Chapter 2

Inferences on inliers in Exponential distribution

2.1. Introduction

Statistical methods in life testing analysis have been developed in the literature primarily for the case of single population. In their classical paper, Epstien and Sobel (1951) postulated a model for life test experiments for ordered observations when underlying distribution was exponential. Interestingly, the observations arising from life testing experiments are generally available in an ordered form. Also, the ordered observations facilitate a decision in a shorter time or with fewer observations. Earlier Walsh (1950) has studied estimates and tests based on r smallest values in a sample of normal distribution. When the failure patterns are subjected to many causes, a single failure time distribution (FTD) may not be a good model for describing the system characteristics. In particular, in tests on electronic components and devices, it has been frequently been observed that the failure rate is initially relatively high, and then actually decreases with increasing age. As the item becomes still older the failure rate either becomes constant or again increases with age depending on the basic failure mechanism involved. This behavior suggests strongly that the population is not homogeneous but rather is made up of several subpopulations mixed in unknown proportions. Finite mixture models have been broadly developed and widely applied to classification, clustering, density estimation and pattern recognition problems, as

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shown by Titterington, Smith and Markov (1985), McLachlan and Basford (1988), Lindsay (1995), B¨ohning (1999) and McLachlan and Peel (2000) etc. With the growing advances of computational methods, especially for the development of Markov chain Monte Carlo (MCMC) techniques, many works are also devoted to Bayesian mixture modeling issues, refer Diebolt and Robert (1994), Escobar and West(1995), Richardson and Green (1997) and Stephens (2000).

The occurrence of *instantaneous* or *early* failures is common in many life testing experiment (See Section 1.1 and 1.2 of Chapter 1 for many such situations, contexts and examples). In the former case, the random variable will have discrete probability mass at the origin (that is life time will be zero) and some positive life times, and in the latter case the failure times may be small in relation to other life times. Exponential distribution has been widely used as model in areas ranging from studies on the lifetimes of manufactured items to research involving survival or remission times in chronic diseases. Early failures (or inliers) are natural occurrences of a life test, where some of the items fails immediately or within in a short time of life test due to inferior quality or faulty construction.

In connection with the early failure study, Mendenhall and Hader (1958) considered a FTD which can be divided into subpopulations each representing a different type or cause of failure and proposed estimation procedures for exponential distribution censored at a predetermined test termination time. The problem was further considered by Miller (1960), where an early failure model is postulated in which one failure rate is assumed to be in effect for an initial time interval $[0,\ T_0]$ and another; lower failure rate is operative thereafter. Estimation for the two failure rates are given for the case T_0 is known and not known exactly but can be assumed to be within a specified interval. Here T_0 can be the censoring time or any time point described by the data set for early failures. If it is assumed that the early failures are reported in a small interval say $[0, \delta]$, δ known and very small, then the failure times during $[0, \delta]$ and the remaining life times will be completely inconsistent with respect to each other. Kale and Muralidharan (2000) have first introduced the term inliers in connection with the estimation of (p, θ) of early failure model with modified failure time distribution (FTD) being an exponential distribution with mean θ assuming δ known. If $\delta = T_0$, then both the problem will be similar, and for $\delta = 0$, the problem reduces to that of Aitchison (1955) for instantaneous failure case. A similar problem was attempted by Lai et al. (2007), wherein they have defined nearly instantaneous through the sample configurations, considering Weibull as the underlying distribution. For a detailed review of inlier prone models and their inferences, refer to Muralidharan (2010).

In this chapter we revisit various early failure models and propose inferential procedures to deal with different life testing situations. Along with the likelihood estimation, we also propose uniformly minimum variance unbiased estimate (UMVUE) for various parametric functions, wherever possible. Throughout the chapter, we consider the FTD as exponential with the following pdf

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \ x \geq 0 \quad . \tag{2.1.1}$$

The maximum likelihood estimator of θ is $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$. The desirable properties of $\hat{\theta}$ is numerous. In particular $\hat{\theta}$ is exactly distributed as $\left(\frac{\theta}{2n}\right)\chi^2_{(2n)}$ and it is a sufficient, efficient and minimum variance estimator of θ. In Section 2.2, we discuss the instantaneous failure model, and obtain ML estimates and unbiased estimates for various parametric functions. The next two Sections 2.3 and 2.4 will discuss early failure models, and their significance in practical studies. The discussion on nearly instantaneous failure model is presented in Section 2.5. We conclude the discussion with simulation studies and some numerical computations in the last two Sections 2.6 and 2.7.

2.2. Analysis of instantaneous failure model

If the underlying distribution is exponential as given in (2.1.1), then according to (1.3.1), the pdf of instantaneous failure model will be

$$g(x;p,\theta)=\begin{cases}1-p, & x=0\\ \frac{p}{\theta}e^{-\frac{x}{\theta}}, & x>0\end{cases} \tag{2.2.1}$$

Let $X_1, X_2, \dots, X_n$ be a random sample from (2.2.1) then the pdf of X_i is

$$g(x_i;p,\theta)=\begin{cases}(1-p)^{I(x_i)}\left(\frac{p}{\theta}e^{-\frac{x}{\theta}}\right)^{1-I(x_i)} & x_i\geq 0, 0<p\leq 1, \theta>0, i=1,2,\dots,n\\ 0, & o.w.\end{cases}$$

where,

$$I(x)=\begin{cases}1, & \text{if } x=0\\ 0, & \text{o.w.}\end{cases} \tag{2.2.2}$$

If $p=P(x>0)$ and further, if we denote $\sum_{i=1}^{n} I(x_i)=n-r$, where r is number of positive observations, then the joint pdf is given by

$$g(\underline{x};p,\theta)=\begin{cases}(1-p)^{n-r}\left(\frac{p}{\theta}\right)^{r}e^{-\frac{1}{\theta}\sum_{i=1}^{n}\left(1-I(x_i)\right)x_i}, & x_i\geq 0, r=0,1,\dots,n\\ 0, & o.w.\end{cases} \tag{2.2.3}$$

The following results are found obvious:

Result 2.2.1. *The joint density function given in (2.2.3) is a two-parameter exponential family of distribution.*

Result 2.2.2. $\left(\sum I(x_i), \sum\left(1 - I(x_i)\right)x_i\right)$ *are jointly sufficient for* p *and* θ.

Result 2.2.3. *The MLE of* p *and* θ *are respectively given by* $\hat{p}_{MLE} = \frac{r}{n}$ *and* $\hat{\theta}_{MLE} = \frac{1}{r}\sum_{x_i>0} x_i$.

Result 2.2.4. $\left(\hat{p}_{MLE}, \hat{\theta}_{MLE}\right)' \sim AN^{(2)}\left[(p,\theta)',\ diag\left(\frac{p(1-p)}{n}, \frac{\theta^2}{np}\right)\right]$.

