CHAPTER - 2

AN INVENTORY MODEL FOR WEIBULL DETERIORATING ITEMS WITH EXPONENTIAL DEMAND AND TIME-VARYING HOLDING COST

2.1 Introduction

In this chapter, an inventory model is developed to minimize the total cost for Weibull deteriorating inventory systems with linear time-varying holding cost and exponentially increasing (or decreasing) demand. The objective of this study is to obtain an optimal order quantity and optimal order cycle such that the total cost becomes minimum.

2.2 Assumptions

- The demand D is a function of time, given by

  D (t) = $ke^{\gamma t}$; $|\gamma| \ll 1$
- The holding cost C_h is a linear function of time, given by

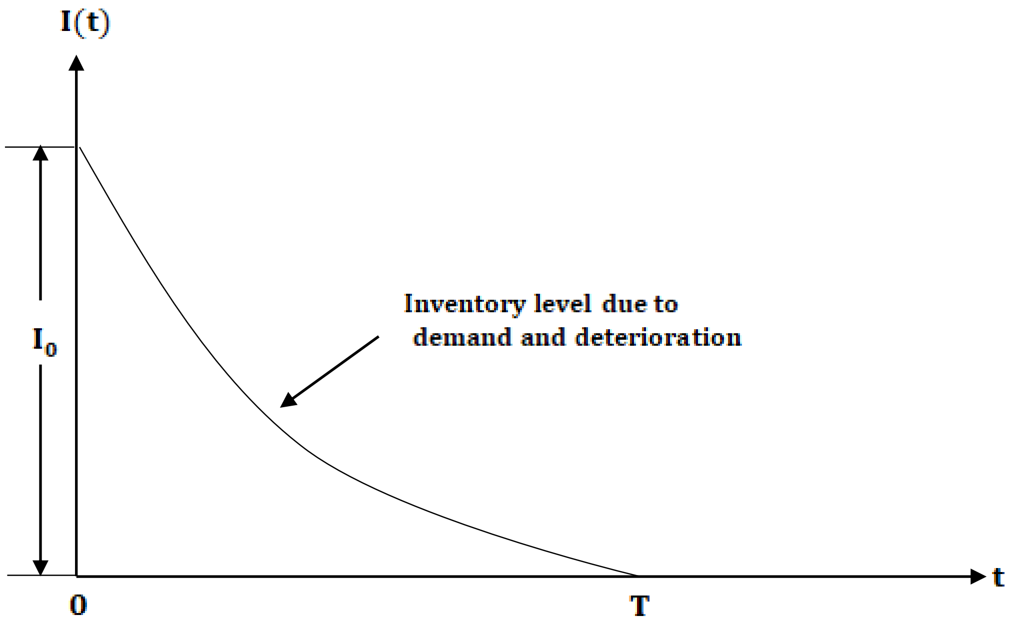
  $C_h(t) = x + yt;$ where, x and y are constants.
- The deterioration rate $\theta(t)$ of an item in the inventory system follows the two parameter Weibull distribution deterioration rate, given by
- $\theta(t) = \alpha\beta t^{\beta-1}$; where $0 \leq \alpha \ll 1,\ \beta > 0.$
- Instant and infinite replenishment rate.
- Shortages are not permitted.
- Lead time is zero.

2.3 Model Development

As shown in figure 2.3.1 at $t = 0$, the initial inventory in the system is I_0. Due to the demand and deterioration, the level of the inventory will decrease continuously with time and become zero at time $T = 0$. The change in inventory level can be described by the differential equation (2.3.1).

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1} I(t) = -ke^{\gamma t}, \qquad 0 \le t \le T \tag{2.3.1}$$

Figure: 2.3.1 Graphical depiction of the inventory level.



