interaction between incident nucleons and a target nucleus can be treated in analogy to the transmission of light through the cloudy refractive medium. It was also refereed as 'cloudy crystal ball model'.

According to this model the elastic scattering may be considered as the refraction of light and all the non-elastic channels(other than elastic) as an absorption of light. Thus, the optical potential consists of two parts that are real and imaginary. Where, real part gives rise to elastic scattering and the imaginary part represents absorption from elastic channel. This was considered as a complex potential to attribute the possibility of non-elastic scattering that goes into the imaginary part.

There are two aspects to the optical model. The first aspect is related to the fundamental one that deals with the nucleon-nucleon interactions. The second is the phenomenological aspect in which parameters have to be properly chosen and the parameters can be varied to obtain a better agreement with the experimental data. In this work the second alternative has been applied for the optical model analysis of the elastic scattering angular distributions.

2.2 Optical potential

The complex form of optical potential can be written as:
$U(r) = -V_\circ f(r, r_v, a_v) - iW_\circ f(r, r_v, a_v)$

Where, $V_\circ$ and $W_\circ$ are the potential depths, r_v and r_v are the radii and a_v, a_w are the diffuseness of the real and imaginary parts of the potential, respectively. Thus, there are six parameters that one has to vary for obtaining the optimum value from the fitted data.

Also, $f(r, r_\circ, a_\circ)$ is,

$$f(r, r_\circ, a_\circ) = \frac{1}{1 + exp(\frac{r - R_r}{a})} \tag{2.1}$$

Where, $R_r = r_\circ A^{1/3}$ This is a two-parameter Fermi form which can be used in the optical potential due to the fact that the range of nuclear force is short, the radial dependence of an optical model potential $U_{opt}(r)$ follows closely the density distribution in a nucleus. This is a well known Woods-Saxon form of potential[1].

To include the effect of Coulomb potential to the scattering the following conditions should be considered,

$$U_c(r) = Z_p Z e^2 \times \begin{cases} (\frac{3}{2} - \frac{r^2}{2R_{Coul}^2})\frac{1}{R_{Coul}} & ; r \leq R_{Coul} \\ \frac{1}{r} & ; r \geq R_{Coul} \end{cases} \tag{2.2}$$

Where, z and Z are the charge numbers for projectile and target nuclei respectively, and R_C is a Coulomb radius.

Moreover, to account for the spin dependences, a term that includes spin effects is,

$$U_{s.o}(r) = \sigma.l \left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r}\left[V_s \frac{d}{dr} f(r, r_{sv}, a_{sv}) + iW_s \frac{d}{dr} f(r, r_{sw}, a_{sw})\right] \tag{2.3}$$

Here, the derivatives of the volume density distributions are used because the radial dependence comes from analogy with the Thomas spin-orbit potential for the force felt by atomic electrons in the Coulomb field of a nucleus.

Thus, the complete form of Phenomenological potential is[1],

$U(r)$=U_{vol}(r) + $\mathrm{U}_{S.O}$(r) + U_C(r)

2.2.1 Dispersion Relation calculation

Dispersion Relation analysis is useful to interpret the energy dependence of potential parameters $V(r;E)$ and $W(r;E)$ at near the Coulomb barrier. This includes the effects of all couplings in a general but implicit way.

From the optical model analysis of elastic scattering angular distributions, it was seen that the potential parameters $V(r;E)$ and $W(r;E)$ show rapid variation in the vicinity of the Coulomb barrier. Particularly, the $W(r;E)$ sharply decreases when E falls below the top of the Coulomb barrier and this decrease is associated with a bell shaped maximum of $V(r;E)$[44]. The real and imaginary parts of complex optical potential $U(r;E) = V(r;E) + iW(r;E)$, are related with the dispersion relation,

$$\Delta V(r;E) = \frac{P}{\pi}\int_0^\infty \frac{W(r;E')}{E'-E} dE' \tag{2.4}$$

