## CHAPTER 5

## ANALOGUES OF GEOMETRIC STABILITY AND RELATED AR PROCESSES

### 5.1. INTRODUCTION

Recall that the exponential AR process EAR of Gaver and Lewis(1980) is defined by

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability } \rho \\ \rho X_{n-1} + Y_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.1.1)$$

where $\rho \in (0, 1)$, $\{Y_n\}$ is a sequence of iid rvs, independent of $X_0$, with exponential($\lambda$) marginals and $X_0$ is distributed as exponential($\lambda$). We have noted earlier that the process $\{X_n\}$ defined above is strictly stationary.

The structure of the EAR process is interesting; especially that the common distribution of innovations is same as the stationary distribution of the process. We have already seen in Section 4.2.2 that this property characterizes EAR process amongst the stationary AR processes given by structure (5.1.1). The ideas developed in this Chapter are the result of considering the following generalization of the structure of EAR process.

$$X_n = \begin{cases} c(\rho) X_{n-1} & \text{with probability } \rho \\ c(\rho) X_{n-1} + Y_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.1.2)$$

where $c(\rho)$ is a function of $\rho \in (0, 1)$, $\{Y_n\}$ is a sequence of nonnegative iid rvs, independent of a nonnegative rv $X_0$, having common distribution G and $X_0$ is distributed as G.

We are interested in obtaining necessary and sufficient conditions for $\{X_n\}$ at (5.1.2) to be stationary for every $\rho \in (0, 1)$. First we note that the process $\{X_n\}$ at (5.1.2) is a special case of the AR process defined by

$$X_n = c(\rho)X_{n-1} + \xi_n, \qquad \ldots(5.1.3)$$

when the common distribution of independent innovations $\xi_n$ assigns a fixed mass $\rho$ at 0. As we noted in Section 1.2.1, the process $\{X_n\}$ defined by (5.1.3) is stationary only if

$$c(\rho) \in [0, 1]. \qquad \ldots(5.1.4)$$

Thus (5.1.4) is a necessary condition for the stationarity of $\{X_n\}$ defined at (5.1.2). Let X and Y be the independent rvs distributed as G. Then $\{X_n\}$ at (5.1.2) is stationary if and only if

$$X \stackrel{d}{=} c(\rho)X + B(\rho)Y\ , \qquad \ldots(5.1.5)$$

where $B(\rho)$ is a $B(1, 1-\rho)$ rv, $B(\rho)$, X and Y are mutually independent.

The solution of stochastic equation (5.1.5) is related to a problem posed by Zolotarev that is described below.

<u>Zolotarev's</u> <u>Problem</u> :

Zolotarev (See Klebanov et al. (1985) ) posed the problem of describing rvs X having the property that for every $\rho \in (0, 1)$ there exists a rv $X(\rho)$ such that

$$X \stackrel{d}{=} X(\rho) + B(\rho)Y , \qquad \ldots(5.1.6)$$

where $B(\rho)$ is a $B(1, 1-\rho)$ rv described above, $Y \stackrel{d}{=} X$ and $X(\rho)$, $B(\rho)$ and $Y$ are mutually independent.

Klebanov et al.(1985) defined geometrically infinitely divisible (GID) distributions and showed that the Zolotarev's problem is same as the problem of describing GID rvs. When for every $\rho \in (0, 1)$, $X(\rho)$ at (5.1.6) is distributed as $c(\rho)X$ then the corresponding rv $X$ is geometrically strictly stable (G(S)S). The G(S)S laws are defined by Klebanov et al. (1985).

This makes it clear that $\{X_n\}$ at (5.1.2) is stationary for *every* fixed $\rho \in (0, 1)$ if and only if $G$ is G(S)S with support $\mathbb{R}^+$. However if stationarity is not insisted upon for every $\rho \in (0, 1)$ then the marginal distribution may not be G(S)S. More precisely, it will be shown that the marginal distribution in this case must be geometrically Semi-stable (A concept introduced by Mohan et al.(1993)).

Analogous to the EAR process, McKenzie (1986) proposed the following stationary AR process with geometric marginals

$$X_n = \begin{cases} \rho * X_{n-1} & \text{with probability } \rho \\ \rho * X_{n-1} + Z_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.1.7)$$

where $\{Z_n\}$ is a sequence of iid rvs, independent of $X_0$, with common distribution Geom(p), $X_0$ is distributed as

Geom(p), and operator $\rho *$ is the thinning operator defined at (1.3.2)

The process $\{X_n\}$ at (5.1.7) is similar, in structure, to EAR process with the only difference that the scalar multiplication is replaced by thinning operator $\rho *$. A generalization, similar to (5.1.2), of $\{X_n\}$ at (5.1.7) will have the following structure.

$$X_n = \begin{cases} c(\rho)*X_{n-1} & \text{with probability } \rho \\ c(\rho)*X_{n-1} + Z_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.1.8)$$

where $c(\rho) \in [0, 1]$ is a function of $\rho \in (0, 1)$, $\{Z_n\}$ is a sequence of nonnegative integer valued iid rvs, independent of $X_0$, having a common distribution G, and $X_0$ is distributed as G.

Let X and Z be the independent rvs having common distribution G. Then $\{X_n\}$ at (5.1.8) is stationary if and only if

$$X \overset{d}{=} c(\rho) * X + B(\rho)Z. \qquad \ldots(5.1.9)$$

Now the problem reduces to that of describing rvs X having property that for every $\rho \in (0, 1)$ there exists a constant $c(\rho) \in [0, 1]$ such that (5.1.9) holds. This problem is similar to the Zolotarev's problem described earlier. To solve this new problem we need to develop the discrete analogue of the concept of geometric stability. We develop this analogous theory in the present Chapter.

*Remark 5.1.1* : It may be noted that the rv that is degenerate at zero provides trivial example for the concepts discussed and developed in this Chapter. Since degenerate rvs are not of interest, we are concerned only with nondegenerate rvs throught this Chapter. Also we restrict our discussion only to nonnegative rvs as such variables are of main interest to us ∎

In Section 5.2 we give some known results about geometrically stable laws. In Section 5.3 discrete geometrically stable laws are introduced and the results analogous to those in classical theory are obtained. It may be recalled that discrete self decomposable and discrete (strictly) stable laws are introduced by Steutel and Van Harn (1979). In Section 5.4 we define discrete versions of Semi-stable and geometrically Semi-stable laws and obtain some interesting results. Analogues of domain of attraction and domain of partial attraction are introduced in Section 5.5. A similar theory in the schemes of maxima/ minima is developed in Section 5.6. In Section 5.7 we propose some more stationary processes that are relevant to the theory developed in this Chapter.