The solution of the linear differential equation (2.3.1) is

$$I(t) = \frac{\int -ke^{\gamma t} \cdot e^{\int \alpha\beta t^{\beta-1} dt} dt + c}{e^{\int \alpha\beta t^{\beta-1} dt}}$$

$$= \frac{\int -ke^{\gamma t} \cdot e^{\alpha t^{\beta}} dt + c}{e^{\alpha t^{\beta}}} \tag{2.3.2}$$

Using the boundary condition $I(T) = 0$, expanding the $e^{(\cdot)}$ terms according to the tailor's series ignoring the higher power terms greater than 1 and then simplifying we get the solution as given in (2.3.3).

$$I(t) = k\left[(T-t) + \frac{\gamma(T^2-t^2)}{2} + \frac{\alpha\left(T^{\beta+1}-t^{\beta+1}\right)}{\beta+1}\right.$$

$$\left. + \frac{\alpha\gamma\left(T^{\beta+2}-t^{\beta+2}\right)}{\beta+2} + \alpha t^{\beta+1} - \alpha T t^{\beta}\right] \quad (2.3.3)$$

Substituting $t = 0$ in (2.3.3), we get the initial order quantity I_0 at $t = 0$.

$$I_0 = I(0) = k\left[T + \frac{\gamma T^2}{2} + \frac{\alpha T^{\beta+1}}{\beta+1} + \frac{\alpha\gamma T^{\beta+2}}{\beta+2}\right] \quad (2.3.4)$$

The ordering cost (OC) per unit time is

$$OC = \frac{C_o}{T} \quad (2.3.5)$$

The total demand per order cycle [0, T] is

$$\int_0^T D(t)\,dt = \int_0^T k\,e^{\gamma t}\,dt$$

$$= \frac{\mathrm{k}}{\gamma}[\mathrm{e}^{\gamma \mathrm{T}} - 1] \quad (2.3.6)$$

The number of deteriorated units during the interval [0, T] is

$$I_0 - \int_0^T D(t)\,dt = k\left[T + \frac{\gamma T^2}{2} + \frac{\alpha T^{\beta+1}}{\beta+1} + \frac{\alpha\gamma T^{\beta+2}}{\beta+2}\right]$$

$$-\frac{k}{\gamma}[e^{\gamma T} - 1] \quad (2.3.7)$$

The deterioration cost per unit time is

$$DC = \frac{\mathrm{kC_d}}{\mathrm{T}}\left[\mathrm{T} + \frac{\gamma \mathrm{T}^2}{2} + \frac{\alpha \mathrm{T}^{\beta+1}}{\beta+1} + \frac{\alpha\gamma \mathrm{T}^{\beta+2}}{\beta+2}\right] - \frac{\mathrm{kC_d}}{\gamma \mathrm{T}}[\mathrm{e}^{\gamma \mathrm{T}} - 1] \quad (2.3.8)$$

Inventory holding cost per unit time is

$$HC = \frac{1}{T}\int_0^T C_h(t)I(t)\ dt$$

$$= \frac{xk}{T}\left[\frac{T^2}{2} + \frac{\gamma T^3}{3} + \frac{\alpha\beta T^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha\gamma T^{\beta+3}}{\beta+3}\right]$$

$$+\frac{yk}{T}\left[\frac{T^3}{6} + \frac{\gamma T^4}{8} + \frac{\alpha\beta T^{\beta+3}}{2(\beta+2)(\beta+3)} + \frac{\alpha\gamma T^{\beta+4}}{2(\beta+4)}\right] \qquad (2.3.9)$$

Total inventory cost per unit time is

$$TC = \text{OC} + \text{DC} + \text{HC}$$

$$= \frac{C_o}{T} + \frac{kC_d}{T}\left[T + \frac{\gamma T^2}{2} + \frac{\alpha T^{\beta+1}}{\beta+1} + \frac{\alpha\gamma T^{\beta+2}}{\beta+2}\right]$$

$$-\frac{kC_d}{\gamma T}[e^{\gamma T} - 1]$$

$$+\frac{xk}{T}\left[\frac{T^2}{2} + \frac{\gamma T^3}{3} + \frac{\alpha\beta T^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha\gamma T^{\beta+3}}{\beta+3}\right]$$

$$+\frac{yk}{T}\left[\frac{T^3}{6} + \frac{\gamma T^4}{8} + \frac{\alpha\beta T^{\beta+3}}{2(\beta+2)(\beta+3)} + \frac{\alpha\gamma T^{\beta+4}}{2(\beta+4)}\right] \qquad (2.3.10)$$

