Chapter 6

Inferences on inliers in Weibull distribution with Type II censored data

6.1. Introduction

In this chapter, we model the inliers situation with two degenerate points. Following are some of the practical contexts, where degeneracy can happen at two discrete points with a mix of positive observations.

1. The size of tumor lesions is of interest to treat *Hematologic malignancy* patients. The measurement effect is zero who have lesions absent (or due to disappearance of tumor during treatment), though who have lesions present at baseline that are evaluable but do not meet the definitions of measurable disease may be considered as measurement 1 otherwise lesions can be accurately measured as longest diameter to be recorded in at least one dimension by chest x-ray, with CT scan or with calipers by clinical exam. Similarly, in studies like Bone lesions, leptomeningeal disease, ascites, pleural/pericardial effusions, lymphangitis cutis/pulmonitis, inflammatory breast disease, and abdominal masses, either the effect is absent or present but not followed by CT or MRI, are considered as non-measurable otherwise accurately measurable on a continuous scale.
2. n the mass production of technological components of hardware, intended to

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function over a period of time, some components may fail on installation and therefore have zero life lengths, some component that does not fail on installation but fails with negligible life (may be coded as one for simplicity), and others that will have a life length that is a positive random variable whose distribution may take different forms.

3. In a clinical trial laboratory, a particular drug is designed and given to certain species of hens so that the new chicks have a weight greater than usual. The possible weight of chicks may be modeled as a continuous distribution, with a discrete mass at ‘zero’ and ‘one’ where zero measures those chicks having no gain of weight, where one measures those chicks with a negligible gain of weight than usual, and a continuous variable having target gain in weight.
4. The rainfall measurement at a place recorded during a season is modeled as a continuous distribution, with a discrete mass at ‘zero’ where zero measures those days having no rainfall, at ‘one’ where one measures those days with no rain but humid and cloudy conditions, and a continuous variable having some positive amount of rain.
5. In the studies of genetic birth defects, children can be characterized by three variables, first discrete variable to indicate if one is affected and born dead, second, one is affected and has neonatal death and third a continuous variable measuring the survival time of affected children born alive. We may consider here a non-standard mixture of the mass point at ‘zero’ (for children born dead), at ‘one’ (for children born and neonatal death) and a non-trivial continuous distribution for other surviving children.
6. For repairable items, some failures result from natural damages of items while the other failures may be caused by inefficient repair of previous failures resulting from the incorrect organization of the repair process. Here intervals between failures of an item may have a non-standard mixture of the mass point at ‘zero’ (for the length of interval zero), at ‘one’ (for the interval of length 1) and a non-trivial continuous distribution for interval length greater than 1.

In situations like above, the inliers model will be a non-standard mixture of distribution with two discrete points, where the random variable X takes non-zero probability p_1 when $X = 0$, probability p_2 when $X = 1$, and probability $(1 - p_1 - p_2)$ when $X > 1$. Such a model can be written as

$$h(p_1, p_2, \underline{\alpha} | x) = \begin{cases} p_1, & x = 0 \\ p_2, & x = 1 \\ (1 - p_1 - p_2) \dfrac{f(x; \underline{\alpha})}{1 - F(1; \underline{\alpha})}, & x > 1 \end{cases} \tag{6.1.1}$$

In this chapter, we study some estimation procedures for the inliers model using the Type II censored lifetime data from the Weibull distribution. In Section 6.2 we present the Type II censored Weibull inlier model. The likelihood estimates and its asymptotic distribution is given in Section 6.3. The uniformly minimum variance unbiased estimation of parameters and parametric functions in Section 6.4. We propose least squares and weighted least squares estimation in Section 6.5 and percentile estimation in Section 6.6. The Monte Carlo simulation study is carried out in Section 6.7, and in last Section 6.8 we carried out a numerical study on two real examples: one on the breast cancer tumor size data from cancer genomic studies and another on mortality data based on NFHS-3 survey.