## 5.2 GEOMETRICALLY STABLE LAWS

The importance of normal distribution is due mainly to the central limit Theorem. Let $X_1$, $X_2$, ...$X_n$ be iid rvs having zero expectation and unit variance. Put

$$S_n(X) = \sum_{1=1}^{n} X_i .$$

The central limit Theorem asserts that the distribution of $n^{-1/2}S_n(X)$ tends to normal as n tends to infinity. Stable distributions play an important role as a natural generalization of normal distribution. The stable laws are defined as follows :

**Definition 5.2.1** : A distribution F is said to be stable (in broad sense) if for each $n \geq 1$, there exist constants $c_n > 0$ and $\gamma_n$ such that

$$c_n^{-1}(S_n(X) - \gamma_n) \stackrel{d}{=} X, \qquad \ldots(5.2.1)$$

and the common distribution F of $X_i$'s and X is not concentrated at origin. F is said to be stable in 'strict sense' if (5.2.1) holds with $\gamma_n = 0$ ∎

It is well established that only norming constants $c_n = n^{1/\alpha}$ are possible, where $\alpha \in (0, 2]$. The constant $\alpha$ is called the *index* or *characteristic exponent* of stable law F.

If we consider a sum $S_n(X)$ of iid rvs having a finite variance, then we have already noted that the limiting distribution is normal, obtained with norming constants $c_n = n^{1/2}$. However if the variance is infinite then the norming constants must be chosen differently, but a limit may still exist. Interestingly all stable distributions and no other distributions occur as such limits. The common distribution of $X_i$'s , in that case, is said to be in the domain of attraction of the limit distribution.

**Definition 5.2.2** : The distribution G of iid rvs $Y_i$ is said to belong to the domain of attraction of a distribution F if for every $n \geq 1$, there exist norming constants $a_n > 0$ and $b_n$ such that the distribution of $a_n^{-1}(S_n(Y) - b_n)$ tends to F as n tends to infinity ∎

A distribution F possesses a nonempty domain of attraction if and only if it is stable. As we are mainly interested in nonnegative rvs, we state below a representation Theorem for the Laplace transforms of stable laws with support $\mathbb{R}^+$ (nonnegative stable laws).

**Theorem 5.2.1** : *The Laplace transform of a nonnegative stable law is of the form*

$$L(t) = e^{-\theta t^{\alpha}}, \quad t \geq 0 \qquad \ldots(5.2.2)$$

*where* $\theta > 0$ *and* $\alpha \in (0, 1)$.

The result in above Theorem is well-known and its proof is based on the following Lemma. The lemma involves the concept of regular variation and is also required for some more results that are presented in this Chapter.

**Definition 5.2.3** : A positive function $L(\cdot)$ defined on $(0, \infty)$ varies slowly at infinity/ zero if for each $x > 0$,

$$\frac{L(tx)}{L(t)} \longrightarrow 1$$

as $t \longrightarrow \infty$ / $t \longrightarrow 0$ ∎

**Definition 5.2.4** : A positive function U defined on $(0, \infty)$ varies regularly at infinity/ zero if

$$U(x) = x^{\rho} L(x),$$

for some $\rho \in \mathbb{R}$ and L varies slowly at infinity/ zero ∎

**Lemma 5.2.1** : (Lemma 2 of VIII.8 in Feller (1965))

*Suppose that* $\frac{\lambda_{n+1}}{\lambda_n} \longrightarrow 1$, *and* $a_n \longrightarrow \infty$. *If* U *is a monotone function such that*

$$\lim \lambda_n U(a_n x) = \chi(x) \le \infty$$

*exists on a dense set, and* $\chi$ *is finite and positive in some interval, then* U *varies regularly at infinity and* $\chi(x) = \theta x^{\alpha}$, *where* $\alpha$ *is a real constant.*

*Remark 5.2.1* : In the above Lemma if $\{a_n\}$ is such that $a_n \longrightarrow 0$, then U varies regularly at zero and $\chi(x)$ is again of the form $\chi(x) = \theta x^{\alpha}$ ∎

*Remark 5.2.2* : It may be recalled that (See Theorem 3.2.2) a Lapace transform must be a completely monotone function. The function $L(t) = \exp(-\theta t^{\alpha})$ is completely monotone if and only if $\alpha \in (0, 1]$. This fact will be required in subsequent results of this Chapter ∎

The general idea of stable distributions was initiated by Lévy in 1924. de Finneti in 1929 introduced the concept of infinitely divisible distributions.

**Definition 5.2.5** : A distribution F of rv X is said to be infinitely divisible if for every n, there exist iid rvs $X_{i,n}$ such that $X \stackrel{d}{=} \sum_{i=1}^{n} X_{i,n}$ ∎

It is clear that stable distributions are infinitely divisible and are distinguished by the fact that the common distribution of $X_{i,n}$'s differ from F only by location and scale parameters.

Klebanov et al. (1985) extended above ideas by replacing ordinary sum of rvs by geometric sums, and introduced the geometrically infinitely divisible (GID) laws and Geometrically strictly stable (G(S)S) laws as follows :

**Definition 5.2.6** : A rv X is GID if for every $\rho \in (0, 1)$ there exists a sequence $\{X_n(\rho),\ n \geq 1\}$ of iid rvs such that

$$X \stackrel{d}{=} \sum_{n=1}^{N(\rho)} X_n(\rho) \qquad \ldots(5.2.3)$$

where $N(\rho)$ is a $\text{Geom}^+(\rho)$ rv independent of the sequence of rvs $\{X_n(\rho)\}$ ∎

*Remark 5.2.3 :* Throughout this Chapter we will use the notation $N(\rho)$ with above meaning. Also, Hereafter, whenever random sums $S_W(X) = \sum_{k=1}^{W} X_k$ are encountered it is understood that the rv W is independent of $\{X_k\}$, and $X_k$ are independent copies of a rv X. Similarly the notation $S_n(X)$ will be used for the sum $\sum_{k=1}^{n} X_k$, where n is a positive integer ∎

That the Zoloratev's problem is same as that of describing GID rvs can be seen as follows.

Suppose X is a rv for which (5.1.6) holds. Let $\phi_X$ and $\phi_{X(\rho)}$ be the cfs of X and $X(\rho)$ respectively. Then

(5.1.6) holds true if and only if $\phi_X$ and $\phi_{X(\rho)}$ satisfy

$$\phi_X(t) = \rho\phi_{X(\rho)}(t) + (1-\rho)\phi_{X(\rho)}(t)\ \phi_X(t).$$

This implies

$$\begin{aligned}\phi_X(t) &= \rho\phi_{X(\rho)}(t)\{1-(1-\rho)\ \phi_{X(\rho)}(t)\}^{-1} \\ &= \rho\phi_{X(\rho)}(t)\sum_{j=0}^{\infty}(1-\rho)^j\ \phi^j_{X(\rho)}(t) \\ &= \sum_{j=1}^{\infty}\rho(1-\rho)^{j-1}\ \phi^j_{\rho}(t)\end{aligned}$$

The expression on right is easily recognized as the cf of Geometric sum $S_{N(\rho)}(X(\rho))$. Thus it follows that X is GID rv. By reversing the above arguments it can be seen that a GID rv satisfies (5.1.6).