Result 2.2.5. *The parameter* p *and* θ *are orthogonal.*

2.2.1. UMVUE based on conditional approach

The UMVUE of survival function using traditional approach of taking conditional expectation of an unbiased estimator given a complete sufficient statistics is given by Muralidharan (2000). True reliability or survival function for this model at time t is given by

$$S(t) = 1 - G(t;p,\theta) = pe^{-\frac{t}{\theta}}, t > 0, \theta > 0 \tag{2.2.4}$$

where

$$G(x;p,\theta) = \begin{cases} 0, & x < 0 \\ 1 - pe^{-\frac{x}{\theta}}, & x > 0 \end{cases}$$

It is obvious that the distribution of R is Binomial with parameters (n,p). Now, we obtain the joint pdf of X, R and Z as follows:

From (2.2.3)

$$g(\underline{x};p,\theta) = \binom{n}{r}(1-p)^{n-r}\left(\frac{p}{\theta}\right)^r e^{-\frac{1}{\theta}\sum_{x_i>0} x_i}\frac{1}{\binom{n}{r}}$$

$$= \sum_{(i_1.i_2,\dots,i_r)} (1-p)^{n-r} \left(\frac{p}{\theta}\right)^r e^{-\frac{1}{\theta}\sum_{x_i>0} x_i} \frac{1}{\binom{n}{r}}$$

where the summation is taken over all values of $i_1.i_2,\dots,i_r$ such that only $x_{i_k} > 0$, $k = 1,2,\dots,r$. The above density can be written as

$$g_{\underline{X},R}(\underline{x};p,\theta) = \sum_{(i_1.i_2,\dots,i_r)} P(X_{j_1} = 0, X_{j_2} = 0, \dots, X_{j_{n-r}} = 0)\; g(x_{i_1}, x_{i_2}, \dots, x_{i_r}; p\,\theta) \qquad (2.2.5)$$

where

$$g(x_{i_1}, x_{i_2}, \dots, x_{i_r}; p\,\theta) = \frac{1}{\theta^r} e^{-\frac{1}{\theta}\sum_{x_i>0} x_i},\ x_{i_k} > 0,\ k = 1,2,\dots,r. \qquad (2.2.6)$$

Making the transformation $X_{i_k} = x_{i_k}$, $k = 1,2,\dots,r-1$ and $Z = \sum_{i=1}^{r} x_{i_k}$, we have the joint pdf of $X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}$ and Z as

$$g(x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}, z; p\,\theta) = \frac{1}{\theta^r} e^{-\frac{z}{\theta}}, z > 0 \qquad (2.2.7)$$

Using (2.2.7) in (2.2.6), we get

$$g_{\underline{X},R,Z}(\underline{x}, r, z; p, \theta) = (1-p)^{n-r} \left(\frac{p}{\theta}\right)^r e^{-\frac{z}{\theta}}, x_i \geq 0; z = \sum_{x_i>0} x_i \qquad (2.2.8)$$

Integrating out $x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}$ from (2.2.6), we get the conditional pdf of z given $R = r$ as

$$g_{Z|R}(z;\theta|r) = \begin{cases} \dfrac{e^{-\frac{z}{\theta}} z^{r-1}}{\Gamma r\, \theta^r}, & z > 0, r > 0 \\ 1, & z = 0, r = 0 \end{cases} \qquad (2.2.9)$$

Again the density (2.2.8) can be written as

$$g_{\underline{X},R,Z}(\underline{x}, r, z; p, \theta) = \binom{n}{r}(1-p)^{n-r}p^r \frac{e^{-\frac{z}{\theta}} z^{r-1}}{\Gamma r \, \theta^r} \frac{\Gamma r}{\binom{n}{r} z^{r-1}}$$

$$= g_R(r; p) g_{Z|R}(z; \theta|r) g\big(x_1, x_2, \dots, x_n; p, \theta|(z, r)\big)$$

Then the pdf of $x_1, x_2, \dots, x_n$ given $(z, r), z > 0, r > 0$ is

$$g\big(x_1, x_2, \dots, x_n|(z, r)\big) = \frac{\Gamma r}{\binom{n}{r} z^{r-1}} x_i \geq 0. \tag{2.2.10}$$

Consider $x_1 > 0$ and using the fact that in the remaining $(n-1)$ x's only $(r-1)$ are positive, we get

$$g\big(x_1|(z, r)\big) = \frac{\Gamma r}{\binom{n}{r} z^{r-1}} \binom{n-1}{r-1} \int_0^{t-x_1} \cdots \int_0^{t-\sum_{i=1}^{r-1} x_i} dx_{r-1} \cdots dx_2$$

Evaluating the integrals successively, we get

$$g\big(x_1|(z, r)\big) = \frac{\Gamma r}{n(r-2)!\, z^{r-1}} (z - x_l)^{r-2}, 0 < x_1 < z, r > 1$$

Hence the pdf of $X_1|(Z, R)$ is

$$g\big(x_1|(z, r)\big) = \begin{cases} 1, & x_1 = 0; z = 0 \text{ (given } r = 0) \\ 1 - \frac{r}{n}, & x_1 = 0; z > 0 \text{ (given } r \geq 1) \\ \frac{r}{n}, & x_1 = z; z > 0 \text{ (given } r = 1) \\ \frac{r(r-1)(z-x)^{r-2}}{n z^{r-1}}, & 0 < x_1 < z; z > 0 \text{ (given } r > 1) \end{cases} \tag{2.2.11}$$

We use (2.2.11) to obtain UMVUE of $S(t)$ when p unknown and known cases below:

Case: p unknown

Define

$$\phi_t(x) = \begin{cases} 1, & x > t \\ 0, & o.w. \end{cases} \tag{2.2.12}$$

Then $E[\phi_t(x)]$ is equal to (2.2.4), which implies that $\phi_t(x)$ is an unbiased estimator of $S(t)$ Using Rao-Blackwell and Lehmann-Scheffe's theorem, the UMVUE of $S(t)$ for p unknown is given by

$$\tilde{S}_1(t) = E[\phi_t(X)|(z,r)]$$

$$= \int_t^z g\big(x|(z,r)\big)\,dx$$

$$= \frac{r}{n}\Big(1 - \frac{t}{z}\Big)^{r-1}, t < z \tag{2.2.13}$$

Corollary 2.2.1. If $r = n$, i.e. $p = 1$, then the UMVUE obtained in (2.2.13) reduces to that of the UMVUE of exponential distribution. (See also Sinha, 1986).

Case: p is known

We have from (2.2.10), the joint density of X_1, and R given $Z = z$ is

$$g(x_1, r|z) = \sum_r g\big(x_1|(z,r)\big) g_R(r)$$

$$g\big(x_1|(z,r)\big)$$
$$=\begin{cases}(1-p)^n, & x_1=0; r=0 \text{ (given } z=0)\\ 1-\frac{r}{n}\binom{n}{r}p^r(1-p)^{n-r}, & x_1=0;\ r\geq 1 \text{ (given } z>0)\\ p(1-p)^{n-1}, & x_1=z; r=1 \text{ (given } z>0)\\ \frac{r(r-1)}{nz}\binom{n}{r}p^r(1-p)^{n-r}\left(1-\frac{x_1}{z}\right)^{r-2}, & 0<x_1<z; z>0 \text{ (given } r>1)\end{cases}$$

Then the conditional density of X_1 given $Z=z$ and $R\geq 1$ is

$$g(x_1|z,r\geq 1)=\begin{cases}(1-p)[1-(1-p)^{n-1}], & x_1=0\\ \frac{(n-1)}{\mathcal{P}z}p^2\left(1-\frac{px_1}{z}\right)^{n-2}, & 0<x_1<z\\ \frac{p(1-p)^{n-1}}{\mathcal{P}}, & x_1=z\end{cases} \quad (2.2.14)$$

where

$$\mathcal{P}=P(R\geq 1)=1-(1-p)^{n-1} \quad (2.2.15)$$

Then, the UMVUE of $S(t)$ for p known is given by

$$\tilde{S}_2(t)=\int_t^z g(x_1|z,r\geq 1)\,dx$$

$$=\int_t^z \frac{(n-1)}{\mathcal{P}z}p^2\left(1-\frac{px_1}{z}\right)^{n-2}dx$$

$$=\frac{p}{\mathcal{P}}\left[\left(1-\frac{pt}{z}\right)^{n-1}-(1-p)^{n-1}\right]. \quad (2.2.16)$$

Corollary 2.2.2. If $p=1$ then $\mathcal{P}=1$, and hence the UMVUE obtained in (2.2.16) reduces to that of the UMVUE of exponential distribution.