This equation refers to the potential at a given radial distance. Here, P is the principle value, $V_\circ$ is the most smoothly dependent on energy and its energy dependence arises due to non-locality due to exchange.

$$V(r;E) = V_\circ(r;E) + \Delta V(r;E), \tag{2.5}$$

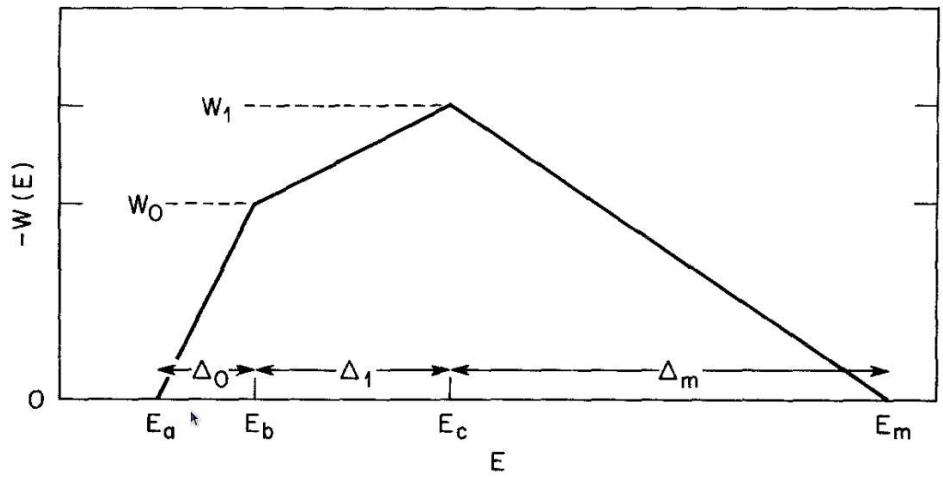

FIGURE 2.1: The linear schematic model for $W(E)$, consisting of three straight-line segments[taken from [44]].

The dispersion term $\Delta V(r,E)$ depends upon $W(r,E)$ at all energies E but the latter is not known so the absolute value of ΔV remains in doubt. However, any localized, rapid variation in the imaginary potential $W(r,E)$ must be accompanied by a similarly localized variation in the real term $\Delta V(r,E)$.

In the present work, linear schematic model have been used to study the energy dependence of the potential parameters. Using this the real part $V(E)$ can be calculated from the $W(E)$ which is composed of three linear segments as shown in Fig.2.1.

The algebraic equation for the $\Delta V(r,E)$ will be,

$$\Delta V(E) = (E - E_b)\frac{P}{\pi}\int_0^\infty \frac{W(E')}{(E' - E_b)(E' - E)}dE' \tag{2.6}$$

Here, E_b is the reference energy. For one more reference energy E_s, $\Delta V(E)$ will be,

$$\Delta V_{Es}(E) = \Delta V(E) - \Delta V(E_s) \tag{2.7}$$

The contribution to $\Delta V(E)$ from the increment (W_{ij}=$W(E_i) - W(E_j)$) of each line segments will be,

$$\Delta V_{ij}(E) = \frac{W_{ij}}{\pi}[\epsilon_i ln|\epsilon_i| - \epsilon_j ln|\epsilon_j|], \tag{2.8}$$

Here, ϵ_i=$\frac{(E-E_i)}{\Delta_{ij}}$, ϵ_j=$\frac{(E-E_j)}{\Delta_{ij}}$, and Δ_{ij}=$(E_j - E_i)$.

The resultant real part will be,

$$\pi\Delta V(E) = W_\circ[\epsilon_a ln|\epsilon_a| - \epsilon_b ln|\epsilon_b|] + (W_1 - W_\circ)[\epsilon_b^{'} ln|\epsilon_b^{'}| - \epsilon_c^{'} ln|\epsilon_c^{'}|] - \tag{2.9}$$

$$W_1[\epsilon_c^{''} ln|\epsilon_c^{''}| - \epsilon_m^{''} ln|\epsilon_m^{''}|] + W_1[\eta ln\eta - (n+1)ln(\eta+1)] \tag{2.10}$$

Where, $W_\circ$, $W_1 \geq 0$ and ϵ_i=$\frac{(E-E_i)}{\Delta_\circ}$, $\epsilon_i^{'}$=$\frac{(E-E_i)}{\Delta_1}$, $\epsilon_i^{''}$=$\frac{(E-E_i)}{\Delta_m}$, η=$\frac{\Delta_1}{\Delta_m}$.