The necessary and sufficient conditions to minimize TC for a given value T are

$\frac{\partial\, TC(T)}{\partial T} = 0$ and $\frac{\partial^2 TC(T)}{\partial T^2} > 0$, respectively.

$$\frac{\partial\, TC(T)}{\partial T} = -\frac{C_o}{T^2}$$

$$+kC_d\left[\frac{\gamma}{2} + \frac{\alpha\beta T^{\beta-1}}{\beta+1} + \frac{\alpha\gamma(\beta+1)T^{\beta}}{\beta+2}\right]$$

$$-\frac{kC_d}{\gamma T^2}[\gamma T e^{\gamma T} - e^{\gamma T} + 1]$$

$$+ xk\,[\frac{1}{2} + \frac{2\gamma T}{3} + \frac{\alpha\beta T^{\beta}}{\beta+2} + \frac{\alpha\gamma(\beta+2)T^{\beta+1}}{\beta+3}]$$

$$+ yk\,[\frac{T}{3} + \frac{3\gamma T^2}{8} + \frac{\alpha\beta T^{\beta+1}}{2(\beta+3)} + \frac{\alpha\gamma(\beta+3)T^{\beta+2}}{2(\beta+4)}] \qquad (2.3.11)$$

$$\frac{\partial^2 \mathrm{TC(T)}}{\partial \mathrm{T}^2} = \frac{2C_o}{\mathrm{T}^3} + \mathrm{kC_d}\left[\frac{\alpha\beta(\beta-1)\mathrm{T}^{\beta-2}}{\beta+1} + \frac{\alpha\gamma\beta(\beta+1)\mathrm{T}^{\beta-1}}{\beta+2}\right]$$

$$-\frac{kC_d}{\gamma}[\left(\frac{T\gamma^2 e^{\gamma T} - \gamma e^{\gamma T}}{T^2}\right) - \left(\frac{\gamma T^2 e^{\gamma T} - 2Te^{\gamma T}}{T^4}\right) - \frac{2}{T^3}]$$

$$+xk\,[\frac{2\gamma}{3} + \frac{\alpha\beta^2 T^{\beta-1}}{\beta+2} + \frac{\alpha\gamma(\beta+1)(\beta+2)T^{\beta}}{\beta+3}]$$

$$+yk\,[\frac{1}{3} + \frac{3\gamma T}{4} + \frac{\alpha\beta(\beta+1)T^{\beta}}{2(\beta+3)} + \frac{\alpha\gamma(\beta+2)(\beta+3)T^{\beta+1}}{2(\beta+4)}] \qquad (2.3.12)$$

If $T^*$is the optimal order cycle, then putting $T = T^*$ in equation (2.3.4), we get the optimal order quantity (Q^*).

$$Q^* = k\left[T^* + \frac{\gamma {T^*}^2}{2} + \frac{\alpha {T^*}^{\beta+1}}{\beta+1} + \frac{\alpha\gamma {T^*}^{\beta+2}}{\beta+2}\right] \qquad (2.3.13)$$

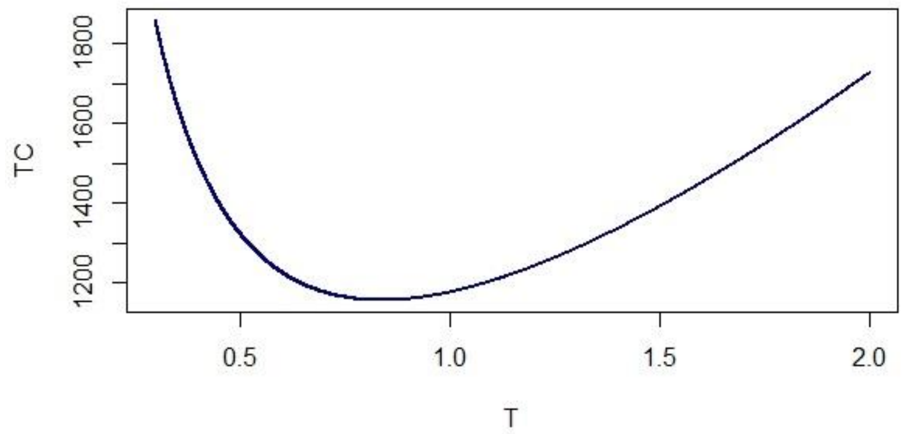
2.4 Examples

Example-1 Consider the following parameter values in proper units.