**Definition 5.2.7** : A rv X is G(S)S if for every $\rho \in (0, 1)$ there exists a constant $c(\rho) > 0$ such that

$$X \overset{d}{=} c(\rho)\ S_{N(\rho)}(X) \quad \blacksquare \qquad \ldots(5.2.4)$$

Clearly, G(S)S laws form a sub class of GID laws, distinguished by the fact that rv $X(\rho)$ is of the form $c(\rho)X$. Consequently, G(S)S rvs are those rvs X possessing the property that for every $\rho \in (0, 1)$ there exist a constant $c(\rho) > 0$ such that

$$X \overset{d}{=} c(\rho)X^* + B(\rho)X \qquad \ldots(5.2.5)$$

where $X \overset{d}{=} X^*$, $B(\rho)$ is as defined earlier and X, $X^*$ and $B(\rho)$ are mutually independent.

We are specifically interested in the sub class of G(S)S laws whose support is $\mathbb{R}^+$. We obtain below the expression for the Laplace transform of a nonnegative G(S)S law ( G(S)S law with support $\mathbb{R}^+$).

**THEOREM 5.2.2** *The Laplace transform of a nonnegative G(S)S law is of the form*

$$L(t) = \frac{1}{1 + \theta t^{\alpha}} \quad \forall t \geq 0, \qquad \ldots(5.2.6)$$

*where* $\theta > 0$ *and* $\alpha \in (0, 1]$.

(Here $\alpha$ will be called the index of the G(S)S law and the law will be denoted by G(S)S($\alpha$, $\theta$)).

**Proof** : From (5.2.5) it follows that the Laplace transform L of a nonnegative G(S)S law must satisfy

$$L(t) = \rho\, L(c(\rho)t) + (1-\rho)\, L(c(\rho)t)\, L(t).$$

That is,

$$L(t) = \frac{\rho L(c(\rho)t)}{1- (1-\rho)L(c(\rho)t)} \quad \forall\, \rho \in (0, 1).$$

Hence $L^*(t) = \exp( 1- 1/L(t))$ must satisfy

$$L^*(t) = \{ L^*(c(\rho)t) \}^{1/\rho} \quad \forall\, \rho \in (0, 1).$$

Putting $\rho = 1/n$ for n= 2, 3, ... and $a_n = c(1/n)$, $L^*(t)$ satisfies

$$L^*(t) = \{L^*(a_n t)\}^n . \qquad \ldots(5.2.7)$$

Now let $t > 0$ and $m > n$. Note that $L^*$ is a monotonically decreasing function taking values in (0, 1). Since $\{L^*(a_n t)\}^n = \{L^*(a_m t)\}^m$ , it follows that $L^*(a_n t) < L^*(a_m t)$, which further implies that $a_n > a_m$.

Thus $\{a_n\}$ is a monotonically decreasing sequence. Now keeping $t \neq 0$ fixed and allowing n to go to infinity in (5.2.7), we get

$$L^*(\lim a_n t = \lim L^*(a_n t) = 1.$$

Thus we conclude that $a_n \downarrow 0$. Re write (5.2.7) as

$$n\{-\log L^*(a_n t)\} = - \log L^*(t)$$

Note that $-\log L^*(t)$ is finite and positive. Then from Remark 5.2.1, it follows that $-\log L^*$ varies regularly at 0, and $L^*(t)$ must be of the form $\exp(-\theta t^\alpha)$ for some $\theta > 0$ and some $\alpha \in \mathbb{R}$. Thus L must be of the form (5.2.6), which is a Laplace transform if and only of $\alpha \in (0, 1]$ (See Remark 5.2.2) ■

*Remark 5.2.4* : It is interesting to note that the case $\alpha$=1 corresponds to a degenerate rv for nonnegative stable laws, whereas for nonnegative geometrically stable laws, $\alpha$=1 corresponds to exponential distribution ■

*Remark 5.2.5* : From the above Theorem, it is clear that the function $c(\rho)$ appearing in Definition 5.2.7 must be of the form $c(\rho) = \rho^{1/\alpha}$ for a nonnegative G(S)S law ■

It is well-known that all stable distributions are absolutely continuous. Even if the iid rvs $X_i$ are discrete in nature, their sum, after normalization, tends to an absolutely continuous rv in distribution whenever the limit exists. Steutel and Van Harn (1979) introduced the concept of discrete stability. They pointed out that a general theory of domain of discrete attraction can be developed on the same lines as in the case of classical setup. Such theory is developed in Section 5.5. A discrete stable rv is defined as follows:

**Definition 5.2.8** A nonnegative integer valued rv X is said to be discrete stable if its APGF P satisfies for some $\alpha \in (0, 1]$,

$$P(s) = P(c^{1/\alpha}s)\ P(\ (1-c)^{1/\alpha}s)$$

for all $c \in (0, 1)$ and $0 < P(1) < 1$ ∎

In terms of rvs, a nonnegative nondegenerate integer valued rv X is discrete stable if there exist $\alpha \in (0, 1]$ such that

$$c^{1/\alpha} * X_1 + (1-c)^{1/\alpha} * X_2 \stackrel{d}{=} X$$

holds for every $c \in (0, 1)$, where $X_1$ and $X_2$ are independent copies of X.

*Remark 5.2.6* : It may be noted that the concept of discrete stability is restricted only to nonnegative integer valued rvs ∎

*Remark 5.2.7* : Recall that the use of APGF is more appealing than pgf. Certainly, the defining equation looks more convenient in terms of APGF than in terms of pgf as given by Steutel and Van Harn (1979) ∎

Steutel and Van Harn (1979) also showed that the APGF of a discrete stable rv must be of the form

$$P(s) = \exp\ (-\theta s^{\alpha})\ ,\ \alpha \in (0, 1],\ \theta > 0. \qquad \ldots(5.2.8)$$

It is interesting to note that the above form of P is same as the form of Laplace transform of a nonnegative stable rv ($L(t) = \exp(-\theta t^{\alpha})$, $\alpha \in (0,1)$, $\theta > 0$ ). In the next Theorem we give an important characterization of a discrete stable law.