2.3 One Dimensional Barrier Penetration Model

The one dimensional barrier penetration model (1DBPM) is used frequently to interpret fusion cross section data. Here, the nuclear fusion is based on the single parameter, radial distance, therefore it is referred as the one dimensional model. Classically, fusion occurs when transmission coefficient (T_l) is one and it doesn't for T_l=0. That means fusion can not take place at below the Coulomb barrier energy. However, quantum mechanically at below the Coulomb barrier energy also, the fusion can occur by tunneling through the barrier[45, 46].

In case of heavy ion reaction the fusion cross sections can be calculated from the relation as given below:

$$\sigma_{fus}(E) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)T_l \tag{2.11}$$

The transmission coefficients due to the inverted parabola, can be written using the Hill-Wheeler formula[47],

$$T_l(E) = \left(1 + exp\left[\frac{2\pi}{\hbar\omega_l}(V_b(l) - E)\right]\right)^{-1} \tag{2.12}$$

where $V_b(l)$ is the barrier height and $\hbar\omega_l$ is the corresponding barrier curvature for the corresponding l^{th} partial wave. It is assumed that the barrier position R_b and the barrier curvature $\hbar\omega_b$ do not change with the angular momentum.

The equation of fusion cross section is reformed by replacing sum by integration:

$$\sigma_{fus}(E) = \frac{\hbar\omega_b R_b^2}{2E} ln\left(1 + exp\left[\frac{2\pi}{\hbar\omega_b}(E - V_b)\right]\right) \tag{2.13}$$

where, V_b, R_b, $\hbar\omega_b$ are the barrier height, position and the curvature for $l = 0$, respectively.

At energies $E \gg V_B$ the cross section can be approximated to the simple classical expression as given below:

$$\sigma_{fus}(E) = \pi R_b^2\left(1 - \frac{V_b}{E}\right) \tag{2.14}$$

For energies $E \ll V_B$ the cross section can be approximated to the,

$$\sigma_{fus}(E) = \frac{\hbar\omega_b R_b^2}{2E} exp\left[\frac{2\pi}{\hbar\omega_b}(E - V_b)\right] \tag{2.15}$$

From above equation it is understood that as the beam energy increases, the fusion cross sections also increases exponentially.

2.4 Decay of Compound Nucleus: Statistical model

The information about the decay of the equilibrated compound nucleus can be derived from the statistical model analysis. The concept of compound nucleus formation was proposed by Niels Bohr in 1936[43]. In the nuclear reaction, when

a projectile x enters into a target nucleus X, an intermediate stage is formed before the production of the final nuclei Y and y.

$$x + X \rightarrow C^* \rightarrow Y + y \tag{2.16}$$

While entering into the target nucleus, projectile quickly dissipates its energy and merges with the closely packed nucleons. In this way all the nucleons in the target nucleus gains additional energy but their random motion gets disturbed. Not a single nucleon of them will have enough energy to come out of the nucleus which is of the order of few million electron volts. However, after a long time when a large number of collision occurs among nucleons, enough energy may be concentrated on one of them enabling to escape from the nucleus which then deexcites or cools off to the ground state. After decaying by emitting nucleons and clusters the remaining residual nucleus is called evaporation residues (ER) which may further decay by emitting γ rays or by β-decay or by fission. After emission of light particles from the compound nucleus if its energy is sufficiently high (above about 10 MeV) then the fusion cross-section can be measured by detecting evaporation residues. This has been the most direct method for light and medium mass systems (A $<$ 80) at excitation energies up to about 2 MeV/nucleon. While, at higher energies it may be necessary to distinguish between evaporation residues and products of deep inelastic scattering. In case of heavy compound nuclei, fission is a competing decay mode where the compound nucleus cross-section is the sum of the cross-sections corresponding either to evaporation residues or to pairs of fission fragments. In the statistical model, all the decay channels of the Compound Nucleus(CN) are taken to be equally probable and depend on the factors such as density of final states and barrier penetration factors [48].

