Chapter 5

Inferences on inliers in Gompertz distribution with Type II censored data

5.1. Introduction

This chapter deals with the inferential studies of inliers in Gompertz distribution with Type II censored data. The Gompertz distribution was first introduced by the British actuary Benjamin Gompertz in 1825 to describe human mortality and establish actuarial tables Gompertz (1825). Since then, many investigators have used the Gompertz distribution or related forms of it in a variety of studies. It has been widely used, especially in actuarial and biological applications and in demography. The Gompertz distribution with shape parameter α and scale parameter θ, have the following pdf

$$f(x;\alpha,\theta) = \begin{cases} \theta e^{\alpha x} e^{-\frac{\theta}{\alpha}(e^{\alpha x}-1)}, & x > 0 \\ 0, & o.w. \end{cases} \tag{5.1.1}$$

The distribution is now applied in many other fields including reliability and life testing studies, epidemiological, biomedical studies and actuarial sciences. This motivated us to study the inferences on parameters of Gompertz distribution, as inliers can be natural occurrences in most of the above practical and experimental studies. For a review of the literature on estimating parameters of the Gompertz distribution, one may refer to Gordon (1990), Chen (1997), Ananda et al. (1996), Walker and

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Adham (2001), Jaheen (2003), Wu et al. (2004), Al-Khedhairi and El-Gohary (2008), Ismile (2010), and many others.

In life testing and reliability, most of the experiments often take a long time to terminate, so the experiment does not observe all failure times due to cost and time considerations. The concept of censoring is introduced to account for these considerations. Censored data means that the information is available only for a part of the sample units, as some failure times may be observed till a fixed period of time or observed till a set of failures are observed. Among the censored data the most commonly used form is of Type II censored samples. In the Type II censoring scheme, n units are placed on the test and instead of continuing experiments until all n units have failed, the experiment is terminated after a prefixed number of failures say $c < n$, so failure times $X_{(1)}, X_{(2)}, \dots, X_{(c)}$ are observed and $X_{(c)} < X_{(n)}$. The remaining $n - c$ items are regarded as censored data.

The organization of the chapter is as follows: The Type II censored Gompertz inlier model present in Section 5.2. The MLE and its asymptotic distribution study in Section 5.3. We propose least squares and weighted least squares procedures for estimating the parameters in Section 5.4 and percentile based estimation procedures for estimating the parameters in Section 5.5. The UMVUE for various parametric functions including pdf and survival function is proposed in Section 5.6. In Section 5.7, we propose tests of the hypothesis of the parameters in the model. The Section 5.8 deals with a simulation study and numerical examples in last Section 5.9.

5.2. The Type II censored Gompertz inliers model

If we use $f(x;\theta)$ as in (5.1.1) in the model (1.3.7), we get the Gompertz inliers model as

$$g(x;p,\alpha,\theta)=\begin{cases}0, & x<d\\ 1-p\,e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}, & x=d\\ p\,\theta\,e^{\alpha x}e^{-\frac{\theta}{\alpha}(e^{\alpha x}-1)}, & x>d\end{cases} \tag{5.2.1}$$

Here d is sufficiently small and assumed known.

The graphical plots of density function $g(x)$ in Figure 5.1 are with 10% censored data for different mixing proportions. For early failure model case, the plot is drawn with a known value of d=10.

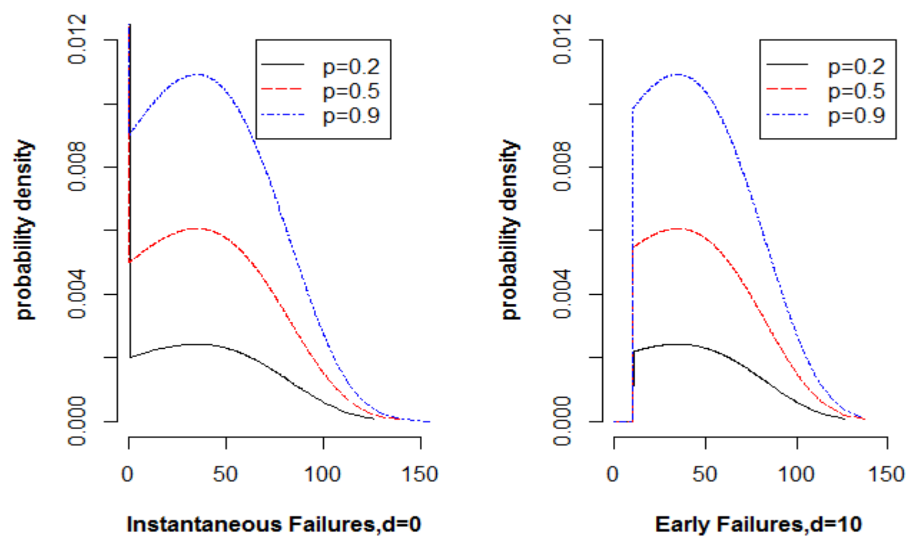

Figure 5.1. *Plots of density function of 10% censored data for varying p, α = 0.02, θ = 0.01*

Similarly, the plot of density functions $g(x)$ in Figure 5.2 are with 10% censored data for various values of α. From the plots it is evident that the density curve of both models gives rise to several different shapes and bumps.

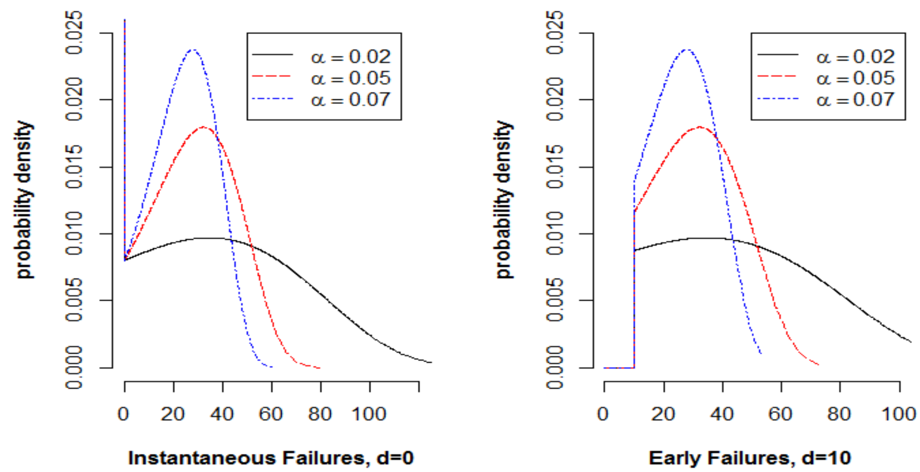

Figure 5.2. *Plots of density function of 10% censored data for varying* α*, p = 0.8 and* θ*=0.01*

In the same manner, we have plotted survival function graphs with 10% censored data for different values of α as shown in Figure 5.3 for instantaneous failures case and for early failures case. The plots shows that, as α decreases survival probability increases. In all figures, the smallest mixing proportion p is given by the solid line.