2.2.2. UMVUE based on exponential family approach

It is observed that, to obtain conditional distribution given the sufficient statistics is bit difficult. There are no closed forms available in some of the cases, and hence an easy approach is by using the method given by Roy and Mitra (1957) in exponential family of distribution is used. Writing (2.2.1) in the form of (1.5.9), then

$$g(x;p,\theta)=\frac{\left[e^{-\frac{1}{\theta}}\right]^{(1-I(x))d(x)}\left[\frac{\theta(1-p)}{p}\right]^{I(x)}}{\left(\frac{\theta}{p}\right)}$$

$$=[a(x)]^{(1-I(x))}[h(\theta)]^{(1-I(x))d(x)}\left[\frac{g(\theta)(1-p)}{p}\right]^{I(x)}\left(\frac{g(\theta)}{p}\right)^{-1} \quad (2.2.17)$$

where $a(x)=1$, $h(\theta)=e^{-\frac{1}{\theta}}$, $d(x)=x$, $g(\theta)=\theta$. The density in (2.2.17) is so obtained is defined with respect to measure $\mu(x)$ which is the sum of Lebesgue measure over $(0,\infty)$ and a singular measure at $\{0\}$, is a well-known form of two parameter exponential family with natural parameters $(\eta_1,\eta_2)=\left(log\left(\frac{\theta(1-p)}{p}\right),log\left(e^{-\frac{1}{\theta}}\right)\right)$ generated by the underlying indexing parameters (p,θ). Here $\left(I(x),\left(1-I(x)\right)x\right)$ is jointly minimal sufficient for (p,θ) as $I(x)$ and $(1-I(x))x$ do not satisfy any linear restriction. Hence the natural parameter space is convex set in E_2 containing a two-dimensional rectangle making (2.2.17) a full rank family. The statistic $\left(I(x),\left(1-\mathrm{I(x)}\right)x\right)$ is thus complete (Lehmann and Casella, 1998, p 42). Kale and Muralidharan (2000) considered the above mixture and obtained optimal estimating equation for θ ignoring p in the case of exponential FTD.

Further, if we denote $z=\sum_{i=1}^{n}[1-I(x_i)]x_i\left(=\sum_{x_{i>0}}x_i\right)$, then the joint density function can be expressed as

$$g(\underline{x};p,\theta)=\binom{n}{r}(1-p)^{n-r}\left(\frac{p}{\theta}\right)^{r}e^{-\frac{z}{\theta}} \tag{2.2.18}$$

Hence $(n-R,Z)$ are jointly complete sufficient for (p,θ). Also, the variable $(Z|R=r,r>0)$ is distributed as a Gamma random variable with parameter $(r,\theta\,)$. Since, $n-R$ is binomial which is same as that of R with parameter (n,p). Hence the joint distribution of $(n-R,Z)$ is

$$g(z,n-r;p,\theta)=P(n-R=n-r)\;g(z;\theta|n-r)$$

$$=P(R=r)\;g(z;\theta|r)$$

$$=\binom{n}{r}(1-p)^{n-r}p^{r}\frac{1}{\Gamma r\;\theta^{r}}z^{r-1}e^{-\frac{z}{\theta}}$$

$$=\begin{cases}(1-p)^{n}, & z=0;\;r=0\\ \binom{n}{r}\dfrac{z^{r}}{\Gamma r}e^{-\frac{z}{\theta}}\left(\dfrac{\theta(1-p)}{p}\right)^{n-r}\left(\dfrac{\theta}{p}\right)^{-n}, & z>0;\;r>0\end{cases}$$

$$=\begin{cases}(1-p)^{n}, & z=0;\;r=0\\ B(z,r,n)[h(\theta)]^{z}\left[\frac{g(\theta)(1-p)}{p}\right]^{n-r}\left(\frac{g(\theta)}{p}\right)^{-n}, & z>0;\;r>0\end{cases} \tag{2.2.19}$$

where

$$B(z,r,n)=\begin{cases}1, & z=0;r=0\\ \binom{n}{r}B(z|r), & z>0;r>0\end{cases} \tag{2.2.20}$$

is such that

$$(1-p)^{n}+\sum_{r=1}^{n}\int_{z>0}\binom{n}{r}B(z|r)\left[e^{-\frac{1}{\theta}}\right]^{z}\left(\frac{\theta(1-p)}{p}\right)^{n-r}\left(\frac{\theta}{p}\right)^{-n}dz=1.$$

and $B(z|r) = \frac{z^{r-1}}{\Gamma r}$. Following Roy and Mitra (1957) and Jani and Singh (1995), it is possible to obtain the uniformly minimum variance unbiased estimates (UMVUE) for some parametric functions. Note that by referring Lemma 1.5.1, the UMVUE's of parametric function $\phi(p,\theta)$exits if and only if $\phi(p,\theta)$can be expressed in the form

$$\phi(p,\theta) = \alpha(0,0,n)(1-p)^n + \sum_{r=1}^{n} \int_{z>0} \frac{\alpha(z,r,n)e^{-\frac{z}{\theta}}\left(\frac{\theta(1-p)}{p}\right)^{n-r}}{\left[\frac{\theta}{p}\right]^n} dz.$$

Thus, using (1.5.15) the UMVUE of a function $\phi(p,\theta)$ of θ and p in $g(x;p,\theta)$ is given by

$$\psi(Z,R,n) = \frac{\alpha(Z,R,n)}{B(Z,R,n)}, B(Z,R,n) \neq 0.$$

Result 2.2.6. *Suppose $X_1, X_2, \dots, X_n$ is a random sample from (2.2.17). For $m \le n$, and using Result 1.5.1, the UMVUE of $(1-p)^m$ is $G_m(Z,R,n)$ as given by (1.5.16).*

Result 2.2.7. *For $m = 1$, in Result 2.2.6, the UMVUE of $(1-p)$ is*

$$G_1(z,r,n) = \begin{cases} \frac{n-r}{n}, & r > 0; z > 0 \\ 1, & r = 0, z = 0 \end{cases}$$

Result 2.2.8. *For $m \le \frac{n}{2}$ using Result 1.5.2 the UMVUE of the variance of $G_m(Z,R,n)$ is given by (1.5.17).*

Result 2.2.9. *For $m = 1$, the UMVUE of the variance of UMVUE of $(1-p)$ is given by*

$$\widehat{var}\,[G_1(z,r,n)] = \begin{cases} \frac{r(n-r)}{n^2(n-1)}, & r = 1, 2, \dots, (n-1) \\ 0, & o.w. \end{cases}$$

Result 2.2.10. *For* $k > 0$ *and using Result 1.5.3, the UMVUE of parametric function* $(1-p)^n + \left(\frac{p}{\theta}\right)^k [1-(1-p)^{n-k}]$ *is given by*

$$H_k(z,r,n) = \begin{cases} \dfrac{(r)_k(r-1)_k}{(n)_k z^k}, & r = 1,2,\ldots,n;\ z > 0 \\ 1, & r = 0;\ z = 0 \end{cases}$$

where $(a)_k = a(a-1)\ldots(a-k+1)$, and $z = \sum_{x_{i>0}} x_i$.