The Statistical model code (PACE)

In the present work, the Statistical model code PACE (The Projection Angular-Momentum Coupled Evaporation) has been used to study the evaporated α particles, which has been developed by A. Gavron[49]. Basically it uses Monte

Carlo simulation method to obtain the decay sequence employing Hauser Feshbech method. From this calculation one can obtain the energy spectra, angular distributions of particles and multi particle correlation in the lab frame. Using this code one can trace different events for the entire decay sequence from compound nucleus to any of the out going channel.

2.5 Fusion cross section calculations

At energies near and below the Coulomb barrier energy, the fusion cross sections are strongly influenced by coupling of the relative motion of the colliding nuclei to several nuclear intrinsic motions. The CCFULL code has been widely used to compute the fusion cross sections and mean angular momenta of the compound nucleus which considers the coupling to all order. This is a FORTRAN 77 program that considers the isocentrifugal approximation and solves the coupled channels equations [50].

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{J(J+1)\hbar^2}{2\mu r^2}+V_N^{(0)}(r)+\frac{Z_P Z_T e^2}{r}+\epsilon_n-E\right]\psi_n(r)+\sum_m V_{nm}(r)\psi_m(r)=0 \tag{2.17}$$

Where, r is the radial component, μ is the reduced mass, E is the energy in the center of mass system, ϵ_n is the excitation energy of the n^{th} channel and V_{nm} are the matrix elements of the coupling Hamiltonian, which consist of Coulomb and nuclear components.

It employs Coulomb excitations and the incoming wave boundary condition inside the Coulomb barrier. According to incoming wave boundary conditions(IWBC), the CCFULL program will solve the coupled channels equations at a minimum position r=r_{min} where inside the Coulomb barrier strong absorption take place. In this method of calculations the numerical solution is matched to a linear combination of incoming and outgoing and Coulomb wave functions at finite distance r_{max} beyond which both the nuclear and the Coulomb coupling are sufficiently small.

The solution of the coupled-channel equations 2.18 with the proper boundary conditions is,

$$\psi_m(r) = \sum_n T_n \chi_{nm}(r) \tag{2.18}$$

From this, the transmission coefficients are obtained and finally the total transmission probability is obtained by summing over the distribution of transmission probabilities.

$$P_J(E) = \sum_n \frac{k_n(r_{min})}{k_\circ} |{T_n}^2| \tag{2.19}$$

The fusion cross section and the mean angular momentum of compound nucleus are thus calculated by,

$$\sigma_{fus}(E) = \sum_J \sigma_J(E) = \frac{\pi}{k_\circ^2} \sum_J (2J+1) P_J(E) \tag{2.20}$$

$$< l >= \sum_J J\sigma_J(E) / \sum_J \sigma_J(E) \tag{2.21}$$

$$= \frac{\pi}{k_\circ^2} \sum_J J(2J+1) P_J(E) / \frac{\pi}{k_\circ^2} \sum_J (2J+1) P_J(E) \tag{2.22}$$

2.6 Coupled Channels calculations

During the interaction process, apart from elastic scattering several other non-elastic collisions may also take place particularly when one or both of the interacting nuclei are deformed. Particle in any of the nuclei may get excited during the reaction from its initial bound state to another state which may be bound or unbound. Also, the nucleon transfer may occur from one nucleus to the other, either singly, or as the simultaneous transfer of two nucleon as a particle cluster. Theoretical models can be used to predict the effects of these multi-step processes to any or all orders, in terms of coupled channels formalism. The calculation that include only bound inelastic states and no transfer or breakup are added, is referred as simple coupled channels calculation. If the transfer channels are added in different partitions then it is termed as coupled reaction channels (CRC) calculation. If the breakup channels in the continuum are included with some discretization method

then it is known as Continuum Discretized Coupled Channel(CDCC) formalism. In the present thesis work, the coupled channels calculations have been done in terms of CRC and CDCC formalism using FRESCO code[42].