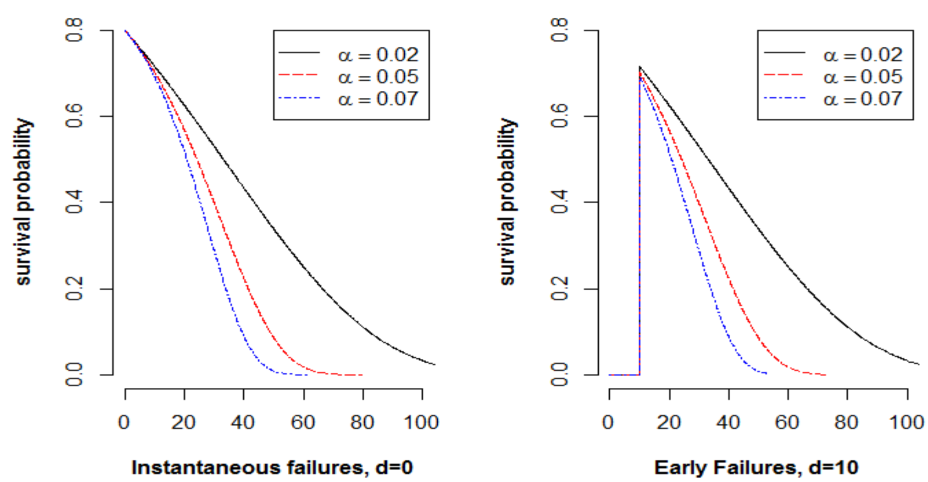

Figure 5.3. *Plots of survival function of 10% censored data for varying* α*, p = 0.8 and* θ*=0.01*

5.3. The likelihood estimates and its asymptotic distribution

The model in (5.2.1) has three parameters p, θ, and α. We assume shape parameter α, as known. To estimate the other parameters of the model we consider maximum likelihood estimation procedure along with its asymptotic distribution below.

5.3.1 Maximum Likelihood Estimation

Suppose n items placed on life test, where $n-r$ items early failures (fails before d) and r survivors. Let $X_1, X_2, \ldots, X_r$ be lifetimes of survivors. By applying the technique of 'Type II censored sample', the experiment terminates at the moment c $(c<r)$ survivors out of r survivors fail. Clearly $r=c$, then the experiment is not terminated and all r lifetimes are observed. Let $c^* = \min(r,c))$ and $X_{(1)}, X_{(2)}, \ldots, X_{(c^*)}$ denote ordered observed failure time of these c^* items from $g \in \mathcal{g}$ as given in (5.2.1). Assuming α known, and referring (1.4.2), the likelihood function is

$$L(p,\alpha,\theta|\underline{x}) = \left(1 - p\, e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right)^{(n-r)} \left(p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right)^{r} \theta^{c^*} e^{\alpha \sum_{i=1}^{c^*} x_{(i)}}$$
$$e^{-\frac{\theta}{\alpha}\left[\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} + (r-c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}\right]} \tag{5.3.1}$$

The log-likelihood function is

$$\begin{aligned} log\, L(p,\alpha,\theta|\underline{x}) &= log\, r! - log(r-c^*)! + (n-r)\, log\left(1 - p\, e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right) \\ &+ r\, log\left(p\, e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right) + c^*\, log\, \theta + \alpha \sum_{i=1}^{c^*} x_{(i)} \\ &- \frac{\theta}{\alpha}\left[\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} + (r-c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}\right] \end{aligned}$$

Referring to 1.5.1, the likelihood equations are

$$\frac{\partial logL(p,\alpha,\theta|\underline{x})}{\partial p} = -\frac{(n-r)e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}{1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} + \frac{re^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}{pe^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} = 0 \qquad (5.3.2)$$

and

$$\frac{\partial log\left(p,\alpha,\theta|\underline{x}\right)}{\partial\theta} = \frac{(n-r)\,p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}\,\frac{\left(e^{\alpha d}-1\right)}{\alpha}}{1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} - \frac{r\,p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}\,\frac{\left(e^{\alpha d}-1\right)}{\alpha}}{p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} + \frac{c^*}{\theta} - \frac{1}{\alpha}\left[\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} + (r-c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}\right] = 0 \qquad (5.3.3)$$

The estimator $\hat{p}_{MLE}$ can be obtained from equation (5.3.2) as

$$\hat{p}_{MLE} = \frac{re^{\frac{\hat{\theta}}{\alpha}\left(e^{\alpha d}-1\right)}}{n} \qquad (5.3.4)$$

Upon using (5.3.4) in (5.3.3), we have the estimator $\hat{\theta}_{MLE}$ in the form

$$\hat{\theta}_{MLE} = \frac{\alpha\, c^*}{\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} + (r-c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}} \qquad (5.3.5)$$

which is free from the parameter p. This is further used in (5.3.4) to get $\hat{p}_{MLE}$.

5.3.2. Asymptotic distribution of MLE

For inlier prone Gompertz distribution $g(x; p, \alpha, \theta)$ given by (5.2.1) with α known,

$$\frac{\partial log\ g(p,\alpha,\theta|x)}{\partial p}=\begin{cases}0, & x<d\\ -\dfrac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}{1-p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}, & x=d\\ \dfrac{1}{p}, & x>d\end{cases}$$

and

$$\frac{\partial log\ g(p,\alpha,\theta|x)}{\partial \theta}=\begin{cases}0, & x<d\\ \dfrac{p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\ \dfrac{(e^{\alpha d}-1)}{\alpha}}{1-p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}, & x=d\\ \dfrac{1}{\theta}-\dfrac{(e^{\alpha x}-1)}{\alpha}, & x>d\end{cases}$$

One can verify that $E\left(\frac{\partial log\ g(p,\alpha,\theta|x)}{\partial p}\right)=0$ and $E\left(\frac{\partial log\ g(p,\alpha,\theta|x)}{\partial \theta}\right)=0$.

Also,

$$\frac{\partial^2\ log\ g(p,\alpha,\theta|x)}{\partial p^2}=\begin{cases}0, & x<d\\ -\dfrac{e^{-\frac{2\theta}{\alpha}(e^{\alpha d}-1)}}{\left[1-p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right]^2}, & x=d\\ -\dfrac{1}{p^2}, & x>d\end{cases}$$

$$\frac{\partial^2 log\ g(p,\alpha,\theta|x)}{\partial \theta^2}=\begin{cases}0, & x<d\\ -\dfrac{p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\left[\dfrac{(e^{\alpha d}-1)}{\alpha}\right]^2}{\left[1-p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right]^2}, & x=d\\ -\dfrac{1}{\theta^2}, & x>d\end{cases}$$

and

$$\frac{\partial^2 log\ g(p,\alpha,\theta|x)}{\partial p\ \partial\theta} = \begin{cases} 0, & x < d \\ \dfrac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\ \dfrac{(e^{\alpha d}\ -\ 1)}{\alpha}}{\left[1-p\ e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right]^2}, & x = d \\ 0, & x > d \end{cases}$$

Hence, the Fisher information is

$$I_{pp} = E\left(-\frac{\partial^2\ log\ g(p,\alpha,\theta|x)}{\partial p^2}\right) = \frac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}{p\ \left(1-pe^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right)} = \frac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}{p\ p^*}$$

$$I_{\theta\theta} = E\left(-\frac{\partial^2\ log\ g(p,\alpha,\theta|x)}{\partial\theta^2}\right) = \frac{(1-p^*)\left\{\theta^2\left[\frac{(e^{\alpha d}\ -\ 1)}{\alpha}\right]^2+p^*\right\}}{\theta^2\ p^*}$$

and

$$I_{p\theta} = E\left(-\frac{\partial^2 log\ g(p,\alpha,\theta|x)}{\partial p\ \partial\theta}\right) = -\frac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\ \frac{(e^{\alpha d}\ -\ 1)}{\alpha}}{p^*}$$

where, $p^* = 1-pe^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}$. Therefore, the Fisher information matrix $I_g(p,\theta)$ is given by

$$I_g(p,\theta) = \begin{bmatrix} I_{pp} & I_{p\theta} \\ I_{\theta p} & I_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}}{p\ p^*} & -\dfrac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\ \frac{(e^{\alpha d}-1)}{\alpha}}{p^*} \\ -\dfrac{e^{-\frac{\theta}{\alpha}(e^{\alpha d}-1)}\ \frac{(e^{\alpha d}-1)}{\alpha}}{p^*} & \dfrac{(1-p^*)\left\{\theta^2\frac{(e^{\alpha d}-1)^2}{\alpha^2}+p^*\right\}}{\theta^2\ p^*} \end{bmatrix}.$$