For various values of $k \geq 1$, one can obtain the UMVUE of the parametric function. Unfortunately, it is impossible to find a unbiased estimate for the parameter θ alone. Aitchison (1955) through the usual classical approach obtain the UMVUE of the parametric function $(1-p)^2\theta^2$ as

$$\varphi(z,r,n) = \begin{cases} \dfrac{(2n-r-1)z^2}{n(n-1)(r+1)}, & r > 0;\ z > 0 \\ 0, & r = 0;\ z = 0 \end{cases}$$

Result 2.2.11. *The UMVUE of the variance of* $H_k(Z,R,n)$, *according to Result 1.5.4 is obtained as*

$$\widehat{var}[H_k(z,r,n)] = \begin{cases} \left[\dfrac{(r)_k(r-1)_k}{(n)_k z^k}\right]^2 - \dfrac{(r)_{2k}(r-1)_{2k}}{(n)_{2k} z^{2k}}, & r = 1,2,\ldots,n;\ z > 0 \\ 0, & o.w. \end{cases}$$

Result 2.2.12. *For fixed* x, *according to Result 1.5.5, the UMVUE of pdf* $g(x;p,\theta)$ *is shown as*

$$\phi_x(z,r,n) = \begin{cases} \dfrac{r(r-1)}{nz}\left(1+\dfrac{x}{z}\right)^{r-2}, & 0 < x < z;\ r = 1,2,\ldots,n \\ \dfrac{n-r}{n}, & x = 0;\ r = 0,1,\ldots,n-1 \\ 0, & o.w. \end{cases}$$

Result 2.2.13. *For $r = n$, that is when all the observations are coming from the density given in (2.1.1), then the UMVUE of the density $f(x;\theta)$ is simplified as*

$$\phi_x(z,r,n) = \begin{cases} \frac{n-1}{z}\left(1+\frac{x}{z}\right)^{n-2}, & 0 < x < z;\ n > 1 \\ 0, & o.w. \end{cases}$$

Result 2.2.14. For fixed x, *according to Result 1.5.6, the UMVUE of variance of pdf $g(x;p,\theta)$ is obtained as*

$$\widehat{var}[\phi_x(z,r,n)]$$
$$= \begin{cases} \left[\frac{r(r-1)}{nz}\left(1-\frac{x}{z}\right)^{r-2}\right]^2 \\ -\frac{r(r-1)^2(r-2)}{n(n-1)z(z-x)}\left(1-\frac{x}{z}\right)^{r-2}\left(1-\frac{x}{z-x}\right)^{r-3}, & 0 < x < z;\ r = 2 \dots, n \\ \left[\frac{r(r-1)}{nz}\left(1-\frac{x}{z}\right)^{r-2}\right]^2, & 0 < x < z;\ r = 2, \dots, n \\ \frac{r(n-r)}{n^2(n-1)}, & x = 0;\ r = 0,1, \dots, n-1 \\ 0, & o.w. \end{cases}$$

For $r = n$, all the results will reduces to that of the estimates of an exponential distribution, without inliers.

Result 2.2.15. *For fixed z and r, the UMVUE of the survival function $S(t) = P(X > t)$, $t \geq 0$ is obtained as*

$$\hat{S}(t) = \begin{cases} \frac{r}{n}\left(1-\frac{t}{z}\right)^{r-1}, & t < z \\ 0, & o.w. \end{cases}$$

Proof: Using Corollary 1.5.1,

$$\hat{S}(t) = \int_{x>t} \phi_x(z, r, n)\, dx$$

$$= \int_t^{\infty} \frac{r(r-1)}{nz}\left(1 - \frac{x}{z}\right)^{r-2} dx$$

$$= \frac{r}{n}\left(1 - \frac{t}{z}\right)^{r-1}, \quad t < z.$$

Hence the proof. ■

Remark: The above result coincides with the result obtained in (2.2.13) under the conditional approach.

Result 2.2.16. *For fixed z and r, the UMVUE of the variance of $\hat{S}(t)$ is obtained as (see also equation (1.5.23))*

$$\widehat{var}\left[\hat{S}(t)\right] = \begin{cases} \left[\frac{r}{n}\left(1 - \frac{t}{z}\right)^{r-1}\right]^2 - \frac{r(r-1)}{n(n-1)}\left(1 - \frac{2\,t}{z}\right)^{r-1}, & z > 2t \\ \left[\frac{r}{n}\left(1 - \frac{t}{z}\right)^{r-1}\right]^2, & t < z < 2t \\ 0, & o.w. \end{cases}$$

For $r = n$, both the above results reduce to the case of an exponential distribution.

2.3. Analysis of early failure model-1

In this section, we consider the early failure model given by Miller (1960) in which one failure rate is assumed to be in effect for an initial time interval $[0, T_0)$ and another lower failure rate is operative thereafter, as given in section 1.3.2. We viewed this model as a model for a shift in the hazard function of exponential distribution, where, we assume that the population of components is composed of two groups with different failure rates, λ_1 and λ_2. Suppose, the early failure group comprises a proportion p of this population, then the situation is modeled using a complete mixture of two distributions in the proportion p and $(1-p)$. Mendehall and Hader (1958) also discussed this type of problem with exponential data. In this context the estimation of λ_1, λ_2, and p is complicated unless each failure is examined to determine to which subpopulation it belongs. For some experimental programs, however, it is not always possible to examine each failure to determine its cause.

Miller (1960) proposed another early failure model, which works in the following manner: Suppose N items are put on test and are tested until failure or time T_1, whichever is sooner, where $T_0 < T_1$. Due to limitations on experimental time, no item is tested past T_1. For this censored test structure, let X^* be the time at which the test terminates for any particular unit. For $X^* = x < T_1$, the value $X^* = x$ denotes the failure of the item at time x. $X^* = T_1$, denotes the time that had not failed when the experiment was terminated at T_1. Then the probability distribution of X^* to be

$$f_{X^*}(x;\ \lambda_1, \lambda_2) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & 0 \le x < T_0 \\ \lambda_2 e^{-\lambda_1 T_0 - \lambda_2 (x - T_0)}, & T_0 \le x < T_1 \\ e^{-\lambda_1 T_0 - \lambda_2 (T_1 - T_0)}, & x = T_1 \end{cases} \qquad (2.3.1)$$

In the event that the experimental procedure is to test all items to failure, the equation (2.3.1) becomes the limiting expressions as $T_1 \to \infty$. The above experiment leads to the estimator of λ as

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i + (N-n)T_1}. \qquad (2.3.2)$$

As $N \to \infty$, $\hat{\lambda}$ will converge in probability to λ. However, if the assumption of a constant failure rate is incorrect and in fact the early failure model pertains, $\hat{\lambda}$ will converge in probability to

$$\Lambda(\lambda_1,\lambda_2,T_0,T_1)=\frac{1-e^{-\lambda_1T_0-\lambda_2(T_1-T_0)}}{\frac{1}{\lambda_1}\left(1-e^{-\lambda_1T_0}\right)+\frac{1}{\lambda_2}e^{-\lambda_1T_0}\left(1-e^{-\lambda_2(T_1-T_0)}\right)}. \tag{2.3.3}$$

By choosing plausible values of λ_1,λ_2, and T_0, the experimenter can determine the magnitude of the error one may be making by not assuming the early failure model. For asymptotic studies under cases T_0 known and unknown, one may refer to Miller (1960) for details. By retaining the model in equation (1.3.3), we slightly modify the sampling plan in this way: Suppose n items are put on test, and $n-r$ of them failed before some time, say δ, which is known and fixed. If we define

$$I(x)=\begin{cases}1, & x\le\delta\\ 0, & x>\delta\end{cases} \tag{2.3.4}$$

then the following results are obvious:

Result 2.3.1. *The MLE of θ_1 and θ_2 are respectively given by*

$$\hat{\theta}_{1MLE}=\frac{1}{n-r}\left(\sum_{x_i<\delta}x_i+r\delta\right)\ and\ \hat{\theta}_{2MLE}=\frac{1}{r}\left(\sum_{x_i<\delta}x_i-\delta\right).$$

Result 2.3.2. $\left(\hat{\theta}_{1MLE},\hat{\theta}_{2MLE}\right)'\sim AN^{(2)}\left[(\theta_1,\theta_2)',diag\left(\frac{\theta_1^2}{n\left(1-e^{-\frac{\delta}{\theta_1}}\right)},\frac{\theta_2^2}{ne^{-\frac{\delta}{\theta_1}}}\right)\right].$

Result 2.3.3. *The parameter θ_1 and θ_2 are independent.*

The above results are comparable with the results obtained by Miller (1960) for $N\to\infty$. Note that, the model discussed in (1.3.2) and (2.3.1) may look like a change point model where changes are observed in the failure rate of two exponential distributions. Instead, if we allow, the failure mechanism to decide the early failures from the available data, then the model can be more viable for easy implementation.