**THEOREM 5.2.3 :** *An* APGF P *with* $0 < P(1) < 1$ *is discrete stable if and only if for every integer* $n \geq 1$, *there exists* $c_n > 0$ *such that*

$$P(s) = \{P(c_n s)\}^n \qquad \ldots(5.2.9)$$

**Proof :** Using arguments similar to those in the proof of Theorem 5.2.2, we conclude that $-\log P$ varies regularly at zero. Hence,

$$-\log\{P(s)\} = \theta s^{\alpha}, \quad \theta > 0, \ \alpha \in \mathbb{R}.$$

That is,

$$P(s) = \exp(-\theta s^{\alpha}),$$

which is an APGF if and only if $\alpha \in (0, 1]$, for otherwise P(s) in not completely monotone and hence can not be an APGF (See Theorem 3.2.1). Thus P is a discrete stable APGF.

Conversely, suppose P is a discrete stable APGF, then P must be of the form (5.2.8), which obviously satisfies (5.2.9) for $n \geq 1$, with $c_n = n^{-1/\alpha}$ ∎

As a consequence of above Theorem, an alternative definition of discrete stable rvs can be given as follows:

**Definition 5.2.9 :** A nonnegative nondegenerate integer valued rv X is said to be discrete stable if for every $n \geq 1$, there exists $c_n \in (0, 1)$ such that

$$c_n * S_n(X) \stackrel{d}{=} X \ ∎$$

*Remark 5.2.8* : Above definition is analogous to the Definition 5.2.1 of stable rvs. It may also be noted that the concept of discrete stability is always in a strict sense ■

*Remark 5.2.9* : Since a discrete stable APGF must be of the form (5.2.8), it follows that it satisfies (5.2.9) even when n is replaced by an arbitrary real number greater than 1. In other words, for every $\rho \in (0, 1)$ there exists $c(\rho) \in (0,1)$ such that $P(s) = \{P(c(\rho)s\}^{1/\rho}$, whenever P is discrete stable. In fact $c(\rho) = \rho^{1/\alpha}$, where $\alpha$ is the parameter of P. ■

In the next Section, we extend the idea of discrete stability to that of discrete geometric stability in a manner that is analogous to that of Klebanov et al. (1985).

## 5.3. DISCRETE GEOMETRIC STABILITY :

We define discrete geometrically stable rv as follows :

**Definition 5.3.1** : An integer valued nonnegative nondegenerate rv X is called discrete geometrically stable (DGS) if for every $\rho \in (0, 1)$, there exist a constant $c(\rho) \in (0, 1)$ such that

$$X \overset{d}{=} c(\rho) * S_{N(\rho)}(X) \blacksquare \qquad \ldots(5.3.1)$$

Note the analogy between Definitions 5.3.1 and 5.2.7. Next we give a representation Theorem for APGF of a DGS distribution.

**THEOREM 5.3.1** : *An* APGF P *is* DGS *if and only if it is of the form*

$$P(s) = (1 + \theta s^{\alpha})^{-1} \ , \ 0 \le s \le 2 \qquad \ldots(5.3.2)$$

*where* $\theta > 0$ *and* $\alpha \in (0, 1]$.

(Here $\alpha$ will be called the exponent of the DGS law, and the law will be denoted by DGS($\alpha$, $\theta$))

**Proof** : Suppose P is DGS. Then from the definition it follows that P satisfies

$$P(s) = \sum_{j=1}^{\infty} \rho \ (1-\rho)^{j-1} \ \{P(c(\rho)s)\}^{j}$$

i.e.,

$$P(s) = \frac{\rho \ P(c(\rho)s)}{1 - (1-\rho)P(c(\rho)s)} \quad \forall \ \rho \in (0, 1).$$

Define $f(s) = \exp\{1 - 1/P(s)\}$. Then $f(s)$ must satisfy

$$f(s) = \{f(c(\rho)s)\}^{1/\rho} \quad \forall \ \rho \in (0, 1).$$

Using arguments similar to those in the proof of Theorem 5.2.2, we conclude that P(s) must be of the form (5.3.2), where $\alpha \in (0, 1]$ and $\theta > 0$.

Conversely, suppose P(s) is given by (5.3.2). Then it can be verified, directly from the definition, that P is DGS ∎

*Remark 5.3.1* : Above result can also be stated as, P is a DGS APGF if and only if exp(1−1/P) is a discrete stable APGF. A similar observation was made by Klebanov et al.(1985) regarding characteristic functions of G(S)S laws and strictly stable laws ∎

*Remark 5.3.2* : From the above Theorem, it is clear that the function $c(\rho)$ appearing in the Definition 5.3.1 must be of the form $c(\rho) = \rho^{1/\alpha}$ ∎

*Remark 5.3.3* : Note that the case $\alpha=1$ for discrete stable law corresponds to Poisson distribution, whereas for discrete geometrically stable law it corresponds to geometric distribution. It may be recalled that for nonnegative geometrically stable law, $\alpha=1$ corresponds to exponential distribution ∎

Following Theorem shows that the condition (5.3.1) for all $\rho \in (0, 1)$ can be relaxed in the Definition 5.3.1. A similar result was proved for G(S)S laws by Klebanov et al.(1985).

**THEOREM 5.3.2** : *An integer valued nonnegative rv* Y *is* DGS *if and only if for some* $\alpha > 0$, *and some* $\rho_1, \rho_2 \in (0, 1)$ *such that* $(\log \rho_1)/\log \rho_2$ *is irrational,*

$$\rho_1^{1/\alpha} * S_{N(\rho_1)}(Y) \overset{d}{=} Y \overset{d}{=} \rho_2^{1/\alpha} * S_{N(\rho_2)}(Y). \quad \ldots(5.3.3)$$

**Proof** : Let P be the APGF of Y. Then (5.3.3) implies that P must satisfy (using arguments made in Theorem 5.3.1)

$$P(s) = \frac{\rho P(\rho_i^{1/\alpha} s)}{1-(1-\rho)P(\rho_i^{1/\alpha} s)}, \quad i = 1, 2.$$

Hence

$\phi = (1/P) - 1$ must satisfy the relation

$$\phi(\rho_i^{1/\alpha} s) = \rho_i \ \phi(s) \quad , 0 \le s \le 1, \ i = 1, 2.$$

Let $\psi(x) = \phi(e^x)e^{-\alpha x}$, and $c_i = \frac{1}{\alpha}\log \rho_i$, $i = 1, 2$. Note that $\psi(x)$ is defined for $x \in (-\infty, 0)$. Then $c_1$ and $c_2$ are periods of $\psi$. Since $\psi$ is continuous and $c_1/c_2$ is irrational, it follows that $\psi$ must be a constant function, Say $\psi \equiv \theta$. Then $\phi$ must be of the form

$$\phi(x) = \theta x^{\alpha} , \; 0 \le x \le 1.$$

Thus P is the APGF of DGS($\alpha$, $\theta$). The converse of the result is obvious from Definition 5.3.1 ∎

Mohan et al. (1993) gave an alternative definition of G(S)S laws, where the authors assume condition (5.2.4) to hold only for numbers $\rho$ of the form $1/n$ instead of all $\rho$ in (0, 1). In the next Theorem we show that the same relaxation can also be made in the definition of DGS laws.