The determinant of $I_g(p,\theta)$ is given by $\Delta = \frac{e^{-\frac{2\theta}{\alpha}(e^{\alpha d}-1)}}{\theta^2\ p^*}$ and hence, the inverse matrix $I_g^{-1}(p,\theta)$ is given by

$$I_g^{-1}(p,\theta) = \begin{bmatrix} \frac{p\left(\theta^2\frac{\left(e^{\alpha d}-1\right)^2}{\alpha^2}+p^*\right)}{e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} & -\frac{\theta^2\left(e^{\alpha d}-1\right)}{\alpha\ e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} \\ -\frac{\theta^2\left(e^{\alpha d}-1\right)}{\alpha\ e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}} & \frac{\theta^2}{1-p^*} \end{bmatrix}.$$

Using the standard result of MLE, we have

$$(\hat{p}_{MLE},\hat{\theta}_{MLE})'\sim AN^{(2)}\left[(p,\theta)',\frac{1}{n}\ I_g^{-1}(p,\theta)\right].$$

The estimated variances van be used for proposing large sample tests for p and θ. The approximate $(1-\alpha)\%$ confidence interval for p and θ are respectively given by

$$\hat{p}_{MLE} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_{MLE}\left(\frac{\hat{\theta}_{MLE}\left(e^{\alpha d}-1\right)^2}{\alpha^2}+1-\hat{p}_{MLE}\ e^{-\frac{\hat{\theta}_{MLE}}{\alpha}\left(e^{\alpha d}-1\right)}\right)}{n\ e^{-\frac{\hat{\theta}_{MLE}}{\alpha}\left(e^{\alpha d}-1\right)}}} \tag{5.3.6}$$

and

$$\hat{\theta}_{MLE} \pm z_{\alpha/2}\sqrt{\frac{\hat{\theta}^2_{MLE}}{n\ \hat{p}_{MLE}\ e^{-\frac{\hat{\theta}_{MLE}}{\alpha}\left(e^{\alpha d}-1\right)}}} \tag{5.3.7}$$

5.4. Least Squares and Weighted Least Squares Estimation

In this section, we provide the regression-based estimators of the unknown parameters, which was originally suggested by Swain et al. (1988) to estimate the parameters of Beta distributions. Hanaa and Abu-Zinadah (2014) studied such estimators for the parameters of Gompertz distributions. Suppose $(X_1, X_2, \dots, X_n)$ is a random sample of size n from a distribution function $G(x; p, \alpha, \theta)$, and $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ denotes the order statistics of the observed sample, then,

$$G(x_{(i)};p,\alpha,\theta)=\begin{cases}0, & x<d\\ 1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha X_{(i)}}-1\right)}, & x\geq d\end{cases} \tag{5.4.1}$$

From equation (1.5.3), for known α, the least squares estimator (LSE) is obtained by minimizing the function

$$\sum_{i=1}^{n}\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}-\frac{i}{n+1}\right)^2 \tag{5.4.2}$$

with respect to unknown parameters. The least squares estimator of (p,θ) is $(\hat{p}_{LSE},\hat{\theta}_{LSE})$, say, obtained by solving the estimating equations (5.4.3) and (5.4.4) simultaneously as given below.

$$\sum_{i=1}^{n}\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}-\frac{i}{n+1}\right)e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}=0 \tag{5.4.3}$$

and

$$\sum_{i=1}^{n}\frac{p}{\alpha}\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}-\frac{i}{n+1}\right)\left(e^{\alpha x_{(i)}}-1\right)e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}=0. \tag{5.4.4}$$

In a similar argument, from the equation (1.5.4), we obtain the weighted least squares estimators (WLSE) of (p,θ) is $(\hat{p}_{WLSE},\hat{\theta}_{WLSE})$, say, obtained by minimizing

$$\sum_{i=1}^{n}w_i\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}-\frac{i}{n+1}\right)^2 \tag{5.4.5}$$

with respect to the unknown parameters p and θ, where $w_i=\frac{1}{v[G(X_{(i)})]}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$.

The minimizing equations in this case are given by

$$\sum_{i=1}^{n}w_i\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}-\frac{i}{n+1}\right)e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}=0 \tag{5.4.6}$$

and

$$\sum_{i=1}^{n} w_i \frac{p}{\alpha}\left(1 - p\, e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right)} - \frac{i}{n+1}\right)\left(e^{\alpha x_{(i)}} - 1\right)e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)} = 0$$

(5.4.7)

respectively. Equations (5.4.6) and (5.4.7) need to solve simultaneously to get $\hat{p}_{WLSE}$ and$\hat{\theta}_{WLSE}$. The estimates are numerically computed for various combinations of parameter values and number of inliers, and are presented in Table 5,1 and 5.4.

5.5. Percentile Estimation

Referring to section 1.5.3, suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from a distribution function as given in (5.4.1) with α known, and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denotes the order statistics of the observed sample. If P_i denotes some estimate of $g(x; p, \alpha, \theta)$ given in (5.2.1), then using equation (1.5.5), the percentile estimate of unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\log P_i - \log G(x_{(i)})\right]^2 \qquad (5.5.1)$$

with respect to p and θ. Here, P_i denotes some estimate of $g(x; p, \alpha, \theta)$. We consider $P_i = \frac{i}{n+1}$, the percentile estimator (PE) of p and θ is $\hat{p}_{PE}$ and $\hat{\theta}_{PE}$; this can be obtained by solving two minimizing equations (5.5.2) and (5.5.3) simultaneously which are as follows:

$$\sum_{i=1}^{n}\left[\log\left(\frac{i}{n+1}\right) - \log\left(1 - p\, e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right)}\right)\right]\left(\frac{e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right)}}{1-p\, e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}} - 1\right)}}\right) = 0$$

(5.5.2)

and

$$\sum_{i=1}^{n}\frac{p}{\alpha}\left[\log\left(\frac{i}{n+1}\right)-\log\left(1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}\right)\right]\left(\frac{\left(e^{\alpha x_{(i)}}-1\right)e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}}{1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha x_{(i)}}-1\right)}}\right)=0. \tag{5.5.3}$$

For further information on percentile estimation, one may refer to Gupta and Kundu (2001), Shawky and Abu-Zinadah (2009) and Shawky and Bakoban (2012) and the details contained therein. Note that, all the above equations are nonlinear and hence, we use some nonlinear method of solving equations. Specifically, we used the *optim function* in R.