2.6.1 Coupled Reaction Channels(CRC) formalism

In the coupled reaction channels (CRC) model of direct reactions, a model of the system wave function is created and Schrödinger's equation is solved as accurately as possible within that model space. Here, P is defined as the projection operator onto the model space. The model used here projects the complete wave function $\overline{\Psi}$ onto a product $\phi_i \equiv \phi_{ip} * \phi_{it}$ of projectile and target states with a wave function $\psi_i(\vec{R}_i)$ describing their relative motion:

$$P\overline{\Psi} \equiv \Psi = \sum_{i}^{N} \phi_i \psi_i(\vec{R}_i) \tag{2.23}$$

Here, the basis states ϕ_{ip} and ϕ_{it} can be bound states of their respective nuclei, or they may be discrete representations of continuum levels for particular i^{th} channel. The states ϕ_i can be in different mass partitions, or they can be different excited states of the projectile and/or the target. In this(CRC) framework there is a finite set (say N) of square-integrable basis states, as this leads to a finite set of equations coupling the channel wave functions $\psi_i(\vec{R}_i)$ as unknowns. In the model space, the Schrödinger's equation $[\overline{\mathcal{H}} - E]\overline{\Psi} = 0$ will be changed to $[\mathcal{H} - E]\Psi = 0$, for a complete Hamiltonian $\overline{\mathcal{H}}$ and total energy E. Here,

$$\mathcal{H} = P\overline{\mathcal{H}}P - P\overline{\mathcal{H}}Q\frac{1}{Q\overline{\mathcal{H}}Q - E - i\epsilon}\, Q\overline{\mathcal{H}}P, \tag{2.24}$$

And $Q \equiv 1 - P$ and ϵ is a positive infinitesimal quantity whose presence ensures that the excluded channels have a time-retarded propagator, and hence removes flux from the model space. Here, the second term as a whole describes the effects of the excluded channels on the model subspace $P\overline{\Psi}$. These effects may have contributions from the compound nucleus formation. Due to the lack of detailed knowledge of these effects, the model Hamiltonian $\mathcal{H}$ uses effective potentials which

approximates the processes in some average manner. The effective potentials are optical potentials with real and imaginary components fitted to some simpler kinds of reactions, and the compound nucleus formation contributes the imaginary component. The projected model Hamiltonian $\mathcal{H}$ for the individual basis states ϕ_i in the CRC system satisfies the relation as given below:

$$H_i - E_i = \langle \phi_i | \mathcal{H} - E | \phi_i \rangle \tag{2.25}$$

Here, E_i is the asymptotic kinetic energy in the i^{th} channel and H_i is the channel-projected Hamiltonian. This will be composed of a kinetic energy term and a diagonal optical potential. The 'interaction potential' V_i is also a part of the $\mathcal{H}$ which is not included in H_i. Now,

$$H_i - E_i + V_i = \mathcal{H} - E. \tag{2.26}$$

This provides V_i which have the vanishing diagonal matrix elements $\langle \phi_i | V_i | \phi_i \rangle = 0$. The model Schrödinger's equation $[\mathcal{H} - E]\Psi = 0$, after projecting separately onto the different basis states ϕ_i, the set of equations:

$$[E_i - H_i]\, \psi_i(\vec{R}_i) = \sum_{j \neq i} \langle \phi_i | \mathcal{H} - E | \phi_j \rangle\, \psi_j(\vec{R}_j). \tag{2.27}$$

These equations couple together the unknown wave functions $\psi_i(\vec{R}_i)$. The matrix element $\langle \phi_i | \mathcal{H} - E | \phi_k \rangle$ has two different forms, depending on whether we expand

$$\begin{aligned} \mathcal{H} - E &= H_i - E_i + V_i \text{ (post form)} \\ &= H_j - E_j + V_j \text{ (prior form).} \end{aligned}$$