We now discuss one such model in the next section. Also note that UMVU estimation is not possible in the above model.

2.4. Analysis of early failure model-2

Now consider the model as given in equation (1.3.4) of Section 1.3.3. If $f(x;\theta)$ is as in (2.1.1), I(x) is the indicator function defined in equation (1.5.10), for $d = \delta$ and $\sum_{i=1}^{n} I(x_i) = n - r$, where r is number of observations greater than δ, then the joint probability correspond to exponential early failure is written as

$$g(\underline{x}; p, \theta) = \left(1 - p\, e^{-\frac{\delta}{\theta}}\right)^{n-r} \left(\frac{p}{\theta}\right)^{r} e^{-\frac{\sum x_{i>\delta} x_i}{\theta}}. \tag{2.4.1}$$

The following results are found obvious:

Result 2.4.1. *The MLE of p and θ are respectively given by*

$$\hat{p}_{MLE} = \frac{r}{n}\, e^{\frac{\delta}{\theta}} \text{ and } \hat{\theta}_{MLE} = \frac{\sum x_{i>\delta}(x_i - \delta)}{r}.$$

Result 2.4.2. $(\hat{p}_{MLE}, \hat{\theta}_{MLE})' \sim \mathrm{AN}^{(2)}\left[(p,\theta)', \frac{1}{n} I_{g_1}^{-1}(p,\theta)\right]$, *where*

$$\frac{1}{n} I_{g_1}^{-1}(p,\theta) = \begin{bmatrix} \frac{p\left(1 - p\, e^{-\frac{\delta}{\theta}}\right)}{n e^{-\frac{\delta}{\theta}}}\left(1 + \frac{\delta^2}{\theta^2\left(1 - p\, e^{-\frac{\delta}{\theta}}\right)}\right) & -\frac{\delta}{n e^{-\frac{\delta}{\theta}}} \\ -\frac{\delta}{n e^{-\frac{\delta}{\theta}}} & \frac{\theta^2}{np\, e^{-\frac{\delta}{\theta}}} \end{bmatrix}$$

Result 2.4.3. *The parameter p is not orthogonal to the parameter θ.*

We now proceed with the UMVU Estimation. Writing (2.4.1), in the form of (1.5.9), we get

$$g_1(x;p,\theta) = (1-pe^{-\frac{\delta}{\theta}})^{I(x)}\left(\frac{p}{\theta}e^{-\frac{x}{\theta}}\right)^{(1-I(x))}$$

$$= [a(x)]^{(1-I(x))}[h(\theta)]^{(1-I(x))d(x)}\left[\frac{g(\theta)(1-pe^{-\frac{\delta}{\theta}})}{pe^{-\frac{\delta}{\theta}}}\right]^{I(x)}\left(\frac{g(\theta)}{pe^{-\frac{\delta}{\theta}}}\right)^{-1} \qquad (2.4.2)$$

where, $a(x)=1$, $h(\theta)=e^{-\frac{1}{\theta}}$, $d(x)=x$, $g(\theta)=\theta e^{-\frac{\delta}{\theta}}$. If we apply the same arguments as before, we can show that (2.4.2) is a two-parameter exponential family with natural parameters $(\eta_1,\eta_2)=\left(log\left[\frac{e^{-\frac{\delta}{\theta}}(1-pe^{-\frac{\delta}{\theta}})}{pe^{-\frac{\delta}{\theta}}}\right], log\left(e^{-\frac{1}{\theta}}\right)\right)$, indexed by the parameters $\left(pe^{-\frac{\delta}{\theta}},\theta\right)$. Hence $\left(I(x),\left(1-I(x)\right)x\right)$ is jointly minimal sufficient for $\left(pe^{-\frac{\delta}{\theta}},\theta\right)$, as $I(x)$ and $\left(1-I(x)\right)x$ do not satisfy any linear restriction. Further, if we denote $z=\sum_{i=1}^{n}[1-I(x_i)]x_i\left(=\sum_{x_i>\delta}x_i\right)$, then the joint density function can be expressed as

$$g_1(z;p,\theta)=\binom{n}{r}\left(pe^{-\frac{\delta}{\theta}}\right)^{r}\left(1-pe^{-\frac{\delta}{\theta}}\right)^{n-r}\frac{\left(e^{-\frac{1}{\theta}}\right)^{z}}{\binom{n}{r}\left(\theta e^{-\frac{\delta}{\theta}}\right)^{r}}$$

$$= P(n-R=n-r)\; g_1(z;\theta\,|n-r). \qquad (2.4.3)$$

Therefore, by Neyman factorization theorem $(n-R,Z)$ are jointly sufficient for $\left(pe^{-\frac{\delta}{\theta}},\theta\right)$. Here r is number of observations greater than δ. Also, $n-R$ is binomial which is same as that of R with parameter $\left(n, pe^{-\frac{\delta}{\theta}}\right)$, and is a complete family. Also the variable $(Z|R = r, r > 0)$ is distributed as a Gamma random variable having density

$$g_1(z;\ \theta|r) = \frac{1}{\Gamma r\left(\theta e^{-\frac{\delta}{\theta}}\right)^r} z^{r-1} e^{-\left(\frac{z}{\theta e^{-\frac{\delta}{\theta}}}\right)},\ z > 0;\ \theta > 0. \tag{2.4.4}$$

Hence $Z|R$ is complete sufficient for $\theta e^{-\frac{\delta}{\theta}}$. This preserves the exponential structure for (2.4.4). Therefore, the joint pdf of complete sufficient statistics can be written as

$$g_1(z, n-r;\ p, \theta) = \begin{cases} \left(1 - pe^{-\frac{\delta}{\theta}}\right)^n, & z = \delta; r = 0 \\ B(z, r, n)\dfrac{[h(\theta)]^z \left[\dfrac{g(\theta)(1 - pe^{-\frac{\delta}{\theta}})}{pe^{-\frac{\delta}{\theta}}}\right]^{n-r}}{\left(\dfrac{g(\theta)}{pe^{-\frac{\delta}{\theta}}}\right)^n}, & z > \delta; r = 1, 2, \dots, n \end{cases} \tag{2.4.5}$$

where

$$B(z, r, n) = \begin{cases} 1, & z = \delta;\ r = 0 \\ \binom{n}{r}\dfrac{z^{r-1}}{\Gamma r}, & z > \delta;\ r = 1,2, \dots, n. \end{cases} \tag{2.4.6}$$

and $B(z|r) = \frac{z^{r-1}}{\Gamma r}$. The UMVUE's of parametric function $\phi(p, \theta)$exits if and only if (p, θ)can be expressed in the form

$$\phi(p, \theta) = \alpha(\delta, 0, n)\left(1 - pe^{-\frac{\delta}{\theta}}\right)^n + \sum_{r=1}^{n} \int_{z>\delta} \alpha(z, r, n)\, e^{-\frac{z}{\theta}} \left[\frac{g(\theta)(1-1-pe^{-\frac{\delta}{\theta}})}{pe^{-\frac{\delta}{\theta}}}\right]^{n-r} \left(\frac{g(\theta)}{pe^{-\frac{\delta}{\theta}}}\right)^{-n} dz.$$

Note that, the UMVUE of function θ alone is not U-estimable. The UMVUE of parametric function can be developed in the same way as we have done in the instantaneous failure case above, where the observations are considered as $X = \delta$ and $X > \delta$ instead of $X = 0$ and $X > 0$ respectively. Hence the expressions are not represented again. The numerical computation of parametric functions are discussed in the last section.