6.2. The Type II censored Weibull inlier model

The Weibull distribution introduced by Waloddi Weibull in 1951 is an important generalization of the exponential distribution with two positive parameters Weibull (1951). It is one of the most popular distributions used for analyzing lifetime data. Much of the popularity of the Weibull distribution is due to the second parameter in the model, which allows great flexibility to model different shapes of the hazard function. The Weibull distribution has the pdf

$$f(x;\underline{\alpha}) = \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{x^{\beta}}{\theta}}, x > 0; \beta > 0, \theta > 0 \qquad (6.2.1)$$

where β is the shape parameter and θ is the scale parameter. If we substitute (6.2.1) in (6.1.1), we get the Weibull inliers model as

$$h(x;\underline{\theta}) = \begin{cases} p_1, & x = 0 \\ p_2, & x = 1 \\ (1 - p_1 - p_2) \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{1}{\theta}(x^{\beta}-1)}, & x > 1 \end{cases} \qquad (6.2.2)$$

where $\underline{\theta} = (p_1, p_2, \beta, \theta)$. The graphical plots of the probability density function and survival function for 30% censored data for different values of the parameters are shown respectively in Figure 6.1 and Figure 6.2.

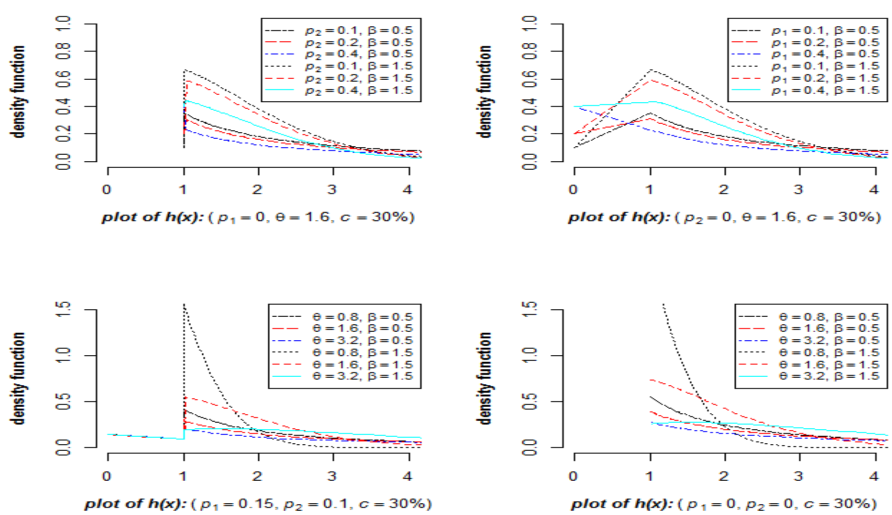

Fig. 6.1. *Density function plots of 30% censored data*

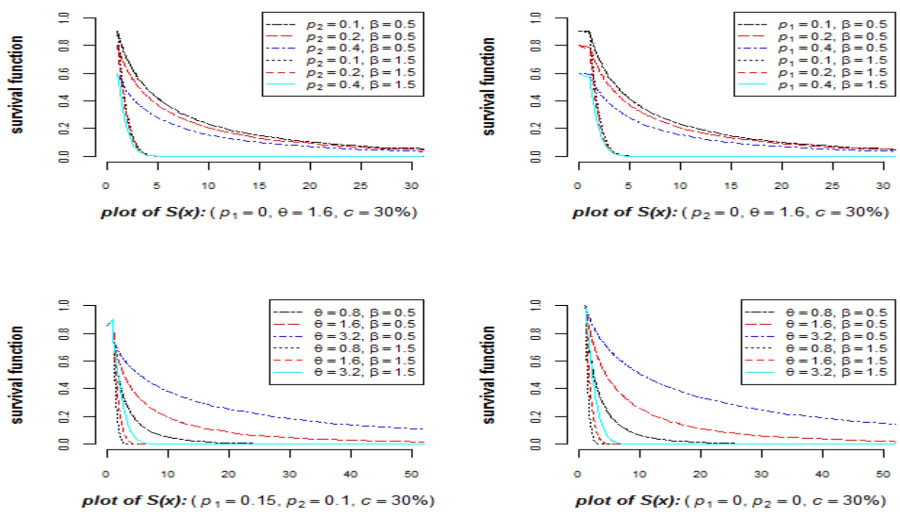

Fig. 6.2. *Survival function plots of 30% censored data*

6.3. The likelihood estimates and its asymptotic distribution

In this section to estimate four parameters p_1, p_2, α, and θ of the model in (6.2.2), we consider maximum likelihood estimation procedure along with its asymptotic distribution below.