$\alpha = 0.04,\ \beta = 2,\ \mathrm{C_d} = 15,\ C_o = 500,\ \gamma = -0.02,\ \mathrm{k} = 250,\ \ \mathrm{x} = 5,\ \mathrm{y} = 0.05$

In R programming, solving the equation (2.3.10) with the above values of parameters, we obtain the optimal order cycle $T^* = 0.837232$ and from equation (2.3.3) and (2.3.10), the optimal order quantity $Q^* = 209.487252$ and the optimal total inventory cost $TC^* = 1155.314$.

Figure 2.4.1 Convexity of the Total cost function (Example-1)

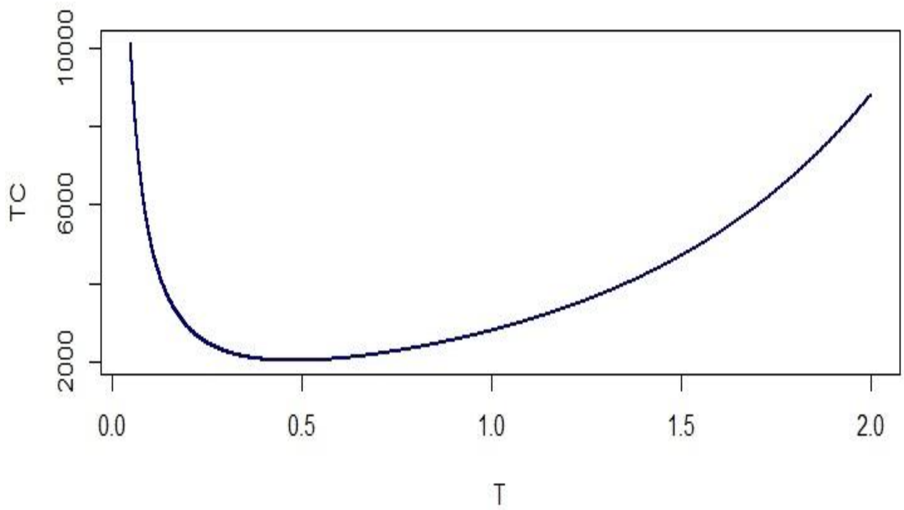


Example-2 Consider the following parameter values in proper units.

$\alpha = 0.08,\ \beta = 4,\ C_d = 20,\ C_o = 500,\ \gamma = 0.1,\ k = 400,\ x = 10,\ y = 0.05$

In R programming, solving the equation (2.3.10) with the above values of parameters, we obtain the optimal order cycle $T^* = 0.4779953$ and from equation (2.3.3) and (2.3.10), the optimum order quantity = 195.933769 and the optimal total inventory cost $TC^* = 2038.264$.

Figure 2.4.2 Convexity of the Total cost function (Example-2).



Figures 2.4.2 and 2.4.3 reveals that the total cost function is convex.