5.6. Uniformly Minimum Variance Unbiased Estimation

By re-writing the pdf $g(x;p,\alpha,\theta)$ in (5.2.1) with α known, in the form of (1.5.9), then

$$g(x;p,\alpha,\theta)=\frac{e^{\alpha x(1-I(x))}\left[e^{-\theta}\right]^{\frac{e^{\alpha d}\left(e^{\alpha(x-d)}-1\right)(1-I(x))}{\alpha}}\left(\frac{1-pe^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}{\theta pe^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}\right)^{I(x)}}{\left(\frac{1}{\theta pe^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}\right)}$$

$$=a(x)^{(1-I(x))}\,\frac{h[\theta]^{d(x)(1-I(x))}}{\left(\frac{g(\theta)}{1-p^*}\right)}\,\left[g(\theta)\left(\frac{p^*}{1-p^*}\right)\right]^{I(x)} \tag{5.6.1}$$

where $a(x)=e^{\alpha x}$, $h(\theta)=e^{-\theta}$, $d(x)=\frac{1}{\alpha}e^{\alpha d}\left(e^{\alpha(x-d)}-1\right)$, $g(\theta)=\frac{1}{\theta}$, and $p^*=1-pe^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}$. Also $a(x)>0$ and $g(\theta)=\int_{x>d}a(x)h[\theta]^{d(x)}\,dx$. The density in (5.6.1) so obtained is defined with respect to a measure $\mu(x)$which is the sum of Lebesgue measure over (d,∞) and a singular measure at d, a well know form of a two parameter exponential family with natural parameters $(\eta_1,\eta_2)=$

$\left(\log\left[\frac{1-p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}{\theta p\,e^{-\frac{\theta}{\alpha}\left(e^{\alpha d}-1\right)}}\right], log\left[e^{-\theta}\right]\right)$ generated by underlying indexing parameters (p^*, θ). Hence $\left(I(x), \frac{e^{\alpha d}}{\alpha}\left(e^{\alpha(x-d)} - 1\right)\left(1 - I(x)\right)\right)$ is jointly minimal sufficient for (p^*, θ), as $I(x)$ and $\frac{e^{\alpha d}}{\alpha}\left(e^{\alpha(x-d)} - 1\right)\left(1 - I(x)\right)$ do not satisfy any linear restriction.

For the Type II censored sample discussed in section 5.3.1, consider the following transformation

$$z_i = (r - i + 1)\left(\frac{e^{\alpha d}}{\alpha}\left(e^{\alpha\left(x_{(i)}-d\right)} - 1\right) - \frac{e^{\alpha d}}{\alpha}\left(e^{\alpha\left(x_{(i-1)}-d\right)} - 1\right)\right), i = 1,2, \dots, c^*;\ x_{(0)} = d \tag{5.6.2}$$

It can be seen that

$$z = \sum_{i=1}^{c^*} z_i = \frac{1}{\alpha}\left[\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} + (r - c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}\right]$$

and

$$|J| = \frac{(r-c^*)!}{r!\, e^{\alpha \sum_{i=1}^{c^*} x_{(i)}}} \tag{5.6.3}$$

Using (5.6.2) and (5.6.3) the joint density function can be expressed as

$$g\left(\underline{x}; p, \alpha, \theta\right) = (p^*)^{n-r}\ (1 - p^*)^r\ \theta^{c^*}\ e^{-\theta z} \tag{5.6.4}$$

$$= \frac{\left[e^{-\theta}\right]^z \left(\frac{p^*}{\theta\,(1-p^*)}\right)^{n-c^*} \left(\frac{(1-p^*)}{p^*}\right)^{r-c^*}}{\left(\frac{1}{\theta\,(1-p^*)}\right)^n} \tag{5.6.5}$$

Also

$$g(\underline{x}; p, \alpha, \theta) = \binom{n}{r} \ (p^*)^{n-r} \ \ (1-p^*)^r \ \ \frac{\theta^{c^*} \, e^{-\theta z}}{\binom{n}{r}} \qquad (5.6.6)$$

$$= P(n-R=n-r) \ \ g(z; \theta \,|n-r).$$

Therefore, by Neyman factorization theorem $(n-c^*,\ R-c^*,\ Z)$ are jointly minimal sufficient for (p^*, θ). Also, $n-R$ is binomial which is the same as that of R with parameter $(n, 1-p^*)$ and is a complete family and the variable $(Z|R=c^*, c^*>0)$ is distributed as a gamma random variable having density function

$$g(z;\ \theta|c^*) = \frac{\theta^{c^*}}{\Gamma c^*} z^{c^*-1} e^{-\theta z},\ z>0;\ \theta>0. \qquad (5.6.7)$$

Hence, $Z|R$ is completely sufficient for θ. This preserves the exponential structure for (5.6.7). Therefore, joint pdf of complete sufficient statistics $(n-c^*,\ R-c^*,\ Z)$ can be written as

$$g(z, r, c^*; p, \alpha, \theta) = \begin{cases} (p^*)^n, & z=d; r=0 \\ B(z, r, c^*, n) \dfrac{e^{-\theta z} \left(\frac{p^*}{\theta(1-p^*)}\right)^{n-c^*} \left(\frac{(1-p^*)}{p^*}\right)^{r-c^*}}{\left(\frac{1}{\theta\,(1-p^*)}\right)^n}, & z>d; r=1,2,\dots,n \end{cases}$$

(5.6.8)

where

$$B(z, r, c^* n) = \begin{cases} 1, & z=d;\ r=0 \\ \binom{n}{r} \dfrac{z^{c^*-1}}{\Gamma c^*}\ , & z>d;\ r=1,2,\dots,n \end{cases} \qquad (5.6.5)$$

is such that

$$(p^*)^n + \sum_{r=1}^{n} \int_{z>d} B(z, r, c^*, n)\, e^{-\theta z} \ \frac{\left(\frac{p^*}{\theta(1-p^*)}\right)^{n-c^*} \left(\frac{(1-p^*)}{p^*}\right)^{r-c^*}}{\left(\frac{1}{\theta\,(1-p^*)}\right)^n} dz = 1.$$

Referring Lemma 1.5.1, the UMVUE of function $\phi(p,\theta)$ in $g(x;p,\alpha,\theta)$ is given by

$$\psi(Z,R,c^*n) = \frac{\alpha(Z,R,c^*,n)}{B(Z,R,c^*n)}, B(Z,R,c^*,n) \neq 0.$$

Note that, the UMVUE of function θ alone is not U-estimable. The UMVUE for some of the parametric functions are given as results below:

Result 5.1. *For the distribution given in (5.6.1) the UMVUE of* $(1-p^*)^m,\ m \leq n$ *is given by*

$$G_m(z,r,c^*,n) = \frac{B(z,r,c^*,n-m)}{B(z,r,c^*,n)} = \begin{cases} \dfrac{\binom{n-m}{r}}{\binom{n}{r}}, r = 0,1,\dots,n-m \\ 0, \qquad o.w. \end{cases}$$

Result 5.2. *If* $m = 1$, *the UMVUE of* $(1-p^*)$, *for distribution (5.6.1) is given by*

$$G_1(z,r,c^*,n) = \frac{B(z,r,c^*,n-1)}{B(z,r,c^*,n)} = \begin{cases} \dfrac{n-r}{r}, r = 0,1,\dots,n-1 \\ 0, \qquad o.w.. \end{cases}$$

Result 5.3. *For* $m \leq \frac{n}{2}$ *the UMVUE of the variance of* $G_m(Z,R,c^*,n)$ *is given by*

$$\widehat{var}\,[G_m(z,r,c^*,n)] = \begin{cases} G_m^2(z,r,c^*,n) - G_{2m}(z,r,c^*,n), r = 1,2,\dots,(n-2m) \\ G_m^2(z,r,c^*,n), \qquad r = (n-2m+1),\dots,(n-m) \\ 0, \qquad o.w. \end{cases}$$

$$
= \begin{cases} \left[\dfrac{\binom{n-m}{r}}{\binom{n}{r}}\right]^2 - \dfrac{\binom{n-2m}{r}}{\binom{n}{r}}, & r = 1,2,\dots,(n-2m) \\ \left[\dfrac{\binom{n-m}{r}}{\binom{n}{r}}\right]^2, & r = (n-2m+1),\dots,(n-m) \\ 0, & o.w. \end{cases}
$$