6.3.1. Maximum Likelihood Estimation

Suppose n items placed on life test, where r_1 items have life zero where as r_2 items have life 1 and remaining $n - r_1 - r_2$ items have life greater than 1, is denoted by $X_1, X_2, \ldots, X_{n-r_1-r_2}$. By applying the technique of 'Type II censored sample', the experiment terminates after prefixed number of failures $n - r_1 - r_2 - c$ out of $n - r_1 - r_2$ items, where $n - r_1 - r_2 - c < n - r_1 - r_2$. Clearly $n - r_1 - r_2 - c = n - r_1 - r_2$, then the experiment is not terminated and all $n - r_1 - r_2$ lifetimes are observed. Let $n - r_1 - r_2 - c^* = \min(n - r_1 - r_2 - c,\ n - r_1 - r_2)$ and

$X_{(1)}, X_{(2)}, \dots, X_{(n-r_1-r_2-c^*)}$ denote ordered observed failure time of these $n - r_1 - r_2 - c^*$ items from $h \in \mathcal{H}$ as given in (6.2.2), then the likelihood equation can be written as

$$L(\underline{\theta}|\underline{x}) = \frac{(n-r_1-r_2)!}{c^*!} \prod_{i=1}^{n-r_1-r_2-c^*} {p_1}^{I_1(x_i)} {p_2}^{I_2(x_i)} \left((1-p_1-p_2) \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{1}{\theta}(x^{\beta}-1)} \right)^{1-I_1(x_i)-I_2(x_i)}$$
$$\left[{p_1}^{I_1(x_i)} {p_2}^{I_2(x_i)} \left((1-p_1-p_2) e^{-\frac{1}{\theta}(x^{\beta}-1)} \right)^{(1-I_1(x_i)-I_2(x_i))} \right]^{c^*}$$

where $I_1(x) = \begin{cases} 1, & x = 0 \\ 0, & o.w. \end{cases}$ and $I_2(x) = \begin{cases} 1, & x = 1 \\ 0, & o.w. \end{cases}$

Here, $\sum_{i=1}^{n} I_1(x_i) = r_1$ and $\sum_{i=1}^{n} I_2(x_i) = r_2$, and therefore

$$L(\underline{\theta}|\underline{x}) = {p_1}^{r_1} {p_2}^{r_2} (1-p_1-p_2)^{(n-r_1-r_2)} \frac{(n-r_1-r_2)!}{c^*!} \left(\frac{\beta}{\theta}\right)^{n-r_1-r_2-c^*} \prod_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta-1}$$
$$e^{-\frac{1}{\theta}\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} - (n-r_1-r_2)\right]} \qquad (6.3.1)$$

Let, the maximum likelihood estimator of parameter $\underline{\theta} = (p_1, p_2, \beta, \theta)$, $\underline{\hat{\theta}}_{MLE} = (\hat{p}_{1MLE}, \hat{p}_{2MLE}, \hat{\beta}_{MLE}, \hat{\theta}_{MLE})$, say. We now investigate the following four possible cases of likelihood estimates:

Case (*i*). $r_2 = 0$, that is $r_1 = n$. The likelihood function simply reduces to $L(\underline{\theta}|\underline{x}) = p_1^n$. Obviously, this is maximum when $p_1 = 1$. This corresponds to the maximum likelihood estimator $\hat{p}_{1MLE} = \frac{r_1}{n}$. Since $L(\underline{\theta}|\underline{x}) = p_1^n$ is free from the other parameters, the maximum likelihood estimator of other parameters does not exist.

Case (*ii*). $r_1 = 0$, that is $r_2 = n$. The likelihood function simply reduces to $L(\underline{\theta}|\underline{x}) = p_2^n$. Obviously, this is maximum when $p_2 = 1$. This corresponds to the maximum likelihood estimator $\hat{p}_{2MLE} = \frac{r_2}{n}$. Since $L(\underline{\theta}|\underline{x}) = p_2^n$ is free from the other parameters, the maximum likelihood estimator of other parameters does not exist.

Case (*iii*). $r_1 < n, r_2 < n$ but $r_1 + r_2 = n$. The likelihood function simply reduces to $L(\underline{\theta}|\underline{x}) = p_1^{r_1} p_2^{r_2}$. Here $p_1 + p_2 < n$. Then the likelihood function $L(\underline{\theta}|\underline{x}) < \left(\frac{x_1}{n}\right)^{r_1} \left(\frac{x_2}{n}\right)^{r_2}$. So $\hat{p}_{1MLE} = \frac{r_1}{n}$ and $\hat{p}_{2MLE} = \frac{r_2}{n}$. The maximum likelihood estimator of other parameters does not exist.