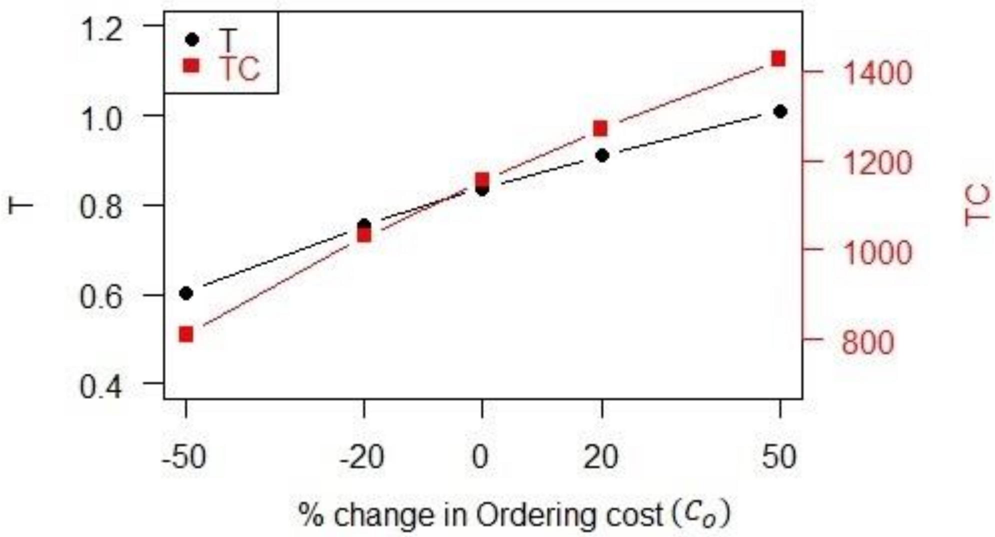
2.5 Sensitivity Analysis

In this section, we will observe the impact of different parameters on the ordering policies by changing the current values (in example-1) by -50%, -20%, 0%, +20%, and +50%, respectively.

Table 2.5.1 Sensitivity of the parameter C_o.

Parameter		Change	T*	Q*	TC*
C_o	750	+50%	1.00998	253.3347	1425.798
	600	+20%	0.91125	228.2302	1269.683
	500	0%	0.83723	209.4873	1155.314
	400	-20%	0.75421	188.5458	1029.665
	250	-50%	0.60384	150.7767	808.9180

Figure 2.5.1 Effect of C_o on T and TC.



Observations from the table 2.5.1 and figure 2.5.1:

As the ordering cost C_o increases,

(a) The length of the order cycle (T) increases.

(b) The order quantity (Q) increases.

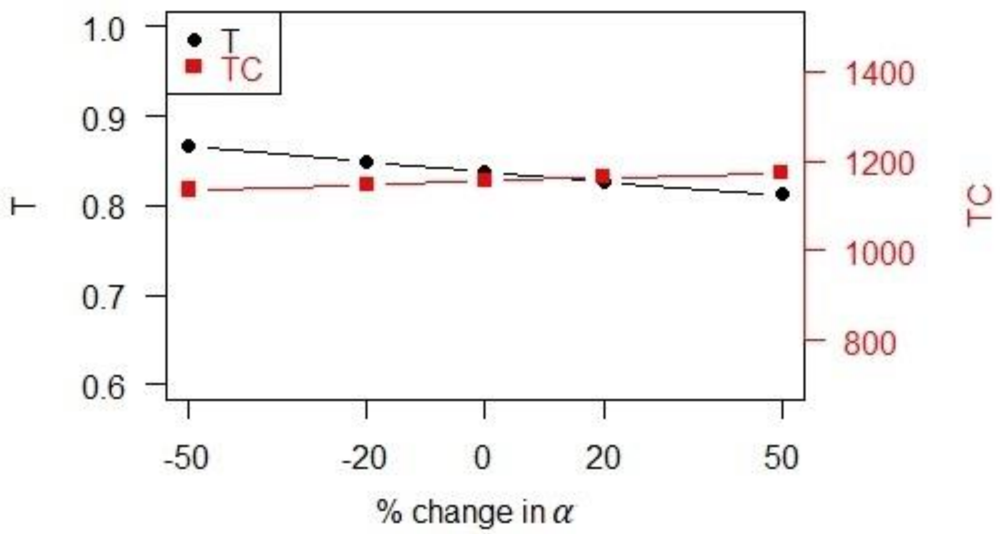
(c) The total cost (TC) increases.

This means when the ordering cost is high the retailers are suggested to increase the order quantity and length of the order cycle.

Table 2.5.2 Sensitivity of the parameter α.