**THEOREM 5.3.3** : *A rv* X *is* DGS *if and only if for every* $n \ge 2$, *there exists* $a_n \in (0, 1)$, such that

$$X \stackrel{d}{=} a_n * S_{N(1/n)}(X). \tag{5.3.4}$$

**Proof** : Let P be an APGF of X. Making the arguments similar to those in Theorem 5.3.1, it follows that $P^* = \exp(1 - 1/P)$ satisfies

$$P^*(s) = \{P^*(a_n s)\}^n , \; n \ge 2$$

Using arguments similar to those in the proof of Theorem 5.2.2, we conclude that $P^*$ must be of the form

$$P^*(s) = \exp(-\theta s^{\alpha}),$$

where $\alpha \in (0, 1]$, $\theta > 0$. This implies $P(s) = \dfrac{1}{1 + \theta s^{\alpha}}$, which is an APGF of DGS law ∎

Next we show that it is sufficient to assume the condition (5.3.4) only for two integers m and n which are relatively prime in order to characterize the DGS laws. First we prove a Lemma which contains a result of general interest.

**Lemma 5.3.1** : *If two integers* m *and* n *are relatively prime then* (log m) /(log n) *is irrational.*

**Proof** : Since m and n are relatively prime, so also are the numbers $m^s$ and $n^k$ for arbitrary positive integers s and k. Obviously, we have $m^s \neq n^k$ for any choice of positive integers s and k. This further implies that

$m \neq n^r$ for any rational r.

$\Rightarrow$ log m/(log n) is an irrational number ■

The following corollary is an immediate consequence of above Lemma and Theorem 5.3.2.

**Corollary 5.3.1** : *If there exist two integers* m *and* n *relatively prime and constants* $a_m$, $a_n$ *in* (0, 1) *such that*

$$a_m * S_{N(1/m)}(X) \overset{d}{=} X \overset{d}{=} a_n * S_{N(1/n)}(X)$$

*then* X *must be a* DGS *rv.*

*Remark 5.3.3* : A similar characterization of geometrically stable law obviously holds, which states that if for two relatively prime integers m and n, there exist constants $a_m$ and $a_n$ such that

$$a_m\, S_{N(1/m)}(X) \overset{d}{=} X \overset{d}{=} a_n\, S_{N(1/n)},$$

then X must be a geometrically stable law ■

## 5.4 DISCRETE SEMI-STABILITY AND GEOMETRIC DISCRETE SEMI-STABILITY

The concept of Semi-stable laws was introduced by Lévy in 1937 (See for example, Lukacs (1970).

**Definition 5.4.1** : A distribution is said to be Semi-Stable if its second characteristic $\varphi(\cdot)$ satisfies

$$\varphi(at) = a^{\alpha}\varphi(t)$$

for some $a \in \mathbb{R}$ such that $a \neq 0$, $a \neq 1$ ∎

It is well-known that $0 < \alpha \leq 2$. Equivalently, a distribution is Semi-stable if its characteristic function $\phi(\cdot)$ satisfies the equation

$$\phi(t) = \{\phi(\beta t)\}^{\gamma}$$

for some $\gamma > 1$ and $\beta \in (0, 1)$, where $\beta^{\alpha}\gamma = 1$. It is known that Semi-stable laws include stable laws as a proper subclass. It has also been established that Semi-stable laws possess many properties of stable laws (See for example, Kruglov (1972)).

Semi-stable laws are obtained as limits of normalized sums $S_n$ of iid rvs, when n does not run through all natural numbers but through a subsequence of natural numbers satisfying certain conditions.

In conformity with Lévy's definition, we define discrete Semi-stable laws as follows.

**Definition 5.4.2** : A nonnegative integer valued rv is said to be discrete Semi-stable if its APGF P satisfies

$$P(s) = \{P(as)\}^{b} \;\forall\; s \in [0, 2], \qquad \ldots(5.4.1)$$

for some $b > 1$ and some $a \in (0, 1)$ ∎

It is clear that there exist unique $\alpha > 0$ such that $a^{\alpha}b = 1$. Also it can be easily shown that $\alpha \in (0, 1]$.

*Remark 5.4.1* : In the above definition b need not be an integer. It may be noted that there exists APGF whose nonintegral powers are also APGFs ■

Compare above definitions with the alternative definition of discrete stable laws given in Remark 5.2.8. It is clear that every discrete stable distribution is discrete Semi-stable.

Following Theorem gives a result that is analogous to a well-known result regarding Semi-stable laws (See Kruglov (1972)).

**THEOREM 5.4.1** *An* APGF P *of a nondegenerate rv is discrete Semi-stable if and only if there exists a sequence* $\{r_k\}$ *of integers,* $r_k \uparrow \infty$, *a sequence* $\{a_k\}$, $a_k \downarrow 0$ *and an* APGF $\zeta$ *such that*

$$\text{(i)} \quad \lim_{k\to\infty} (r_{k+1}/\ r_k) = b < \infty$$

$$\text{and (ii)} \quad \lim_{k\to\infty} \{\zeta(a_k s)\}^{r_k} = P(s).$$

*Moreover,* P *is discrete stable if and only if* (i) *and* (ii) *hold with* b = 1.

**Proof** : Suppose P is discrete Semi-stable. Then there exists $a \in (0, 1)$ and $b > 1$ such that (5.4.1) holds. Then for every positive integer k, we have

$$P(s) = \{P(a^k s)\}^{b^k}.$$

Let $r_k = [b^k]$ and $a_k = a^k$, $k \geq 1$, where $[x]$ denotes the integer part of $x$. Then $r_k \uparrow \infty$, and (i) and (ii) hold with $\zeta = P$.

Conversely, suppose there exist a sequence $\{r_k\}$ of integers, $r_k \uparrow \infty$, a sequence $\{a_k\}$, $a_k \downarrow 0$ and an APGF $\zeta$ such that (i) and (ii) hold.

Case 1 : Let $b > 1$. Consider the identity,

$$\{\zeta(a_{k+1}s)\}^{r_k} = \{\zeta^{r_{k+1}}(a_{k+1}s)\}^{r_k/r_{k+1}}.$$

Then from (i) and (ii), $\zeta^{r_k}(a_{k+1}s) \longrightarrow \{P(s)\}^{1/b}$, and from the "Convergence of types" lemma it follows that $a_{k+1}/a_k \longrightarrow a > 0$, and

$$\{P(s)\}^{1/b} = P(as). \qquad \ldots(5.4.2)$$

Since $a_k \downarrow 0$, we have $a \in (0, 1]$. If $a = 1$, then (5.4.2) implies that $P(s) \equiv 1$, which is not possible for a nondegenerate APGF. Thus we conclude that $a \in (0, 1)$, and hence P is discrete Semi-stable.