Result 5.4. *For $m = 1$, the UMVUE of the variance of UMVUE of $(1 - p^*)$ is given by*

$$
\widehat{var}\,[G_1(z,r,c^*,n)] = \begin{cases} \dfrac{r(n-r)}{n^2(n-1)}, & r = 0,1,\dots,n-1 \\ 0, & o.w. \end{cases}
$$

Result 5.5. *For k > 0 the UMVUE of parametric function*

$$
(1-p^*)^n + (\theta p^*)^k[1-(1-p^*)^{n-k}], \theta > 0; 0 < p^* < 1
$$

is obtained as

$$
H_k(z,r,c^*,n) = \frac{B(z, r-[1-I(z)]k, c^*-[1-I(z)]k, n-k)}{B(z,r,c^*,n)}
$$

$$
= \begin{cases} \dfrac{(r)_k\ (c^*-1)_k}{(n)_k\ z^k}, & z > 0;\ r = 1,2,,\dots,n \\ 1, & z = 0; r = 0 \end{cases}
$$

where $(a)_b = a(a-1)(a-2)\dots(a-b+1)$ *and* $z = \frac{\sum_{i=1}^{c^*} e^{\alpha x_{(i)}} - (r-c^*)e^{\alpha x_{(c^*)}} - re^{\alpha d}}{\alpha}$.

For various values of k, one can obtain the UMVUE of the parametric function.

Result 5.6. *The UMVUE of the variance of $H_k(Z,R,n)$ is obtained as*

$$\widehat{var}\,[H_k(z,r,c^*,n)] = H_k^2(z,r,c^*,n) - H_{2k}(z,r,c^*,n)$$

$$= \begin{cases} \left[\dfrac{(r)_k\ (c^*-1)_k}{(n)_k\ z^k}\right]^2 - \dfrac{(r)_{2k}\ (c^*-1)_{2k}}{(n)_{2k}\ z^{2k}}, & z>0; r=1,2,\dots,n \\ 0, & o.w. \end{cases}$$

Result 5.7. *For fixed x, the UMVUE of the pdf g(x; p,α, θ) given in (5.6.1), is obtained as*

$\phi_x(z,r,c^*,n)$

$$= \begin{cases} a(x)\dfrac{B(z-d(x),r-1,c^*-1,n-1)}{B(z,r,c^*,n)}, & x>d; z>d(x); r=2,\dots,n \\ \dfrac{B(z,r,c^*,n-1)}{B(z,r,c^*,n)}, & x=d; r=0,1,\dots,n-1 \\ 0, & o.w. \end{cases}$$

$$= \begin{cases} \dfrac{r(c^*-1)}{n\ z}e^{\alpha x}\left(1-\dfrac{(e^{\alpha x}\ -\ e^{\alpha d})}{\alpha\, z}\right)^{c^*-2}, & x>d; z>\dfrac{(e^{\alpha x}\ -\ e^{\alpha d})}{\alpha}; r=2,\dots,n \\ \dfrac{n-r}{n}, & x=d; r=0,1,\dots,n-1 \\ 0, & o.w. \end{cases}$$

Result 5.8. *For $c^*=r=n$, that is when all the observations are coming from the pdf in (5.1.1), then the UMVUE of the density $f(x;\theta)$ is simplified as*

$\phi_x(z,r,n)$

$$= \begin{cases} \dfrac{(n-1)}{z}e^{\alpha x}\left(1-\dfrac{(e^{\alpha x}\ -\ 1)}{\alpha\, z}\right)^{n-2}, & x>0; z>\dfrac{(e^{\alpha x}\ -\ 1)}{\alpha};\ n>1 \\ 0, & o.w. \end{cases}$$

Result 5.9. *The UMVUE of the variance of $\phi_x(Z,R,n)$ is obtained as*

$$\widehat{var}[\phi_x(z,r,c^*,n)] = \begin{cases} \phi_x^2(z,r,c^*,n) - \phi_x(z,r,c^*,n)\phi_x(z-d(x),r-1,c^*-1n-1), & x > d; z > 2x;\ r = 2,3,\dots,n \\ \phi_x^2(z,r,c^*,n), & x > d; x < z < 2x \\ \phi_x^2(z,r,c^*,n) - \phi_x(z,r,c^*,n)\phi_x(z,r,c^*n-1), & x = d; r = 0,1,\dots,n-1 \\ 0, & o.w. \end{cases}$$

$$= \begin{cases} \left[\frac{r(c^*-1)}{n\ z} e^{\alpha x}\left(1-\frac{(e^{\alpha x}-e^{\alpha d})}{\alpha z}\right)^{c^*-2}\right]^2 - \frac{r(r-1)(c^*-1)(c^*-2)}{n(n-1)\,z\left(z-\frac{(e^{\alpha x}-1)}{\alpha}\right)}\left(1-\frac{(e^{\alpha x}-e^{\alpha d})}{\alpha z}\right)^{c^*-2} e^{2\alpha x}\left(1-\frac{(e^{\alpha x}-e^{\alpha d})}{\alpha z-(e^{\alpha x}-e^{\alpha d})}\right)^{c^*-3}, & x > 0;\ z > \frac{2\,(e^{\alpha x}-1)}{\alpha};\ r = 2,3,\dots,n \\ \left[\frac{r(c^*-1)}{n\ z} e^{\alpha x}\left(1-\frac{(e^{\alpha x}-1)}{\alpha z}\right)^{c^*-2}\right]^2, & x > 0;\ \frac{(e^{\alpha x}-1)}{\alpha} < z < \frac{2(e^{\alpha x}-1)}{\alpha} \\ \frac{r(r-1)}{n^2(n-1)}, & x = 0; r = 0,1,\dots,n-1 \end{cases}$$

For $c^* = r = n$ the above two results reduce to the case of Gompertz distribution with no inliers.

Result 5.10. *For fixed z, r and c^*, the UMVUE of the survival function $S(t) = P(X > t)$, $t \geq d$ is obtained as*

$$\hat{S}(t) = \begin{cases} \frac{r}{n}\left(1-\frac{(e^{\alpha t}-e^{\alpha d})}{\alpha z}\right)^{c^*-1}, & z > \frac{e^{\alpha t}-e^{\alpha d}}{\alpha} \\ 0, & o.w. \end{cases}$$

Proof: We have

$$\hat{S}(t) = \int_{x>t} \phi_x(z,r,n)\, dx$$

$$= \frac{r(c^*-1)}{n}\int_t^{\infty} \frac{1}{z} e^{\alpha x}\left(1-\frac{(e^{\alpha x}-e^{\alpha d})}{\alpha z}\right)^{c^*-2} dx$$

$$= \frac{r}{n}\left(1 - \frac{(e^{\alpha t} - e^{\alpha d})}{\alpha\, z}\right)^{c^*-1}$$

Hence the proof. ■

Result 5.11. *The UMVUE of the variance of* $\hat{S}(t)$ *obtained as*

$$\widehat{var}\left[\hat{S}(t)\right] = \begin{cases} \hat{S}^2(t) - 2\int_{x>t}\int_{y>t} a(x)a(y)\dfrac{B(z-d(x)-d(y), r-2, c^*-2, n-2)}{B(z,r,c^*,n)}\,dx\,dy, & z > 2d(t);\ r = 3,4,\dots,n \\ \hat{S}^2(t), & d(t) < z \le 2d(t);\ r = 2,3,\dots,n \\ 0, & o.w. \end{cases}$$

$$= \begin{cases} \left[\dfrac{r}{n}\left(1 - \dfrac{(e^{\alpha t} - e^{\alpha d})}{\alpha\, z}\right)^{c^*-1}\right]^2 - \dfrac{r(r-1)}{n(n-1)}\left(1 - \dfrac{2\,(e^{\alpha t} - e^{\alpha d})}{\alpha\, z}\right)^{c^*-1}, & z > \dfrac{2\,(e^{\alpha t} - e^{\alpha d})}{\alpha};\ r = 3,4,\dots,n \\ \left[\dfrac{r}{n}\left(1 - \dfrac{(e^{\alpha x} - e^{\alpha d})}{\alpha\, z}\right)^{c^*-1}\right]^2, & \dfrac{(e^{\alpha t} - e^{\alpha d})}{\alpha} < t < \dfrac{2\,(e^{\alpha t} - e^{\alpha d})}{\alpha};\ r = 2,\dots,n \\ 0, & o.w. \end{cases}$$

For $c^* = r = n$, both the above results reduce to the case of the Gompertz distribution with no inliers.