2.5. Nearly instantaneous failure model

It is seen that the models described above are represented as a mixture of a singular distribution at zero and exponential distribution in different proportion. Because of the singular nature of the distribution, it is unable to define the failure rate function meaningfully. Lai et.al. (2007) have studied a flexible model as a mixture of two continuous distributions. This modification allows establishing and studying the failure rate function via mixture distribution.

Instead of assuming an instant or an early failure to occur at a particular time point as in the original model, the model is represented as a mixture of a generalized Dirac delta function and $f(x;\ \theta)$ as given in (2.1.1). Thus the resulting modification gives rise to a density function

$$f(x; p, \theta) = (1-p)\Delta_\delta(x - x_0) + \frac{p}{\theta} e^{-\frac{x}{\theta}} \tag{2.5.1}$$

where

$$\Delta_\delta(x - x_0) = \begin{cases} \frac{1}{\delta}, & x_0 \le x \le x_0 + \delta \\ 0, & o.w. \end{cases} \tag{2.5.2}$$

for sufficiently small δ. We note that $\Delta(x-x_0)=\lim_{\delta\to 0}(x-x_0)$, where $\Delta(.)$ is the Dirac delta function that is well known in mathematical analysis. We may view the Dirac delta function as a normal distribution having a zero mean and standard deviation that tends to 0. For a fixed value of δ, equation (2.5.2) denotes a uniform distribution over an interval $[x_0, x_0+\delta]$ so the modified model is now effectively a mixture of a continuous FTD with a uniform distribution. Also note that from (2.5.1) and (2.5.2), we see that the mixture density function is not continuous at x_0 and $x_0+\delta$. However, both the survival function and failure rate function are continuous. They are respectively given as

$$S(x)=\begin{cases} 1-p+pe^{-\frac{x}{\theta}}, & 0\le x<x_0 \\ \frac{(1-p)(\delta+x_0-x)}{\delta}+pe^{-\frac{x}{\theta}}, & x_0\le x\le x_0+\delta \\ pe^{-\frac{x}{\theta}}, & x>x_0+\delta \end{cases} \qquad (2.5.3)$$

and

$$h(x)=\begin{cases} \frac{1}{\theta}\left(\frac{pe^{-\frac{x}{\theta}}}{1-p+pe^{-\frac{x}{\theta}}}\right), & 0\le x<x_0 \\ \frac{1-p+\frac{1}{\theta}\left(\delta pe^{-\frac{x}{\theta}}\right)}{(1-p)(\delta+x_0-x)+\delta pe^{-\frac{x}{\theta}}}, & x_0\le x\le x_0+\delta \\ \frac{1}{\theta}, & x>x_0+\delta. \end{cases} \qquad (2.5.4)$$

Also, note that, for $x_0=0$, the model reduces to the case of instantaneous failures and for (small) $x_0\neq 0$, the model reduces to the case of early failures. We consider a special case of the model (2.5.1) where $x_0=0$. The model may be called the exponential with “nearly instantaneous failure” model. The snapshots are taken of some possible shapes from this model, as it is important to identify whether the model is useful for specific datasets for which empirical plots are available. In Figure 2.1, three density functions with p = 0.2, 0.5 and 0.8 are plotted. In all figures, largest mixing proportion p is given by the solid line. The survival functions are given by

Figure 2.2, which corresponds to the density functions in Figure 2.1. The failure rate function is given in Figure 2.3.

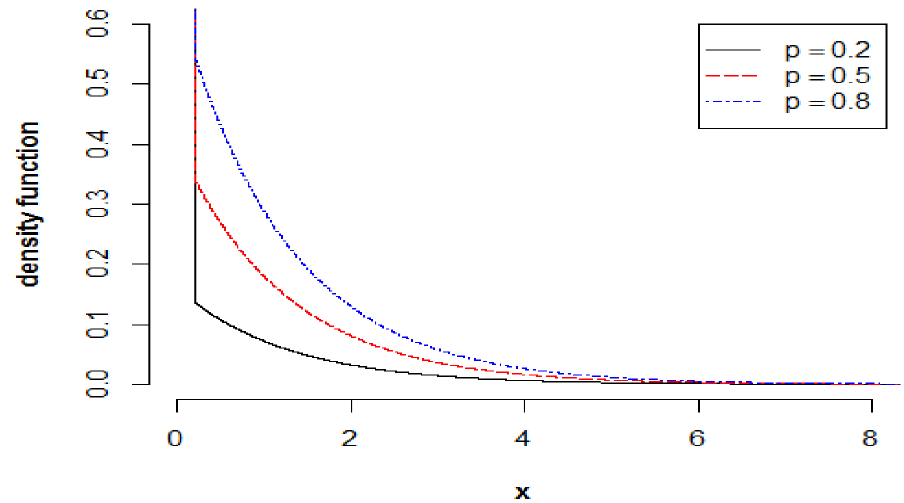

Figure 2.1. *Plots of density function* $f(x; p, \theta)$: $\theta = 0.8, \delta = 0.2, x_0 = 0$

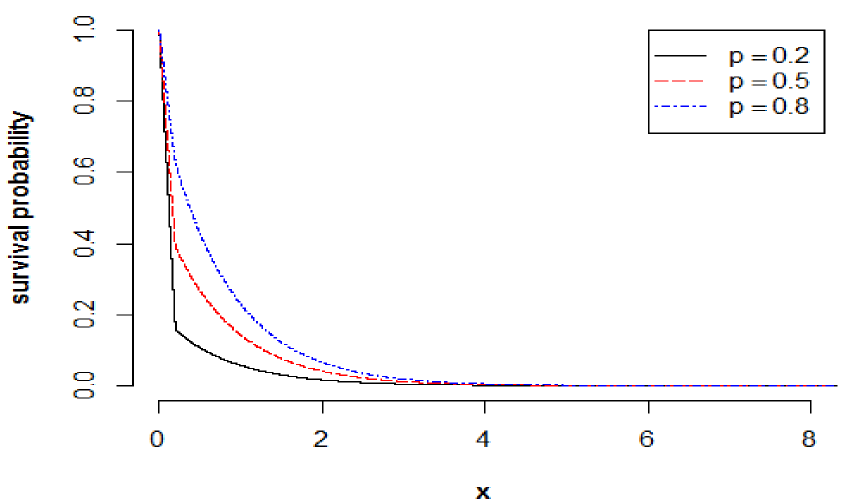

Figure 2.2. *Plots of survival function* $S(t)$: $\theta = 0.8, \delta = 0, x_0 = 0$