So,

$$\begin{aligned} \langle \phi_i | \mathcal{H} - E | \phi_j \rangle &= V_{ij}^{\text{post}} + [H_i - E_i] K_{ij}(\text{post}) \\ \text{or} &= V_{ij}^{\text{prior}} + K_{ij}[H_j - E_j](\text{prior}) \end{aligned} \tag{2.28}$$

where,

$$V_{ij}^{\text{post}} \equiv \langle \phi_i | V_i | \phi_j \rangle, \quad V_{ij}^{\text{prior}} \equiv \langle \phi_i | V_j | \phi_j \rangle, \quad K_{ij} \equiv \langle \phi_i | \phi_j \rangle. \tag{2.29}$$

The overlap function $K_{ij} = \langle \phi_i | \phi_j \rangle$ in equation 2.28 arises from the well-known non-orthogonality between the basis states ϕ_i and ϕ_j if these are in different mass partitions. The more details are given in the Ref.[51].

2.6.2 Continuum Discretized Coupled Channel(CDCC) formalism

A theoretical formalism called the Continuum Discretized Coupled Channel(CDCC) method is used to study the projectile breakup effects on different reaction processes such as the inelastic scattering and transfer reaction through the various multi step processes. It was first introduced in the study of deuteron breakup process. The CDCC method can be generalized to include the target excitation channels as well as the projectile-target mutual excitation channels. Thus, the inelastic scattering and inelastic breakup reactions can also be studied together with their elastic counterparts. For the reactions which involve weakly bound nuclei, this method is currently most reliable and practical theoretical tool to study the breakup process and its role on other reactions. In the present thesis work CDCC calculations have been carried out for ^{6,7}Li projectiles. In this method, the breakup continuum states are described in terms of a finite number of discrete states. In the calculation the continuum discretized states are treated exactly in a coupled channels framework. In order to have finite representation of original continuum states three standard ways:

- *The mid point method, which consists of $U_{kp}(r)$ for a discrete set $p = p_0 ... p_n$ of scattering energies such as $\epsilon_p = \epsilon_{min} + (p - p_0)\Delta\epsilon$, where ϵ_{min} and $\epsilon_{min} + (p_n - p_0)\Delta\epsilon$, the minimum and maximum excitation energies to be included in the model space.*

- *The second one is pseudo states which are simply the eigen state of the internal Hamiltonian H_{int} on some convenient square integrable basis. This pseudo state decay to zero at large distances and have no simple relation to the $v + c$ scattering solutions $U_k(r)$.*

- *The third one is the continuum or average method, where the scattering wave functions are averaged over k to be made square integrable.*

The radial functions $U_p(r)$ for each 'bin' states are superposition of the scattering eigen states for some weight function $g_p(k)$ as,

$$U_p(r) \;=\; \sqrt{\frac{2}{\pi N_p}} \;\; \int_{k_{p-1}}^{k_p} g_p(k) U_k(r) dk \tag{2.30}$$

With normalization constant,

$$N = \int_{k_{p-1}}^{k_p} |g_p(k)|^2 dk \tag{2.31}$$

Such that $U_p(r)$ form an orthonormal set when all the (k_{p-1}, k_p) are non overlapping continuum intervals. Suppose to have amplitude for a cross section $a(k) = \langle u_k(r)|\Omega(r)\rangle$, for some source term $\Omega(r)$. If we insert a set of bin functions,

$$a(k) = \langle u_k(r)|\Omega(r)\rangle \equiv \sum_p \langle u_k(r)|U_p(r)\rangle\langle U_p(r)|\Omega(r)\rangle \tag{2.32}$$