5.7. Test of hypothesis for parameters

Since p is the mixing parameter and involve only the number of observations correspond to the positive part, the most powerful test for testing $H_0\!: p = 1$ against

$H_1: p < 1$ of size α' is similar to the one obtained in (1.6.1), with power function $\beta(p) = 1 - (1 - \alpha')p^n$. The power function can be computed numerically for any combination of n, p and α'.

The LMP test of size α' for testing $H_0: p = 1$ against $H_1: p < 1$ for θ known based on n iid observations from the density $g(x; p, \theta)$ is given by

$$\Phi_2(x) = \begin{cases} 1, & np - r\, e^{\frac{\theta}{\alpha}(e^{\alpha d}-1)} < c_\alpha\left(p^2 - pe^{\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right) \\ \gamma, & np - r\, e^{\frac{\theta}{\alpha}(e^{\alpha d}-1)} = c_\alpha\left(p^2 - pe^{\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right) \\ 0, & np - r\, e^{\frac{\theta}{\alpha}(e^{\alpha d}-1)} > c_\alpha\left(p^2 - pe^{\frac{\theta}{\alpha}(e^{\alpha d}-1)}\right) \end{cases}$$

where c_α and γ are such that $E_{H_0}[\Phi_2(x)] = \alpha'$.

The most powerful test for $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, $(\theta_1 > \theta_0)$ for p known and shape parameter $\alpha_0 = \alpha_1 = \alpha$ according to (1.6.3) is given by

$$\Phi_3(x) = \begin{cases} 1, & \sum_{x_i > d}(e^{\alpha x_i} - 1) > \frac{\alpha\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{-\frac{\theta_0}{\alpha}(e^{\alpha d}-1)}\right) - log\left(1 - pe^{-\frac{\theta_1}{\alpha}(e^{\alpha d}-1)}\right)\right] + r(log\,\theta_0 - log\,\theta_1)\right\}}{(\theta_0 - \theta_1)} \\ \gamma, & \sum_{x_i > d}(e^{\alpha x_i} - 1) = \frac{\alpha\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{-\frac{\theta_0}{\alpha}(e^{\alpha d}-1)}\right) - log\left(1 - pe^{-\frac{\theta_1}{\alpha}(e^{\alpha d}-1)}\right)\right] + r(log\,\theta_0 - log\,\theta_1)\right\}}{(\theta_0 - \theta_1)} \\ 0, & \sum_{x_i > d}(e^{\alpha x_i} - 1) < \frac{\alpha\left\{c_\alpha + (n-r)\left[log\left(1 - pe^{-\frac{\theta_0}{\alpha}(e^{\alpha d}-1)}\right) - log\left(1 - pe^{-\frac{\theta_1}{\alpha}(e^{\alpha d}-1)}\right)\right] + r(log\,\theta_0 - log\,\theta_1)\right\}}{(\theta_0 - \theta_1)} \end{cases}$$

where c_α and γ are such that $E_{H_0}[\Phi_3(x)] = \alpha'$, where α' is level of significance. In a similar argument, the LMP test of size α' for testing $H_0: \theta \le \theta_0$ against $H_1: \theta > \theta_0$ for p known based on n iid observations from the density $g(x; p, \theta)$ according to (1.6.4) is given by

$$\Phi_4(x) = \begin{cases} 1, & \sum_{x_i>d}(e^{\alpha x_i}-1) > -\alpha c_\alpha + \dfrac{(n-r)p\left(e^{\alpha d}-1\right)e^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}}{\left(1-pe^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}\right)} \\ \gamma, & \sum_{x_i>d}(e^{\alpha x_i}-1) = -\alpha c_\alpha + \dfrac{(n-r)p\left(e^{\alpha d}-1\right)e^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}}{\left(1-pe^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}\right)} \\ 0, & \sum_{x_i>d}(e^{\alpha x_i}-1) < -\alpha c_\alpha + \dfrac{(n-r)p\left(e^{\alpha d}-1\right)e^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}}{\left(1-pe^{-\frac{\theta_0}{\alpha}\left(e^{\alpha d}-1\right)}\right)} \end{cases}$$

where c_α and γ are such that $E_{H_0}[\Phi_4(x)] = \alpha'$.

5.8. Simulation study

In this section, keeping $c^* = r$, we conduct simulation experiments to check the performance of estimators under different combination of (n, r). Each estimate is based on a simulation of 1000 random samples of size $n = 50$ with different choices of inliers $n - r$. For all models, the value of θ = 0.01 and $\alpha = 0.02$ is assumed for the Gompertz distribution and for early failure models, the value of d is set at 2.5. Table 5.1 presents various estimates of parameters along with their standard error of estimate which is shown in brackets and a 95% confidence interval for instantaneous and early failure models.

Table 5.1. *Summary of estimates of models*

(n,r)	Parameter	Instantaneous Failure Model				Early Failure model			
		MLE	LSE	WLSE	PE	MLE	LSE	WLSE	PE
(50,05)	$\hat{p}$	0.90026 (0.00159)	0.89232 (0.02843)	0.92962 (0.00107)	0.94172 (0.00261)	0.88631 (0.05056)	0.89791 (0.03977)	0.86720 (0.00315)	0.95178 (0.00557)
	$\hat{\theta}$	0.01381 (7.35e-05)	0.01663 (0.00145)	0.01710 (1.842e-05)	0.02533 (0.00093)	0.01366 (0.00209)	0.01497 (0.00152)	0.01393 (9.415e-05)	0.016354 (0.00116)
	95% CI for p	(0.89714, 0.90338)	(0.83660, 0.94804)	(0.89022, 0.89442)	(0.92407, 0.93430)	(0.78722, 0.98540)	(0.81997, 0.97586)	(0.89174, 0.90408)	(0.94087, 0.96270)
	95% CI for θ	(0.01366, 0.01395)	(0.01341, 0.01909)	(0.01701, 0.01713)	(0.01529, 0.01891)	(0.00956, 0.01776)	(0.01199, 0.01795)	(0.01479, 0.01515)	(0.01408, 0.01862)
(50,15)	$\hat{p}$	0.69638 (0.06422)	0.79727 (0.02275)	0.87711 (0.00097)	0.86096 (0.00130)	0.69254 (0.06835)	0.69745 (0.04567)	0.73253 (0.00211)	0.71717 (0.02546)
	$\hat{\theta}$	0.01418 (0.00241)	0.02115 (0.00117)	0.02360 (3.50e-05)	0.03576 (0.00211)	0.01387 (0.00241)	0.01491 (0.00230)	0.01578 (0.00012)	0.01768 (0.00183)
	95% CI for p	(0.57051, 0.82225)	(0.75268, 0.84185)	(0.87520, 0.87901)	(0.85840, 0.86352)	(0.55858, 0.82650)	(0.60794, 0.78696)	(0.69332, 0.70159)	(0.66728, 0.76706)
	95% CI for θ	(0.00945, 0.01890)	(0.01885, 0.02345)	(0.02353, 0.02367)	(0.03162, 0.03990)	(0.00915, 0.01859)	(0.01039, 0.01943)	(0.01468, 0.01514)	(0.01408, 0.02128)
(50,25)	$\hat{p}$	0.49524 (0.07001)	0.71665 (0.01939)	0.82986 (0.00909)	0.81796 (0.00391)	0.49227 (0.07261)	0.50694 (0.05537)	0.65321 (0.00187)	0.51059 (0.03466)
	$\hat{\theta}$	0.01438 (0.00292)	0.042192 (0.00399)	0.06648 (0.00060)	0.04971 (0.00339)	0.01429 (0.00296)	0.01523 (0.00410)	0.02482 (0.00052)	0.01575 (0.00320)
	95% CI for p	(0.35802, 0.63246)	(0.67865, 0.75465)	(0.82808, 0.83164)	(0.82220, 0.83752)	(0.34996, 0.63459)	(0.39842, 0.61546)	(0.50328, 0.51060)	(0.44266, 0.57853)
	95% CI for θ	(0.00866, 0.02010)	(0.03437, 0.05001)	(0.06530, 0.06766)	(0.05983, 0.07313)	(0.00848, 0.02009)	(0.00719, 0.02328)	(0.01421, 0.01626)	(0.00948, 0.02202)