Case (*iv*). $r_1 + r_2 < n$. The log-likelihood function is given by

$$\begin{aligned}\log L(\underline{\theta}|\underline{x}) &= r_1 \log p_1 + r_2 \log p_2 + (n - r_1 - r_2)\log(1 - p_1 - p_2) \\ &\quad + \log(n - r_1 - r_2)! - \log c^*! + (n - r_1 - r_2 - c^*)[\log\beta - \log\theta] \\ &\quad + (\beta - 1) \sum_{i=1}^{n-r_1-r_2-c^*} \log x_{(i)} \\ &\quad - \frac{1}{\theta}\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} - (n - r_1 - r_2)\right] \qquad (6.3.2)\end{aligned}$$

The maximum likelihood estimator of parameter $\underline{\theta}$ is obtained by solving the following likelihood equations:

$$\frac{\partial \log L(\underline{\theta}|\underline{x})}{\partial p_1} = \frac{r_1}{p_1} - \frac{n-r_1-r_2}{1-p_1-p_2} = 0 \qquad (6.3.3)$$

$$\frac{\partial \log L(\underline{\theta}|\underline{x})}{\partial p_2} = \frac{r_2}{p_2} - \frac{n-r_1-r_2}{1-p_1-p_2} = 0 \qquad (6.3.4)$$

$$\begin{aligned}\frac{\partial \log L(\underline{\theta}|\underline{x})}{\partial \beta} &= \frac{n-r_1-r_2}{\beta} + \sum_{i=1}^{n-r_1-r_2-c^*} \log x_{(i)} \\ &\quad - \frac{1}{\theta}\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} \log x_{(i)} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} \log x_{(n-r_1-r_2-c^*)}\right]\end{aligned} \qquad (6.3.5)$$

and

$$\frac{\partial \log L(\underline{\theta}|\underline{x})}{\partial\theta} = -\frac{n-r_1-r_2-c^*}{\theta} + \frac{\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} - (n-r_1-r_2)\right]}{\theta^2} = 0 \tag{6.3.6}$$

Solving (6.3.3) and (6.3.4) simultaneously, we get

$$\hat{p}_{1MLE} = \frac{r_1}{n} \tag{6.3.7}$$

and

$$\hat{p}_{2MLE} = \frac{r_2}{n} \tag{6.3.8}$$

From (6.3.6), the estimate of θ is

$$\hat{\theta}_{MLE} = \frac{\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} - (n-r_1-r_2)\right]}{n-r_1-r_2-c^*} \tag{6.3.9}$$

Using (6.3.9) in (6.3.5) we get

$$\frac{n-r_1-r_2}{\beta} + \sum_{i=1}^{n-r_1-r_2-c^*} \log x_{(i)} - \frac{(n-r_1-r_2-c^*)\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} \log x_{(i)} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} \log x_{(n-r_1-r_2-c^*)}\right]}{\left[\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} - (n-r_1-r_2)\right]} = 0 \tag{6.3.10}$$

Equation (6.3.10) is solved using iterative procedures to get $\hat{\beta}$ and then solve (6.3.9) to get $\hat{\theta}_{MLE}$. In the case of β known, the maximum likelihood estimator of p_1, p_2 and θ respectively is $\hat{p}_{1MLE} = \frac{r_1}{n}$, $\hat{p}_{2MLE} = \frac{r_2}{n}$, and $\hat{\theta}_{MLE}$ simplifies to

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta} \log x_{(i)} + c^* x_{(n-r_1-r_2-c^*)}^{\beta} \log x_{(n-r_1-r_2-c^*)}}{n-r_1-r_2-c^*}.$$

6.3.2. Asymptotic distribution of MLE

For $h(x;\underline{\theta})$ as given in (6.2.3)

$$\frac{\partial ln\, h(x;\underline{\theta})}{\partial p_1} = \begin{cases} \frac{1}{p_1}, & x = 0 \\ 0, & x = 1 \\ -\frac{1}{(1-p_1-p_2)}, & x > 1 \end{cases}$$

$$\frac{\partial ln\, h(x;\underline{\theta})}{\partial p_2} = \begin{cases} 0, & x = 0 \\ \frac{1}{p_2}, & x = 1 \\ -\frac{1}{(1-p_1-p_2)}, & x > 1 \end{cases}$$

$$\frac{\partial ln\, h(x;\underline{\theta})}{\partial \beta} = \begin{cases} 0, & x = 0 \\ 0, & x = 1 \\ \frac{1}{\beta} + \log x - \frac{x^{\beta}\log x}{\theta}, & x > 1 \end{cases}$$

and

$$\frac{\partial ln\, h(x;\underline{\theta})}{\partial \theta} = \begin{cases} 0, & x = 0 \\ 0, & x = 1 \\ -\frac{1}{\theta} - \frac{(x^{\beta}-1)}{\theta^2}, & x > 1 \end{cases}$$