So, to have the $\langle u_k(r)|U_p(r)\rangle$ overlap of bin wave function and true scattering wave function $u_k(r)$, $g_p(k)$ should be chosen to be able to reproduce $a(k)$.For the bins far from the resonance state large r should be chosen. Thus, $u_k(r)$ α $e^{i\delta_l(k)} sin(kr - l\pi/2 + \delta l(k))$. Where, $\delta l(k)$ are the phase shifts of the scattering states. From the equation 2.32, the $a(k)$ will be,

$$e^{-i\delta_l(k)} \langle \sin(kr - l\pi/2 + \delta_l(k))|\Omega(r)\rangle \tag{2.33}$$

The resonant behavior of $a(k)$ can be reproduced by setting
$g_p(K) = e^{-i\delta_l(k)} \sin\delta_l(k)$. The inclusion of $e^{-i\delta_l(k)}$ factor in the $g_p(K)$, make the

wave function real valued. Same as the average method the pseudo state method also based on square integrable functions, however the mid point method does not use square integrable basis state.

2.7 Extraction of fusion barrier distributions

In the present thesis work fusion barrier distributions have been carried out from quasi-elastic as well as fusion excitation functions. A brief introduction to the fusion barrier distributions is given in the first chapter of this thesis.

2.7.1 Barrier distribution from quasi-elastic excitation function

Classically, the derivation of single barrier from the quasi-elastic scattering is already given in the section 2.7. Several barriers may be described by a coupled channel model which considers coupling to a finite number of states but neglects their excitation energies (adiabatic approximation) and uses the isocentrifugal approximation as described in the section 2.5.

Considering the form factor $F(r)$ of the coupling interaction as channel independent, the coupled-channel equations will be,

$$(T + V(r) + \Lambda_\alpha F(r) - E)\Psi_\alpha = 0 \tag{2.34}$$

where, E is the incident energy and Λ_α represents diagonal elements of coupling matrix for channel (α). T and V are respectively, the kinetic and potential energies. And, the Ψ_α are obtained such that the coupling interaction V_{ij}=$F(r)\Gamma_{ij}$ becomes diagonal. Γ_{ij} represents coupling matrix.

$$\Psi_\alpha = U^\dagger_{\alpha i}\phi^{phys}_i \tag{2.35}$$

Where, $U_{\alpha i}$ is defined for unitary transformation and Ψ_α is the eigen channel wave function. ϕ_i^{phys} is the nuclear wave functions in the physical channels i. So, that

$$U_{\alpha i}\Gamma_{ij}U_{j\beta}^{\dagger} = \Lambda_{\alpha i}\delta_{\alpha\beta} \tag{2.36}$$

The physical S-matrix elements will be[24],

$$S_{i;0}^{phys} = U_{i\alpha}U_{0\alpha}S_\alpha \tag{2.37}$$

Now, putting this values in the equation of total fusion cross section will be,

$$\sigma^f = \frac{\pi}{k^2}\sum_{l,i,\alpha\beta}(2l+1)(1 - U_{i\alpha}U_{0\alpha}S_\alpha^l U_{i\beta}^* U_{0\alpha}^* S_\beta^{l*}) \tag{2.38}$$

From the relation

$$\sum_i U_{i\alpha}U_{i\beta}^* = \sum_i U_{\beta i}^{\dagger}U_{i\alpha} = \delta_{\alpha\beta} \tag{2.39}$$

The equation 2.38 will be ,

$$\sigma^f = \frac{\pi}{k^2}\sum_{l,\alpha} W_\alpha(2l+1)(1 - |S_\alpha^l|^2) = \sum_\alpha W_\alpha\sigma_\alpha^f \tag{2.40}$$

Now, from the equation 1.6,

$$D^f = \frac{dT}{dE} = \frac{1}{\pi r^2}\times\frac{d^2(E.\sigma^f)}{dE^2} \quad = \sum_\alpha W_\alpha\frac{d^2(E.\sigma_\alpha^f)}{dE^2} \quad = \sum_\alpha W_\alpha D_\alpha^f \tag{2.41}$$