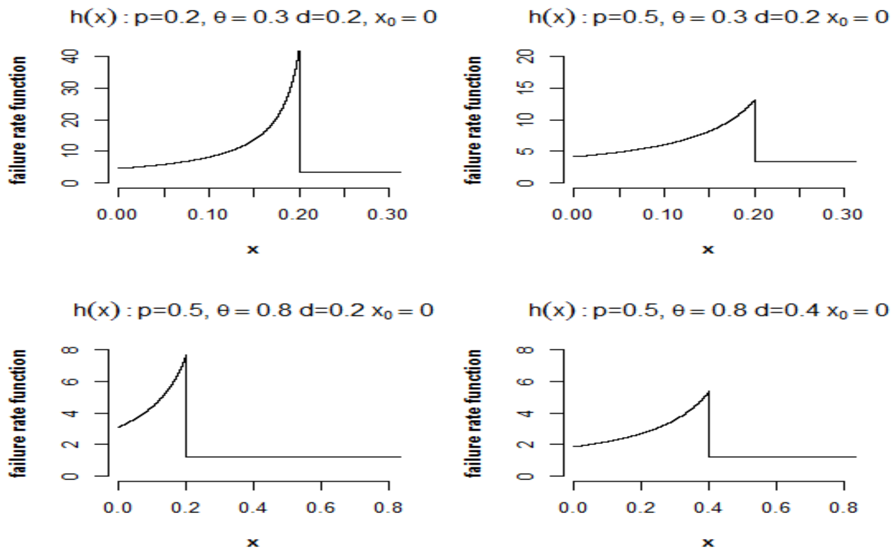

Figure 2.3. *Plots of failure rate function*

The parameter estimates are not available in closed form, but can be estimated numerically. This and other comparative study on all the model parameter estimates and confidence intervals are presented in the last section.

2.6. Simulation study

In this section, we conduct simulation experiments to check the performance of estimators under different combination of (n, r). We present the estimates of the parameters and parametric functions for various choices of inliers using simulated data. Each estimate is based on a simulation of 1000 random samples of size n=50 with different choices of r. For all models, a value of θ =12 is assumed for exponential distribution, and for early failure models, the value of δ is set at 3. In Table 2.1, we present the estimates of the parameters and parametric functions for various choices of early failures. The standard error of the estimate is shown in the bracket. For early failure model-1, we estimated both the exponential parameters of failure rate. Note that the estimate of p and is θ is comparable in all cases.

Table 2.1. *Summary of estimates of models*

Model	(n, r)	Parameter	Estimates (se)	Confidence interval
Instantaneous failure	(50,15)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.30132 (0.06413) 11.84763 (3.10818)	(0.17563, 0.42701) (5.75571, 17.93956)
	(50,25)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.49904 (0.07002) 11.94607 (2.40912)	(0.36180, 0.63628) (7.22428, 16.66786)
Nearly instantaneous failure	(50,15)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.30916 (0.07346) 11.92468 (2.98543)	(0.16518, 0.45314) (6.07334, 17.77602)
	(50,25)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.50780 (0.06922) 12.06702 (2.56342)	(0.37213, 0.64347) (7.04281, 17.09123)
Early failure-1 (δ=3)	(50,15)	$\hat{\theta}_{1MLE}$ $\hat{\theta}_{2MLE}$	2.59393 (0.44694) 11.76792 (3.03974)	(1.44270, 3.74516) (3.93808, 19.59778)
	(50,25)	$\hat{\theta}_{1MLE}$ $\hat{\theta}_{2MLE}$	4.36831 (0.88836) 11.91585 (2.41742)	(2.08006, 6.65656) (7.08942, 16.85781)
Early failure-2 (δ =3)	(50,15)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.31351 (0.05083) 12.03409 (0.35732)	(0.21388, 0.41313) (5.03070, 19.03748)
	(50,25)	$\hat{p}_{MLE}$ $\hat{\theta}_{MLE}$	0.50290 (0.06306) 11.92112 (2.75176)	(0.37930, 0.62649) (6.52778, 17.31447)

Table 2.2 presents the UMVU estimates of parametric functions and its variance for instantaneous and early failure model-2. It is seen that the standard error is very small for every combination of (n, r) and k.

Table 2.2. *Summary of estimates of parametric functions and its estimate of the variance*

		UMVUE of parametric function and its variance	
		Instantaneous failure model	Early failure model-2 (δ =3.0)
(n, r)	k	$(1-p)^n + \left(\frac{p}{\theta}\right)^k [1-(1-p)^{n-k}]$	$(1-pe^{-\delta/\theta})^n + \left(\frac{p}{\theta}\right)^k \left[1-\left(1-pe^{-\delta/\theta}\right)^{n-k}\right]$
(50,15)	1	2.531e-02 (7.822e-05)	1.530e-02 (3.690e-05)
	2	6.381e-04 (2.291e-07)	2.261e-04 (3.489e-08)
	3	1.603e-05 (3.415e-08)	3.253e-06 (4.122e-09)
(50,25)	1	4.179e-02 (1.113e-04)	2.526e-02 (5.590e-05)
	2	1.746e-03 (8.576e-07)	6.255e-04 (1.432e-07)
	3	7.306e-05 (4.022e-09)	1.518e-05 (2.077e-10)
(50,40)	1	6.672e-02 (1.392e-04)	4.120e-02 (7.801e-05)
	2	4.458e-03 (2.660e-06)	1.675e-03 (5.326e-07)
	3	2.986e-04 (3.015e-08)	6.717e-05 (2.027e-09)
(50,50)	1	8.360e-02 (1.456e-04)	5.170e-02 (8.638e-05)
	2	6.991e-03 (4.305e-06)	2.648e-03 (9.330e-07)
	3	5.849e-04 (7.372e-08)	1.343e-04 (5.622e-09)

It is observed from Table 2.2 that as the value of k increases, the estimate of the parametric function increases and variance of the estimate goes down. For early failure model case the reduction in variance is drastic for each combination of n and r. It is also observed that the UMV estimates are increasing function of the sample sizes. The UMV estimate of probability density function and reliability function for the above two models along with their standard errors are shown in Table 2.3. A value of δ =3 is assumed for early failure model-2. The entries in brackets are the estimate of variances.

Table 2.3. *Summary of estimates of pdf and survival functions*

(n, r)	k	UMVUE of PDF		UMVUE of survival function	
		Instantaneous failure model	Early failure model-2	Instantaneous failure model	Early failure model-2
(50,15)	2	0.02142 (4.424e-05)	0.01352 (2.450e-05)	0.25469 (0.00315)	0.20924 (0.00288)
	4	0.01812 (2.525e-05)	0.01194 (1.622e-05)	0.21524 (0.00253)	0.18381 (0.00238)
	6	0.01533 (1.477e-05)	0.01054 (1.076e-05)	0.18187 (0.00214)	0.16136 (0.00203)
	8	0.01296 (9.041e-06)	0.00929 (7.239e-06)	0.15364 (0.00185)	0.14155 (0.00178)
	10	0.01096 (5.920e-06)	0.00819 (4.970e-06)	0.12978 (0.00161)	0.12409 (0.00158)
(50,25)	2	0.03534 (6.171e-05)	0.02222 (3.629e-05)	0.42209 (0.00379)	0.33774 (0.00375)
	4	0.02989 (3.420e-05)	0.01954 (2.346e-05)	0.35702 (0.00317)	0.29604 (0.00313)
	6	0.02528 (1.922e-05)	0.01717 (1.518e-05)	0.30199 (0.00280)	0.25938 (0.00272)
	8	0.02138 (1.124e-05)	0.01508 (9.927e-06)	0.25544 (0.00252)	0.22717 (0.00243)
	10	0.01808 (7.078e-06)	0.01324 (6.635e-06)	0.21608 (0.00228)	0.19888 (0.00220)
(50, 40)	2	0.05647 (7.303e-05)	0.03614 (4.885e-05)	0.67773 (0.00262)	0.54433 (0.00379)
	4	0.04780 (3.714e-05)	0.03170 (3.025e-05)	0.57370 (0.00259)	0.47658 (0.00331)
	6	0.04046 (1.840e-05)	0.02779 (1.860e-05)	0.48564 (0.00269)	0.41718 (0.00305)
	8	0.03425 (9.140e-06)	0.02436 (1.146e-05)	0.41110 (0.00276)	0.36509 (0.00288)
	10	0.02899 (4.907e-06)	0.02135 (7.197e-06)	0.34800 (0.00274)	0.31944 (0.00274)
(50,50)	2	0.07073 (7.126e-05)	0.04534 (5.260e-05)	0.84603 (0.00041)	0.68298 (0.00285)
	4	0.05983 (3.214e-05)	0.03975 (3.136e-05)	0.71578 (0.00118)	0.59800 (0.00269)
	6	0.05062 (1.277e-05)	0.03485 (1.833e-05)	0.60558 (0.00239)	0.52351 (0.00270)
	8	0.04283 (4.065e-06)	0.03054 (1.056e-05)	0.51234 (0.02390)	0.45821 (0.00276)
	10	0.03623 (8.409e-07)	0.02676 (6.101e-06)	0.43347 (0.02660)	0.40098 (0.00278)