Case 2 : Let $b = 1$. In this case for every $\nu \in (0, 1)$ there exists a subsequence $\{k_j\}$ of positive integers such that $r_{k_j} < r_j$, and $\lim_{j\to\infty} (r_{k_j} / r_j) = \nu$. Then again using similar arguments as in Case 1, we observe that there exists $a=a(\nu)$ such that $\{P(s)\}^{\nu} = P(as)$. Since $\nu \in (0, 1)$ is arbitrary, from Theorem 5.2.3 it follows that P is a discrete stable APGF.

To prove the second statement, in view of Case 2 considered above, it is sufficient to note that (i) and

(ii) obviously hold for discrete stable laws by taking $r_k = k$ for $k \geq 1$ ■

As a consequence of Theorem 5.4.1, an alternative definition of discrete Semi-stable laws can be given in terms of rvs as follows:

**Definition 5.4.3** : A nonnegative integer valued rv X is said to be discrete Semi-stable if there exists a nondecreasing sequence $\{r_k\}$ of integers with $\lim_{k\to\infty} \frac{r_{k+1}}{r_k} < \infty$, and a sequence $\{a_k\}$ of normalizing constants such that

$$X \stackrel{d}{=} a_k * S_{r_k}(X) \text{ ■}$$

Mohan et al.(1993) introduced the concept of geometrically Semi-stable laws. Their definition is given below.

**Definition 5.4.4** : A rv X is said to be geometrically Semi-stable if there exist a $p \in (0, 1)$ and $\{a_n\}$, $a_n > 0$, such that

$$X \stackrel{d}{=} a_n S_{N(p^n)}, \ n \geq 1 \text{ ■} \qquad \ldots(5.4.3)$$

They showed that a geometrically strictly stable law is necessarily a geometrically Semi-stable law. We show that in the definition of geometrically Semi-stable laws given by Mohan et al.(1993), it is sufficient to assume condition (5.4.3) only for one value of n. This is shown in the following Theorem.

**Theorem 5.4.2** : *If there exist* $p \in (0, 1)$ *and* $a > 0$ *such that*

$$X \stackrel{d}{=} a\, S_{N(p)}(X) \qquad \ldots(5.4.4)$$

*then* X *is a geometrically Semi-stable law.*

**Proof** : Let X be a rv for which (5.4.4) holds. Then

$$Y = S_{N(p)}(X) \stackrel{d}{=} a^{-1}X . \qquad \ldots(5.4.5)$$

Consider the geometric sum $S_{N(p)}(Y)$, where Y is as defined above. Then we have from (5.4.5),

$$S_{N(p)}(Y) \stackrel{d}{=} a^{-1}S_{N(p)}(X) \stackrel{d}{=} a^{-2}X . \qquad \ldots(5.4.6)$$

Also we have

$$S_{N(p)}(Y) = \sum_{j=1}^{N(p)} S_{N_j(p)}(X) = S_{N(p^2)}(X), \qquad \ldots(5.4.7)$$

where $N_j(p)$ are independent copies of N(p).

The last equality follows from the fact that $\text{Geom}^+(p)$ compound of $\text{Geom}^+(p)$ distribution is $\text{Geom}^+(p^2)$ distribution. Now (5.4.6) and (5.4.7) imply that

$$S_{N(p^2)}(X) \stackrel{d}{=} a^{-2}X.$$

This argument can be used repeatedly to establish

$$S_{N(p^n)}(X) \stackrel{d}{=} a^{-n}X \quad \text{for all } n \geq 1.$$

Thus X is geometrically Semi-stable according to Definition 5.4.3 (with $a_n = a^n$) ∎

Thus we can define geometrically Semi-stable laws as follows :

**Definition 5.4.5** : A rv X is said to be geometrically Semi-stable if there exist a $p \in (0, 1)$ and $a > 0$ such that

$$X \stackrel{d}{=} a\, S_{N(p)}(X) \blacksquare$$

Next we define discrete geometrically Semi-stable (DGSS) laws in conformity with the definition of geometrically Semi-stable laws given above.

**Definition 5.4.6** : A nonnegative integer valued rv X is said to be discrete geometrically Semi-stable if there exist positive numbers $\rho$, $c < 1$ such that

$$X \stackrel{d}{=} c * S_{N(\rho)}(X) \blacksquare$$

Next Theorem can be easily proved using arguments similar to those made in the proof of Theorem 5.3.1.

**THEOREM 5.4.3** : *An* APGF P *is* DGSS *if and only if* $\exp(1 - 1/P)$ *is a discrete Semi-stable* APGF.

In the next section we introduce the domain of discrete attraction, domain of discrete partial attraction, geometric domain of discrete attraction and geometric domain of discrete partial attraction. The results analogous to those in classical theory are obtained.

## 5.5 DOMAIN OF DISCRETE ATTRACTION AND GEOMETRIC DOMAIN OF DISCRETE ATTRACTION.

A natural question that follows the concept of stability is that of the domain of attraction. It was pointed out by Steutel and Van Harn (1979) that a theory of domain of discrete attraction can be developed on the same lines as in the classical setup. In this section we develop such a theory.

**Definition 5.5.1** : An APGF $\zeta$ is said to belong to the domain of discrete attraction of an APGF P, if there exists a sequence $\{a_n\}$ in (0, 1), $a_n \downarrow 0$, such that

$$a_n * S_n(Y) \longrightarrow X \text{ in distribution}, \qquad \ldots(5.5.1)$$

where Y and X have APGFs $\zeta$ and P respectively ■

Next we show that only discrete stable laws can occur as such limits.

**Theorem 5.5.1** : *Suppose an* APGF $\zeta$ *belongs to the domain of discrete attraction of an* APGF P, *then* P *must be discrete stable.*

**Proof** :Let $\{a_n\}$ be the sequence for which (5.5.1) holds. Then APGFs $\zeta$ and P must satisfy

$$\lim_{n\to\infty} \{\zeta(a_n)\}^n = P(s)$$

Taking logarithms on both the sides, we get

$$\lim_{n\to\infty} -n\log \zeta(a_n s) = -\log P(s).$$

Then from Remark 5.2.1 it follows that $-\log \zeta$ varies regularly at 0, and $-\log P$ must be of the form

$$-\log P(s) = \theta s^{\alpha} ,$$

for some real $\alpha$ and $\theta > 0$,

or $\quad P(s) = \exp\{-\theta s^{\alpha}\}.$

Since P is an APGF if and only if $\alpha \in (0, 1]$, it follows that P is an APGF of a discrete stable law ■

From Theorem 5.2.3 it follows that every discrete stable distribution belongs to its own domain of discrete attraction. Thus it follows that every discrete stable distribution and no other distribution has a nonempty domain of discrete attraction.