One can verify that

$$E\left(\frac{\partial ln\, h(x;\underline{\theta})}{\partial p_1}\right) = E\left(\frac{\partial ln\, h(x;\underline{\theta})}{\partial p_2}\right) = E\left(\frac{\partial ln\, h(x;\underline{\theta})}{\partial \beta}\right) = E\left(\frac{\partial ln\, h(x;\underline{\theta})}{\partial \theta}\right) = 0.$$

Also,

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1^2} = \begin{cases} -\frac{1}{p_1^2}, & x = 0 \\ 0, & x = 1 \\ -\frac{1}{(1-p_1-p_2)^2}, & x > 1 \end{cases}$$

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2^2} = \begin{cases} 0, & x = 0 \\ -\frac{1}{p_2^2}, & x = 1 \\ -\frac{1}{(1-p_1-p_2)^2}, & x > 1 \end{cases}$$

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial \beta^2} = \begin{cases} 0, & x = 0 \\ 0, & x = 1 \\ \frac{1}{\beta^2} - \frac{x^\beta (\log x)^2}{\theta}, & x > 1 \end{cases}$$

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial \theta^2} = \begin{cases} 0, & x = 0 \\ 0, & x = 1 \\ \frac{1}{\theta^2} - \frac{2(x^\beta - 1)}{\theta^3}, & x > 1 \end{cases}$$

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial p_2} = \begin{cases} 0, & x = 0 \\ 0, & x = 1 \\ -\frac{1}{(1-p_1-p_2)^2}, & x > 1 \end{cases}$$

$$\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial \beta} = \frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2\, \partial \beta} = \frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial \theta} = \frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2\, \partial \theta} = 0 \;\; \forall\, x.$$

Hence, the Fisher information is

$$I_{p_1 p_1} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1^2}\right) = \frac{1-p_2}{p_1(1-p_1-p_2)} = \frac{1-p_2}{p_1\, p^*}$$

$$I_{p_2 p_2} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2^2}\right) = \frac{1-p_1}{p_2(1-p_1-p_2)} = \frac{1-p_1}{p_2\, p^*}$$

$$I_{\beta\beta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial \beta^2}\right) = \frac{p^*}{\beta^2}\left[2A + 2e^{\frac{1}{\theta}} S - 1\right]$$

$$I_{\theta\theta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial\theta^2}\right) = \frac{1-p_1-p_2}{\theta^2} = \frac{p^*}{\theta^2}$$

$$I_{p_1p_2} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial p_2}\right) = \frac{1}{1-p_1-p_2} = \frac{1}{p^*}$$

$$I_{p_1\beta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial\beta}\right) = 0,\ I_{p_1\theta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_1\, \partial\theta}\right) = 0$$

$$I_{p_2\beta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2\, \partial\beta}\right) = 0,\ I_{p_2\theta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial p_2\, \partial\theta}\right) = 0$$

and

$$I_{\beta\theta} = E\left(-\frac{\partial^2 ln\, h(x;\underline{\theta})}{\partial\beta\theta}\right) = -\frac{p^*}{\beta\theta}A$$

where, $p^* = 1-p_1-p_2$, $S = \sum_{j=0}^{\infty}\frac{(-1)^j}{j^2\ j!\theta^j}$ and $A = e^{\frac{1}{\theta}}E_1\left(\frac{1}{\theta}\right)+1$.

The Fisher information matrix $I_h(\underline{\theta})$ is given by

$$I_h(\underline{\theta}) = \begin{bmatrix} I_{p_1p_1} & I_{p_1p_2} & I_{p_1\beta} & I_{p_1\theta} \\ I_{p_2p_1} & I_{p_2p_2} & I_{p_2\beta} & I_{p_2\theta} \\ I_{\beta p_1} & I_{\beta p_2} & I_{\beta\beta} & I_{\beta\theta} \\ I_{\theta p_1} & I_{\theta p_2} & I_{\theta\beta} & I_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{1-p_2}{p_1\, p^*} & \frac{1}{p^*} & 0 & 0 \\ \frac{1}{p^*} & \frac{1-p_1}{p_2\, p^*} & 0 & 0 \\ 0 & 0 & \frac{p^*}{\beta^2}\left[2A+2e^{\frac{1}{\theta}}S-1\right] & -\frac{p^*}{\beta\theta}A \\ 0 & 0 & -\frac{p^*}{\beta\theta}A & \frac{p^*}{\theta^2} \end{bmatrix} \tag{6.3.11}$$