Parameter		**% change**	**T***	**Q***	**TC***
α	0.06	+50%	0.812102	204.02205	1174.388
	0.048	+20%	0.826713	207.20167	1163.092
	0.04	0%	0.837230	209.48725	1155.314
	0.032	-20%	0.848469	211.92560	1147.321
	0.02	-50%	0.866897	215.91716	1134.884

Figure 2.5.2 Effect of α on T and TC



Observations from the table 2.5.2 and figure 2.5.2:

As α increases,

(a) The length of the order cycle (T) decreases.

(b) The order Quantity (Q) decreases.

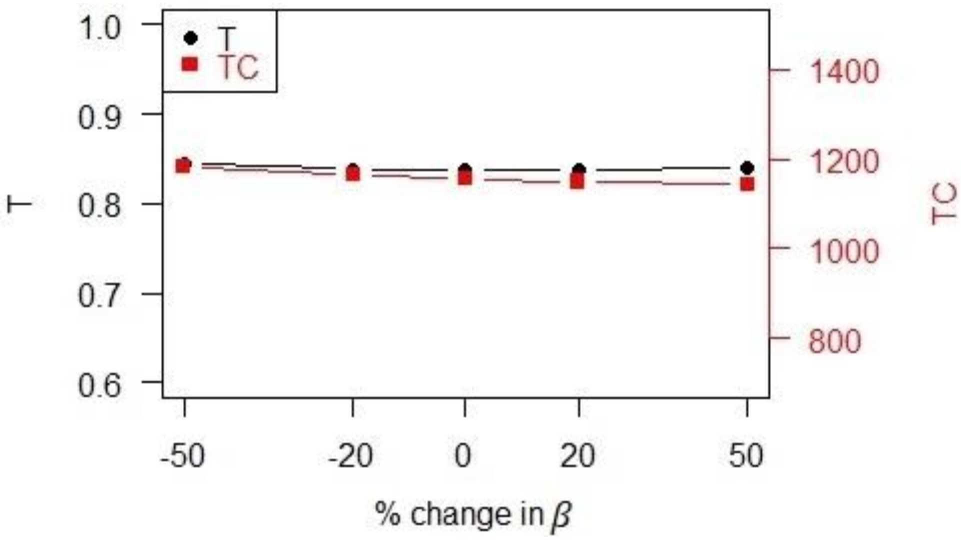
(c) The total cost (TC) increases.

When α increases the deterioration rate increases and hence the deterioration cost increases, so the total cost also increases. For a greater value of α, the deterioration will be high and so the length of the order cycle should be smaller to decrease the deterioration cost.

Table 2.5.3 Sensitivity of the parameter β.

Parameter		% change	T*	Q*	TC*
β	3	+50%	0.839592	209.36129	1141.236
	2.4	+20%	0.837651	209.24816	1148.644
	2	0%	0.837230	209.48725	1155.314
	1.6	-20%	0.838150	210.18216	1164.078
	1	-50%	0.844641	212.90362	1183.654

Figure 2.5.3 Effect of β on T and TC.



Observations from the table 2.5.3 and figure 2.5.3:

As β increases,

(a) The length of the order cycle (T) decreases for $\beta \leq 2$ and increases for $\beta > 2$.

(b) The effect of β on order Quantity (Q) is not linear.