2.7. Numerical examples

Here we discuss two datasets given in the Appendix. Dataset A.2 is based on the complete failure times in weeks of 50 items as considered by Murthy et al. (2004)

involves early failures and other dataset A.1 is based on Vannman's (1991) data on drying of woods under schedules 1 of Experiments 3 involves both instantaneous and early failures. Both the datasets supports exponential distribution for the positive failures times. The plots along with the parameter estimates are presented below.

The graphical plot of these examples is given in Figure 2.4 and Figure 2.5 below.

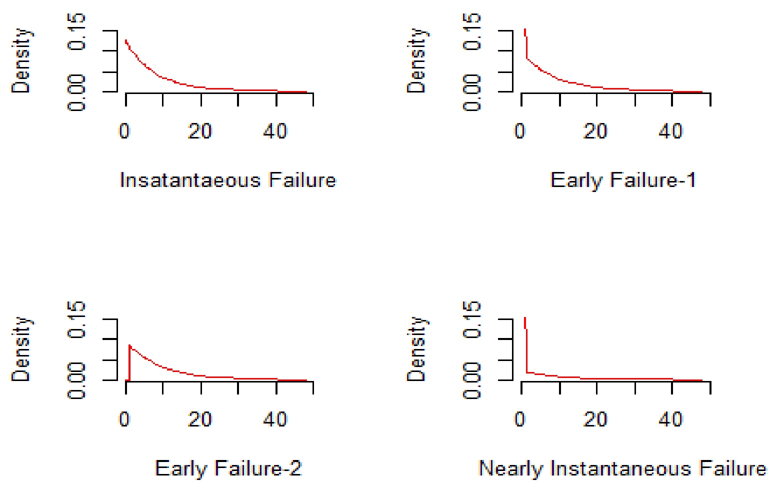

Figure 2.4. *Example-Murthy et al. (2004) data* $(\delta = 1.0, p = 0.8)$

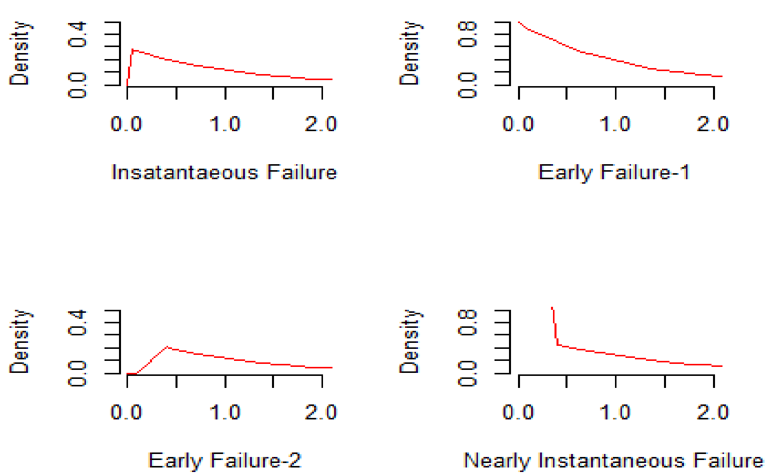

Figure 2.5. *Example-Vannman's (1991) data* $(\delta = 0.2, p = 0.3)$

The summary of estimates of models for some selected value of parameters of the above two examples are shown in Table 2.4. The entry shown in bracket is the estimate of standard error.

Table 2.4. *Summary of estimates of models*

Model	Parameter/parametric function	Estimates (SE)	
		Murthy et al. (2004)	Vannman's (1991)
Instantaneous failure	$\hat{p}_{MLE}$	1.00000 (0.00000)	0.30000 (0.07246)
	$\hat{\theta}_{MLE}$	7.82102 (1.10606)	1.01686 (0.29354)
	95% Confidence Interval for p	(-)	(0.15799, 0.44201)
	95% Confidence Interval for θ	(5.65318, 9.98886)	(0.44153, 1.59219)
	UMVUE of $(1-p)$	-	0.70000 (0.07335)
	UMVUE of $(1-p)^n + \frac{p}{\theta}[1-(1-p)^n]$	0.12530 (0.01789)	0.27044 (0.10310)
	UMVUE of pdf at x =2.0	0.09797 (0.01049)	0.04515 (0.01703)
	UMVUE survival function at x =2.0	0.77783 (0.02793)	0.04187 (0.02608)
Nearly instantaneous failure		δ =1.0	δ =0.0135
	$\hat{p}_{MLE}$	0.78000 (0.00654)	0.30406 (0.01022)
	$\hat{\theta}_{MLE}$	9.92026 (1.05647)	1.00336 (0.28964)
	95% Confidence Interval for p	(0.67822, 0.96684)	(0.28404, 0.32409)
	95% Confidence Interval for θ	(6.12068, 11.71984)	(0.43566, 1.57105)
Early failure-1		δ =1.0, p =0.80	δ =0.2, p =0.30
	$\hat{\theta}_{1MLE}$	3.92373 (1.16990)	0.07120 (0.01161)
	$\hat{\theta}_{2MLE}$	8.92026 (1.43296)	1.00664 (0.64841)
	95% Confidence Interval for $\hat{\theta}_1$	(1.63078, 6.21668)	(0.04843, 0.09395)
	95% Confidence Interval for $\hat{\theta}_2$	(6.11170, 11.72881)	(0.00000, 2.27750)
Early failure-2		δ =1.0	δ =0.20
	$\hat{p}_{MLE}$	0.87253 (0.04812)	0.30495 (0.04520)
	$\hat{\theta}_{MLE}$	8.47973 (1.42838)	1.00664 (0.31833)
	95% Confidence Interval for p	(0.77822, 0.96684)	(0.21636, 0.39353)
	95% Confidence Interval for θ	(6.12068, 11.71984)	(0.38273, 1.63055)
	UMVUE of $(1-pe^{-\delta/\theta})$	0.22000 (0.05918)	0.75000 (0.06935)
	UMVUE of $\left(1-pe^{-\delta/\theta}\right)^n + \frac{p}{\theta}\left[1-\left(1-pe^{-\delta/\theta}\right)^n\right]$	0.07661 (0.01369)	0.18647 (0.07899)
	UMVUE of pdf at x =2.0	0.06324 (0.00955)	0.04375 (0.01449)
	UMVUE of survival function at x =2.0	0.64056 (0.05273)	0.04893 (0.03922)