In this way for multiple fusion barriers in the adiabatic and isocentrifugal approximation, a representation $D^f(E)$ of the fusion barrier distribution is given by the second differential of $E.\sigma^f$ with respect to the energy. The scattering amplitudes in the physical channels as:

$$f_i(\theta) = \frac{1}{2ik}\sum_i(2l+1)P_l(\cos\theta)exp(2i\sigma_\alpha^l)S_i^l \tag{2.42}$$

$$= \frac{1}{2ik}\sum_i(2l+1)P_l(\cos\theta)exp(2i\sigma_\alpha^l)U_{i\alpha}U_{0\alpha}S_\alpha^l \tag{2.43}$$

Here, σ_α^l are the Coulomb phases. After summing over the squares of the elastic and inelastic amplitudes yields using equation 2.39, the quasi-elastic scattering differential cross section will be,

$$\frac{\sigma^{qel}}{\sigma^R} = \frac{\sigma^{el}}{\sigma^R} + \sum_{inel} \frac{\sigma^{inel}}{\sigma^R} = \sum_\alpha W_\alpha \frac{\sigma_\alpha^{el}}{\sigma_R} \tag{2.44}$$

Hence, the quasi-elastic differential cross section is a weighted sum of the eigen channel elastic differential cross sections.

$$D^{qel(E)} = \frac{d}{dE}(\frac{\sigma^{qel}}{\sigma^R}) = \sum_\alpha W_\alpha D_\alpha^{el} \tag{2.45}$$

This reflects the distribution of the barrier weights W_α.

2.7.2 Barrier distribution from fusion excitation function

The representation of fusion barrier distribution can also be obtained from the measured fusion cross sections due to the fact that $T_0 + R_0 = 1$. The coupling of various reaction channels can reveal the distributions in barrier heights. The equation of total fusion cross-section for the l^{th} partial wave can be written as,

$$\sigma_{fus}(E) = \frac{\pi}{k_\circ^2} \sum_l (2l+1) T_l(E) \tag{2.46}$$

The scattering matrix from the transmission probability,

$$T_l(E) = 1 - |S_l|^2 \tag{2.47}$$

Inserting this value in the equation 2.46,

$$\sigma_{fus}(E) = \frac{\pi}{k_\circ^2} \sum_l (2l+1)(1 - |S_l|^2) \tag{2.48}$$

The transmission functions $T_l(E)$, in case of inverted parabola is,

$$T_l(E) = 1 + exp[\frac{2\pi}{\hbar\omega_l}(B_l - E)] \tag{2.49}$$

Now, equation 2.48 will be,

$$\sigma_{fus}(E) = \frac{\hbar\omega R^2}{2E} ln(1 + exp[\frac{2\pi}{\hbar\omega}(E - B)]) \tag{2.50}$$

Here, B is the barrier height and $\hbar\omega_0$ is the barrier curvature for $l = 0$. R is the barrier position. Now, the classical expression for a single barrier is:

$$\sigma(E, B) = \pi R^2(1 - \frac{B}{E})(E > B) \tag{2.51}$$

$$= 0(E < B). \tag{2.52}$$

The one-dimensional Coulomb barrier can be replaced by a continuous distribution of fusion barriers $D(B)$ by the σ_f,

$$\sigma_{fus}(E) = \int_0^\infty \sigma(E, B)D(B)dB \tag{2.53}$$

where $\sigma(E, B)$ is the cross-section, summed over all partial waves l, for a single barrier B and $\int D(B)dB = 1$ [52, 53]. The double derivative of transmission co-efficient T_l (1.5) gives delta function as given by,

$$\frac{dT(E)}{dE} = \frac{1}{\pi r^2}\frac{d^2(E.\sigma_{Fus})}{dE^2} = \delta(E - B) \tag{2.54}$$

Inserting this equation into equation 2.53, the total fusion cross-section will become,

$$\frac{dT(E)}{dE} = \frac{1}{\pi r^2}\frac{d^2(E.\sigma_{Fus})}{dE^2} = D(E - B) \tag{2.55}$$

This equation can be used to obtain the barrier distribution directly from the measured fusion cross-sections.