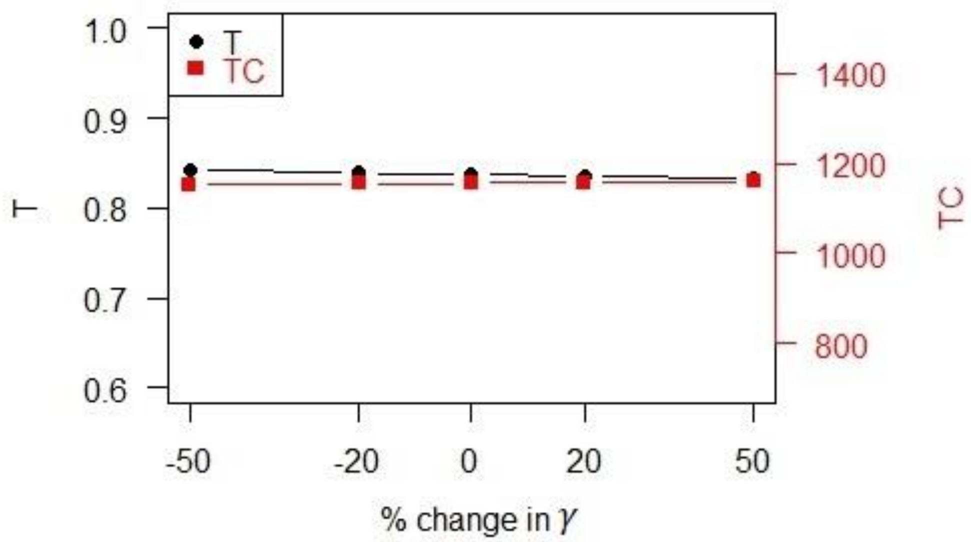
(c) The total cost (TC) decreases.

For $\beta = 2$ the deterioration rate $\theta(t) = \alpha\beta t^{\beta-1}$ is constant, for $\beta < 2$ the deterioration rate decreases and for $\beta > 2$ the deterioration rate increases. Hence, the effect of β on T is nonlinear. It is observed that the effect of β is very mild on T and TC.

Table 2.5.4 Sensitivity of the parameter γ.

Parameter		% change	T*	Q*	TC*
γ	-0.01	+50%	0.832727	211.58570	1158.625
	-0.016	+20%	0.835402	210.31658	1156.653
	-0.02	0%	0.837230	209.48725	1155.314
	-0.024	-20%	0.839101	208.67100	1153.955
	-0.03	-50%	0.841979	207.47011	1151.877

Figure 2.5.4 Effect of γ on T and TC.



Observations from the table 2.5.4 and figure 2.5.4:

As γ increases,

(a) The length of the order cycle (T) decreases.

(b) The order Quantity (Q) increases.

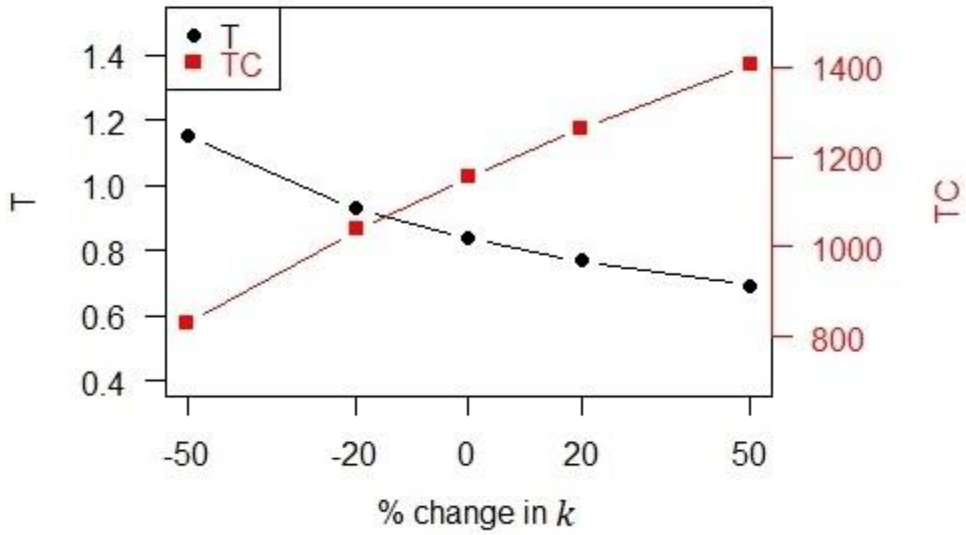
(c) The total cost (TC) increases.

Since, the demand function is $D(t) = ke^{\gamma t}$, when $\gamma < 0$ the demand will exponentially decrease, when $\gamma > 0$ the demand will increase exponentially and when $\gamma = 0$ the demand will be constant (k). When the value of gamma increases, the demand and total cost increases.

Table 2.5.5 Sensitivity of the parameter k.