Note that the estimate of p and is θ is comparable in all cases. It is also seen that the standard error is very small for every combination of n and $n-r$. The LSE is close to ML estimators, whereas, the WLSE and PE overestimate the parameters slightly when the number of inliers increases for the instantaneous failure model. As the number of inliers decreases, all estimators are quite close for both the models.

Table 5.2 presents the UMVU estimates of parametric functions and UMVUE of its variance for instantaneous and early failure model for different values of k and different combination of n and $n-r$.

Table 5.2. *Summary of estimates of parametric functions and its estimate of the variances*

$(n, n-r)$	k	UMVUE of parametric function and its variance	
		Instantaneous failure model	Early failure model (d =2.5)
(50,05)	1	0.01216 (3.837e-06)	0.01101 (3.445e-06)
	2	1.509e-04 (2.653e-09)	1.234e-04 (1.948e-09)
	3	1.910e-06 (0.147e-12)	1.410e-06 (6.982e-13)
(50,15)	1	0.00959 (3.701e-06)	0.00868 (3.213e-06)
	2	9.451e-05 (1.668e-09)	7.728e-05 (1.166e-09)
	3	9.564e-07 (4.862e-13)	7.039e-07 (2.678e-13)
(50,25)	1	0.00680 (3.072e-06)	0.00624 (2.741e-06)
	2	4.730e-05 (6.905e-10)	4.024e-05 (5.491e-10)
	3	3.347e-07 (9.859e-14)	2.679e-07 (7.273e-14)

It may be observed from Table 5.2 that for each combination of n and $n-r$, the UMVU estimates are decreasing function of k. It is also observed that as the number of inliers increases, the UMVU estimate of the parametric function decreases and UMVUE of variance of the estimate decreases. The UMVU estimate of probability density function and survival function for the above two models for various combinations are shown in Table 5.3. The entries in brackets are the UMVUE of variances of estimates.

Table 5.3 shows that as the value of k increases, the estimate of the pdf and survival function decreases and UMVUE of the variance of the estimate become very small for every combination of n and $n-r$. The UMVU estimates of the pdf and survival function increases as the number of inliers decreases.

Table 5.3. *Summary of estimates of pdf and Reliability functions*

$(n, n-r)$	k	UMVUE of PDF		UMVUE of survival function	
		Instantaneous failure model	Early failure model	Instantaneous failure model	Early failure model
(50,05)	2	0.01230 (3.714e-06)	0.01116 (3.359e-06)	0.87580 (0.00171)	0.83363 (0.00235)
	4	0.01244 (3.574e-06)	0.01129 (3.258e-06)	0.85105 (0.00166)	0.81118 (0.00226)
	6	0.01256 (3.419e-06)	0.01142 (3.144e-06)	0.82606 (0.00163)	0.78847 (0.00220)
	8	0.01266 (3.249e-06)	0.01153 (3.016e-06)	0.80083 (0.00163)	0.76552 (0.00215)
	10	0.01275 (3.067e-06)	0.01163 (2.876e-06)	0.77542 (0.00165)	0.74235 (0.00213)
(50,15)	2	0.00970 (3.597e-06)	0.00879 (3.148e-06)	0.67709 (0.00401)	0.65080 (0.00422)
	4	0.00980 (3.479e-06)	0.00890 (3.070e-06)	0.65759 (0.00381)	0.63311 (0.00402)
	6	0.00988 (3.348e-06)	0.00900 (2.980e-06)	0.63791 (0.00364)	0.61522 (0.00384)
	8	0.00996 (3.205e-06)	0.00908 (2.879e-06)	0.61807 (0.00350)	0.59715 (0.00369)
	10	0.01002 (3.052e-06)	0.00915 (2.768e-06)	0.59809 (0.00337)	0.57891 (0.00355)
(50,25)	2	0.00688 (3.002e-06)	0.00632 (2.688e-06)	0.48155 (0.00474)	0.46200 (0.00473)
	4	0.00695 (2.920e-06)	0.00639 (2.626e-06)	0.46772 (0.00449)	0.44930 (0.00450)
	6	0.00700 (2.829e-06)	0.00645 (2.554e-06)	0.45375 (0.00427)	0.43646 (0.00428)
	8	0.00707 (2.728e-06)	0.00651 (2.474e-06)	0.43967 (0.00406)	0.42349 (0.00408)
	10	0.00711 (2.618e-06)	0.00656 (2.387e-06)	0.42550 (0.00388)	0.41042 (0.00390)

5.9. Numerical Example

We consider the Vanmann's data of Schedule 1 of Experiment 2 on one batch of 37 boards of dataset A.1 from appendix. There are 13 instantaneous failures in this Schedule. The density and survival plots for various combinations of parameters along with the parameter estimates are presented below.

In Figures 5.4 three density functions $g(x)$ with $c^* = r$ and different value of p are plotted for instantaneous failure model and for early failure model keeping d=1.0.

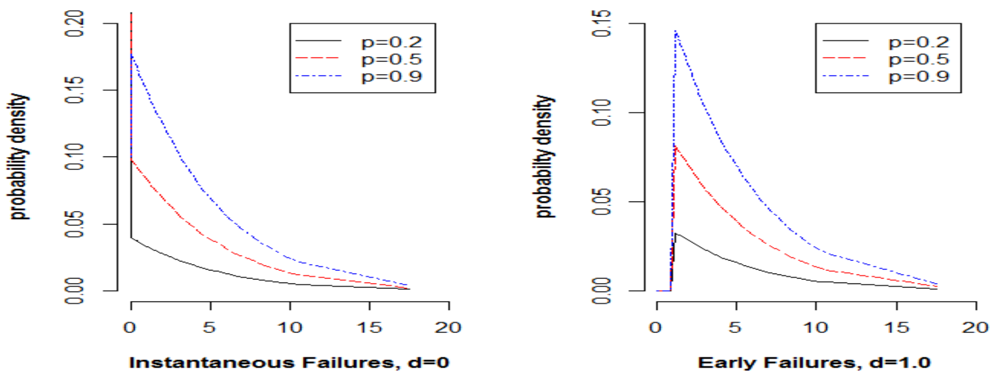

Figure 5.4. *Plots of density functions at varying p for Vannman's dataset*

Figures 5.5 show three reliability functions curves for different value of α plotted for instantaneous failure model and for early failure model keeping d=1.0.