Next we define the domain of discrete partial attraction.

**Definition 5.5.2** : An APGF $\zeta$ is said to be in the domain of discrete partial attraction of P if there exists a sequence $\{n_j\}$ of positive integers such that $n_j \uparrow \infty$ and a sequence $\{a_j\}$ of reals in (0, 1) $a_j \downarrow 0$ such that

$$a_j * S_{n_j}(Y) \longrightarrow X \text{ in distribution,}$$

where Y and X have APGFs $\zeta$ and P respectively ■

From Theorem 5.4.1 it is clear that every discrete Semi-stable distribution belongs to its own domain of discrete partial attraction.

The geometric domain of attraction/ partial attraction was defined by Mohan et al.(1993) as follows.

**Definition 5.5.3:** A distribution function (df) H is said to belong to the geometric domain of attraction of df F if there exist $\{a_n\}$ and $\{b_n\}$ of real constants, $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that

$$\lim_{n\to\infty} P(a_n^{-1}\{S_{N(1/n)}(X) - b_n\} \le x) = F(x)$$

at all continuity points x of F, where X is a rv having distribution H ∎

They showed that the class of limit laws, when $b_n=0$, in above is same as the class of G(S)S laws.

**Definition 5.5.4** : A df H is said to belong to the geometric domain of partial attraction of F if there exists real constants $\{p_n\}, \{a_n\}$ and $\{b_n\}$, $p_n \in (0, 1)$, $p_n \longrightarrow 0$ and $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that

$$\lim_{n\to\infty} P\, [a_n^{-1}\ \{S_{N(p_n)}(X) - b_n\} \le x\ ] = F(x)$$

at all continuity points of F, where X is a rv having distribution H ∎

It was shown by them that the class of partial limit laws of the above type, when $b_n = 0$, is same as the class of all gid laws.

Next we give discrete analogues of above concepts.

**Definition 5.5.5** : An APGF $\zeta$ is said to be in the geometric domain of discrete attraction of APGF P, if there exists a sequence $\{a_n\}$ in (0, 1), $a_n \downarrow 0$, such that

$$a_n * S_{N(1/n)}(Y) \longrightarrow X \text{ in distribution,} \qquad \ldots(5.5.2)$$

where Y and X have APGFs $\zeta$ and P respectively ∎

From Theorem 5.3.3, it is clear that a DGS law belongs to its own geometric domain of discrete attraction.

**Theorem 5.5.2** *An* APGF $\zeta$ *belongs to the geometric domain of discrete attraction of an* APGF P *if and only if it belongs to the domain of discrete attraction of* $\exp(1 - 1/P)$.

**Proof** : Suppose $\zeta$ belongs to the geometric domain of discrete attraction of P. Then there exists $\{a_n\}$, $a_n \downarrow 0$, such that (5.5.2) holds. Then $\zeta$ and P must satisfy

$$\lim_{n\to\infty} \frac{\frac{1}{n}\,\zeta(a_n s)}{1-(1-\frac{1}{n})\zeta(a_n s)} = P(s)$$

or equivalently,

$$P(s) = \lim_{n\to\infty} \frac{\zeta(a_n s)}{\zeta(a_n s) + n\{1 - \zeta(a_n s)\}}$$

$$= \lim_{n\to\infty} \frac{1}{1 + n\{1 - \zeta(a_n s)\}}$$

This implies that $\lim_{n\to\infty} [\, n\, \{1 - \zeta(a_n s)\}] = h(s)$ exists. Then we have

$$P(s) = \{1 + h(s)\}^{-1}$$

or $\quad h(s) = (1/P) - 1.$

Also we have

$$\log(\lim_{n\to\infty} \zeta^n(a_n s)) = \lim_{n\to\infty} \log \zeta^n(a_n s)$$

$$= \lim_{n\to\infty} \{n \log \zeta(a_n s)\}$$

$$= \lim_{n\to\infty} \{n(\zeta(a_n s)-1)\} = -h(s).$$

That is, $\quad \lim_{n\to\infty} \{\zeta(a_n s)\}^n = \exp\{-h(s)\}.$

Thus $\zeta$ belongs to the domain of discrete attraction of $\exp(1 - 1/P)$. Reversing the above arguments we get the converse result ∎

As an important consequence of above Theorem and Theorem 5.5.1, we get the following corollary (Also See Remark 5.3.1).

**COROLLARY 5.5.1** : *Every DGS distribution, and no other distribution has nonempty geometric domains of discrete attraction.*

Next we define the geometric domain of discrete partial attraction.

**Definition 5.5.6** : An APGF $\zeta$ is said to be in the geometric domain of discrete partial attraction of APGF P if there exists a sequence $\{n_j\}$ of positive integers such that $n_j \uparrow \infty$, and a sequence $\{a_j\}$ in (0, 1), $a_j \downarrow 0$, such that

$$a_j * S_{N(1/n_j)}(Y) \longrightarrow X \text{ in distribution}$$

where Y and X have APGFs $\zeta$ and P respectively ∎

Making arguments similar to those in Theorem 5.5.2, it can be easily proved that an APGF $\zeta$ belongs to the geometric domain of discrete partial attraction of P if and only if it belongs to the domain of discrete partial attraction of $\exp(1 - \frac{1}{P})$.

**5.6 SIMILAR CONCEPTS IN THE SCHEME OF MAXIMA / MINIMA :**

While developing the concepts for the scheme of maxima/ minima, one has to simply replace the roles of thinning operation and APGF by scalar multiplication and distribution/ survival function respectively. Then analogous results can be obtained very easily. Since every result in the schemes of maxima and minima can be obtained as mentioned above, we omit the details. We only give the definitions of geometrically strictly max-stable, max-semi-stable and geometrically max-semi-stable laws. It may be recalled that max-stable and min-stable laws are same as extreme value distributions in the respective schemes. In the following definitions $\{Y_j\}$ is a sequence of iid rvs as before and $Y_j \stackrel{d}{=} Y$.