Parameter		% change	T*	Q*	TC*
k	375	+50%	0.692127	259.39180	1406.233
	300	+20%	0.768798	230.66288	1261.862
	250	0%	0.83723	209.48725	1155.314
	200	-20%	0.928626	186.10622	1037.487
	125	-50%	1.151517	144.78302	828.5111

Figure 2.5.5 Effect of k on T and TC.



Observations from the table 2.5.5 and figure 2.5.5:

As k increases,

(a) The length of the order cycle (T) decreases.

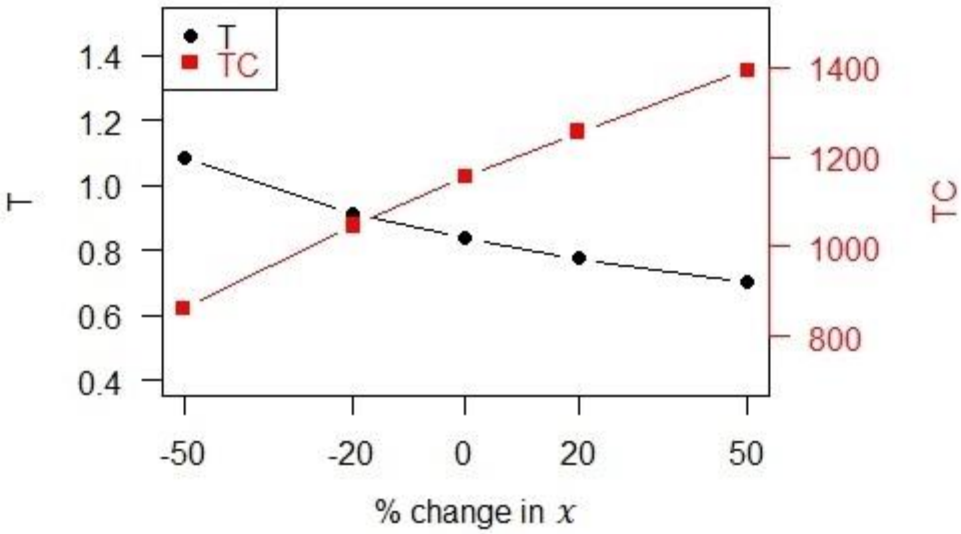
(b) The order Quantity (Q) increases.

(c) The total cost (TC) increases.

Table 2.5.6 Sensitivity of the parameter x.

Parameter		% change	T*	Q*	TC*
x	7.5	+50%	0.703641	175.82148	1393.743
	6	+20%	0.775622	193.93878	1255.751
	5	0%	0.83723	209.48725	1155.314
	4	-20%	0.915131	229.20865	1046.177
	2.5	-50%	1.081919	271.70633	860.2226

Figure 2.5.6 Effect of x on T and TC.



Observations from the table 2.5.6 and figure 2.5.6:

As x increases,

(a) The length of the order cycle (T) decreases.

(b) The order Quantity (Q) decreases.

(c) The total cost (TC) increases.

It is obvious that the total cost is very sensitive with respect to the parameter x.

Table 2.5.7 Sensitivity of the parameter y.

Parameter		% change	T*	Q*	TC*
y	0.075	+50%	0.836283	209.24744	1156.040
	0.06	+20%	0.836852	209.39123	1155.605
	0.05	0%	0.837230	209.48725	1155.314
	0.04	-20%	0.837613	209.58353	1155.023
	0.025	-50%	0.838170	209.72430	1154.586

In table 2.5.7, it is obvious that as the parameter y changes there is no significant change in the value of T*, Q*, and TC*. This is due to the small value of y. But in long run, the holding cost will increase significantly because the holding cost is a linear function of time ($h(t) = x + yt$) and at the beginning of every order cycle the intercept parameter value (i.e. the value of x) will increase.

2.6 Conclusion

In this chapter we attempted to obtain the optimal solution for an inventory system of Weibull deteriorating items with exponential demand and linear holding cost. The convexity of the total variable cost function (TC) is shown graphically. The sensitivity analysis shows how the total cost increases/decreases w.r.t different cost parameters.