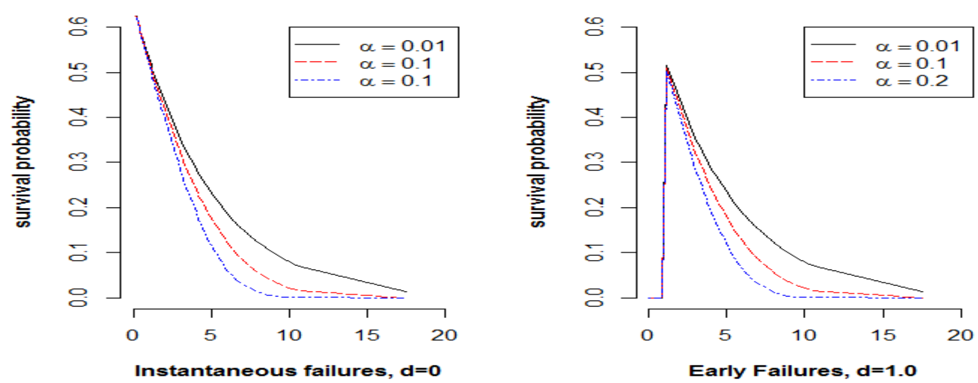

Figure 5.5. *Plots of survival functions with varying α for Vannman's dataset*

The summary of estimates of models for some selected values of parameters of the above example when $c^* = r$ is shown in Table 5.4. The entry in brackets show the standard error of the estimate.

Table 5.4. *Summary of estimates of models for example in section 5.9*

	Estimator	Estimates (SE)	
		Instantaneous Failure model, d=0	Early Failure Model, d=1.0
α not known	$\hat{p}_{MLE}$ & $\widehat{SE}(\hat{p}_{MLE})$	0.64865 (0.07845)	0.54995 (0.09689)
	$\hat{\theta}_{MLE}$ & $\widehat{SE}(\hat{\theta}_{MLE})$	0.17304 (0.05629)	0.11770 (0.05099)
	$\hat{\alpha}_{MLE}$ & $\widehat{SE}(\hat{\alpha}_{MLE})$	0.03733 (0.05203)	0.08128 (0.05736)
	95% CI for p	(0.49482, 0.80247)	(0.36002, 0.73988)
	95% CI for θ	(0.06272, 0.28336)	(0.01776, 0.21764)
	95% CI for α	(0.00000, 0.13930)	(0.00000, 0.19370)
	$\hat{p}_{LSE}$ & $\widehat{SE}(\hat{p}_{LSE})$	0.58984 (0.05172)	0.49294 (0.10580)
	$\hat{\theta}_{LSE}$ & $\widehat{SE}(\hat{\theta}_{LSE})$	0.13740 (0.07050)	0.06657 (0.07539)
	$\hat{\alpha}_{LSE}$ & $\widehat{SE}(\hat{\alpha}_{LSE})$	0.04528 (0.11897)	0.16899 (0.20072)
	95% CI for p	(0.48847, 0.69122)	(0.28559, 0.70029)
	95% CI for θ	(0.00000, 0.27558)	(0.00000, 0.21442)
	95% CI for α	(0.00000, 0.27846)	(0.00000, 0.56240)
	$\hat{p}_{WLSE}$ &$\widehat{SE}(\hat{p}_{WLSE})$	0.58934 (0.00390)	0.48808 (0.00791)
	$\hat{\theta}_{WLSE}$ & $\widehat{SE}(\hat{\theta}_{WLSE})$	0.13212 (0.00473)	0.06193 (0.00493)
	$\hat{\alpha}_{WLSE}$ & $\widehat{SE}(\hat{\alpha}_{LSE})$	0.06081 (0.00734)	0.18368 (0.01269)
	95% CI for p	(0.58189, 0.59697)	(0.47258, 0.50359)
	95% CI for θ	(0.12285, 0.14139)	(0.05227, 0.07160)
	95% CI for α	(0.04641, 0.07520)	(0.15880, 0.20856)
	$\hat{p}_{PE}$ & $\widehat{SE}(\hat{p}_{PE})$	0.59182 (0.03144)	0.49476 (0.06292)
	$\hat{\theta}_{PE}$ & $\widehat{SE}(\hat{\theta}_{PE})$	0.15254 (0.05366)	0.06813 (0.05207)
	$\hat{\alpha}_{PE}$ & $\widehat{SE}(\hat{\alpha}_{LSE})$	0.01650 (0.08849)	0.16503 (0.14700)
	95% CI for p	(0.53020, 0.65344)	(0.37144, 0.61808)
	95% CI for θ	(0.04737, 0.25771)	(0.00000, 0.17019)
	95% CI for α	(0.00000, 0..09497)	(0.00000, 0.45314)
$\alpha = 0.02$ (known)	$\hat{p}_{MLE}$ & $\widehat{SE}(\hat{p}_{MLE})$	0.64865 (0.07848)	0.57678 (0.10006)
	$\hat{\theta}_{MLE}$ & $\widehat{SE}(\hat{\theta}_{MLE}$	0.18785 (0.03834)	0.16855 (0.03972)
	95% CI for p	(0.49482, 0.80247)	(0.37957, 0.77398)
	95% CI for θ	(0.11270, 0.26300)	(0.09070, 0.24640)
	$\hat{p}_{LSE}$ & $\widehat{SE}(\hat{p}_{LSE})$	0.75330 (0.00078)	0.72486 (0.00007)
	$\hat{\theta}_{LSE}$ & $\widehat{SE}(\hat{\theta}_{LSE})$	0.21538 (0.00100)	0.19319 (0.00072)
	95% CI for p	(0.75177, 0.75483)	(0.72472, 0.72500)
	95% CI for θ	(0.21342, 0.21734)	(0.19178, 0.19460)
	$\hat{p}_{WLSE}$ &$\widehat{SE}(\hat{p}_{WLSE})$	0.84339 (2.241e-06)	0.82500 (2.089e-06)
	$\hat{\theta}_{WLSE}$ & $\widehat{SE}(\hat{\theta}_{WLSE})$	0.23783 (4.479e-06)	0.21655 (3.039e-06)
	95% CI for p	(0.84339, 0.84339)	(0.82499, 0.82501)
	95% CI for θ	(0.23782, 0.23784)	(0.21654, 0.21656)
	$\hat{p}_{PE}$ & $\widehat{SE}(\hat{p}_{PE})$	0.84185 (2.704e-05)	0.78913 (3.186e-05)
	$\hat{\theta}_{PE}$ & $\widehat{SE}(\hat{\theta}_{PE})$	0.57070 (0.00390)	0.25593 (6.261e-04)
	95% CI for p	(0.84180, 0.84190)	(0.78907, 0.78919)
	95% CI for θ	(0.56306, 0.57843)	(0.25470, 0.25716)
	$G_1(z,r,n)$	0.3513514 (0.00633)	0.51351 (0.00633)
	$H_1(z,r,n)$	0.12220 (0.00086)	0.06932 (0.00042)
	UMVUE of pdf		
	$\phi_{47}(z,r,n)$	6.774e-08 (6.773e-08)	1.543e-06 (1.543e-06)
	$\phi_{48}(z,r,n)$	1.464e-08 (1.464e-08)	1.004e-06 (1.000e-06)
	UMVUE of survival function		
	$\hat{S}(47)$	1.143e-07(1.143e-07)	3.365e-06 (3.364e-06)
	$\hat{S}(48)$	6.250e-08(6.251e-08)	2.109e-06 (2.109e-06)