**Definition 5.6.1** A rv Y is said to be geometrically max-stable (in strict sense) (GMS) if for every $\rho \in (0, 1)$ there exists a constant $c(\rho) > 0$ such that

$$Y \stackrel{d}{=} c(\rho) \bigvee_{j=1}^{N(\rho)} Y_j \quad \blacksquare$$

**Definition 5.6.2** A rv Y, with d.f. F, is said to be max-semi-stable if there exists a > 0 and b > 0 such that

$$F(x) = \{F(ax)\}^b \quad \blacksquare$$

**Definition 5.6.3** A rv Y is said to be geometrically max-semi-stable if there exist $\rho \in (0, 1)$ and a c > 0 such that

$$Y \stackrel{d}{=} c \bigvee_{j=1}^{N(\rho)} Y_j \quad \blacksquare$$

Results analogous to Theorem 5.3.1 in the schemes of maxima as well as minima show that Pareto (of type III considered by Yeh et al. (1988)) is the only geometrically strictly max-stable / min-stable distribution, whereas the results analogous to Theorem 5.4.2 show that the class of geometrically max-semi-stable /min-semi-stable distributions is same as the class of Semi-Pareto distributions introduced by Pillai (1991). This characterization of Semi-Pareto family was proved independently by Sreehari (1995).

The concepts of geometric domain of max attraction/ min attraction as well as geometric domain of max-partial attraction / min- partial attraction can be defined in an analogous way. The analogous results also hold. It may be recalled that the domain of max- partial attraction was defined by Green (1976), who showed that every probability distribution has a nonempty domain of max- partial attraction. Interestingly this is also true for geometric domains of max partial attraction.

## 5.7. STATIONARY AUTOREGRESSIVE PROCESSES

We have noted in the introduction that the process $\{X_n\}$ defined at (5.1.2) is stationary for <u>all</u> $\rho \in (0, 1)$ if and only if the common distribution G of $Y_n$'s and $X_0$ is G(S)S. From the Definition 5.4.5 of geometrically Semi-stable laws, it follows that the process $\{X_n\}$ of structure at (5.1.2) is stationary for <u>some</u> $\rho \in (0, 1)$ if and only if G is geometrically Semi-stable.

From the definitions of DGS and DGSS laws, it is clear that the discrete process $\{X_n\}$ defined at (5.1.8) is stationary for <u>all</u> $\rho \in (0, 1)$ if and only if the common distribution G of $Z_n$'s and $X_0$ is DGS , whereas the stationarity holds for <u>some</u> $\rho \in (0, 1)$ if and only if G is DGSS.

Consider the maximum process $\{U_n\}$ defined by

$$U_n = \begin{cases} c(\rho)U_{n-1} & \text{with probability } \rho \\ \max(c(\rho)U_{n-1}, W_n) & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.7.1)$$

where $\{W_n\}$ is a sequence of iid rvs, independent of $U_0$, having a common distribution G, $U_0$ is distributed as G and $c(\rho) \in [0, 1]$ is a function of $\rho \in (0, 1)$.

From the concepts introduced in Section 5.6, and making analogous arguments as in the above two cases, it becomes clear that $\{U_n\}$ defined at (5.7.1) is stationary for <u>all</u> $\rho \in (0, 1)$ if and only if G is geometrically strictly max-stable (Pareto), whereas stationarity holds for <u>some</u> $\rho \in (0, 1)$ if and only if G is geometrically max-semi-stable. (Semi-Pareto).

It may be noted that in all the three cases viz., G(S)S laws, DGS laws and geometrically strictly max-stable laws, the function $c(\rho)$ must be of the form $c(\rho) = \rho^{1/\alpha}$. Thus stationary autoregressive processes with GSS($\alpha$, $\theta$), DGS($\alpha$, $\theta$) and GMS($\alpha$, $\theta$) (GMS distribution with survival function $\bar{F}(x) = (1+\theta x^{\alpha})$ ) marginals are respectively as follows.

$$X_n = \begin{cases} \rho^{1/\alpha} X_{n-1} & \text{with probability } \rho \\ \rho^{1/\alpha} X_{n-1} + \xi_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.7.2)$$

$$Y_n = \begin{cases} \rho^{1/\alpha} * Y_{n-1} & \text{with probability } \rho \\ \rho^{1/\alpha} * Y_{n-1} + Z_n & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.7.3)$$

and

$$U_n = \begin{cases} \rho^{1/\alpha}\, U_{n-1} & \text{with probability } \rho \\ \max(\rho^{1/\alpha}\, U_{n-1}, W_n) & \text{with probability } 1-\rho \end{cases}, \qquad \ldots(5.7.4)$$

where the common distribution of $\xi_n$'s and $X_0$ is G(S)S($\alpha$, $\theta$), the common distribution of $Z_n$'s and $Y_0$ is DGS($\alpha$, $\theta$) and the common distribution of $W_n$'s and $U_0$ is GMS($\alpha$, $\theta$). Here $\xi_n$'s , $Z_n$'s and $W_n$'s are independent of $X_0$, $Y_0$ and $U_0$ respectively.

The stationary minification process with GMS($\alpha$, $\theta$) marginals will be of the form

$$U'_n = \begin{cases} \rho^{-1/\alpha}\, U'_{n-1} & \text{with probability } \rho \\ \min(\rho^{-1/\alpha}\, U'_{n-1}, W_n) & \text{with probability } (1-\rho) \end{cases} \qquad \ldots(5.7.5)$$

*Remark 5.7.1* : EAR(1) process of Gaver and Lewis (1980) is a particular case of the G(S)S process (5.7.2), and geometric process of McKenzie(1986) is a particular case of DGS process (5.7.3). GMS process (5.7.5) is same as the Pareto process ARP(1) of Yeh et al. (1988) ∎

From the above discussions it is clear that the processes (5.7.2) and (5.7.3), with $\alpha = 1$, are stationary for every $\rho \in (0, 1)$ if and only if $\{\xi_n\}$ and $\{Z_n\}$ have exponential and geometric marginals respectively. These characterizations of exponential process and geometric process were given in Sections 4.2.2 and 4.3.1 respectively.

SOME NEW PROCESSES :

Let F be a geometrically infinite divisible distribution. Then for every $\rho \in (0, 1)$, there exists a rv $X(\rho)$ with distribution $F_{(\rho)}$ such that (5.1.6) holds. This fact can be exploited to construct a new stationary process $\{X_n\}$, with marginals F, as follows :

$$X_n = \begin{cases} \xi_n & \text{with probability } \rho \\ X_{n-1} + \xi_n & \text{with probability } (1-\rho) \end{cases}, \qquad \ldots(5.7.6)$$

where $\{\xi_n\}$ is a sequence of iid rvs, independent of $X_0$, with common distribution $F_{(\rho)}$, and $X_0$ is distributed as F.

Clearly the process defined at (5.7.6) is stationary for every $\rho \in (0, 1)$ if and only if F is geometrically infinitely divisible.

A process similar to (5.7.6) can be constructed with the addition replaced by max/ min operation. This process will be stationary for every $\rho \in (0, 1)$ if and only if F is geometrically max / min infinite divisible. But as noted in Section 5.6, geometric max infinite divisibility is a trivial concept, it follows that every distribution can be the marginal of such a process.

The structure of the process defined at (5.7.6) is interesting and may be useful to model some real processes.