The determinant of $I_h(\underline{\theta})$ is given by $\frac{p^*}{p_1p_2\beta^2\theta^2}\left[2e^{\frac{1}{\theta}}S-(A-1)^2\right]$ and hence, the inverse matrix $I_h^{-1}(\underline{\theta})$ is given by

$$I_h^{-1}(\underline{\theta}) = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & 0 & 0 \\ -p_1p_2 & p_2(1-p_2) & 0 & 0 \\ 0 & 0 & \dfrac{\beta^2}{p^*\left[2A + 2e^{\frac{1}{\theta}}S - 1\right]} & -\dfrac{\beta\theta^2 A}{p^*\left[2e^{\frac{1}{\theta}}S - (A-1)^2\right]} \\ 0 & 0 & -\dfrac{\beta\theta^2 A}{p^*\left[2e^{\frac{1}{\theta}}S - (A-1)^2\right]} & \dfrac{\theta^2}{p^*} \end{bmatrix}$$

(6.3.12)

Using the standard result of MLE, we have

$$\left(\underline{\hat{\theta}}_{MLE}\right)' \sim AN^{(4)}\left[(\underline{\theta})', \frac{1}{n} I_h^{-1}(\underline{\theta})\right] \quad (6.3.13)$$

Using the estimated variances, one can also propose large sample tests for p_1, p_2, β and θ. The approximate $(1-\alpha)\%$ confidence interval for p_1, p_2, β and θ are respectively given by

$$\hat{p}_{1MLE} \pm z\alpha/_2 \sqrt{\frac{\hat{p}_{1MLE}(1-\hat{p}_{1MLE})}{n}} \quad (6.3.14)$$

$$\hat{p}_{2MLE} \pm z\alpha/_2 \sqrt{\frac{\hat{p}_{2MLE}(1-\hat{p}_{2MLE})}{n}} \quad (6.3.15)$$

$$\hat{\beta}_{MLE} \pm z\alpha/_2 \sqrt{\frac{\hat{\beta}^2_{MLE}}{n\,\hat{p}^*\left[2\,e^{\frac{1}{\theta}}\,(S-(A-1)^2)\right]}} \quad (6.3.16)$$

and

$$\hat{\theta}_{MLE} \pm z\alpha/_2 \sqrt{\frac{\hat{\theta}^2_{MLE}}{(n-c^*)\ \hat{p}^*_{MLE}}} \quad (6.3.17)$$

where $\hat{p}^*_{MLE} = 1 - \hat{p}_{1MLE} - \hat{p}_{2MLE}$.

6.4. Uniformly Minimum Variance Unbiased Estimation

Assuming β known and using (1.5.24), the model in (6.2.2) can be expressed as

$$h(x;\underline{\theta}) = {p_1}^{I_1(x)}{p_2}^{I_2(x)}\left((1-p_1-p_2)\beta x^{\beta-1}\frac{e^{-\left(\frac{x^\beta-1}{\theta}\right)}}{\theta}\right)^{(1-I_1(x)-I_2(x))}$$

$$=\frac{\left(\beta x^{\beta-1}\right)^{(1-I_1(x)-I_2(x))}\left(\frac{\theta\, p_1}{1-p_1-p_2}\right)^{I_1(x)}\left(\frac{\theta\, p_2}{1-p_1-p_2}\right)^{I_2(x)}\left(e^{-\frac{1}{\theta}}\right)^{(x^\beta-1)(1-I_1(x)-I_2(x))}}{\left(\frac{\theta}{1-p_1-p_2}\right)}$$

$$=(a(x))^{(1-C_1(x)-C_2(x))}\frac{\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{C_i(x)}}{g(\underline{\theta})} \qquad (6.4.1)$$

where $(X)=\beta X^{\beta-1}$, $h_1(\underline{\theta})=\frac{\theta\, p_1}{1-p_1-p_2}$, $h_2(\underline{\theta})=\frac{\theta\, p_2}{1-p_1-p_2}$, $h_3(\underline{\theta})=e^{-\frac{1}{\theta}}$, $g(\underline{\theta})=\frac{\theta}{1-p_1-p_2}$, $C_1(X)=I_1(X)$, $C_2(X)=I_2(X)$, and $C_3(X)=\left(X^\beta-1\right)\left(1-I_1(X)-I_2(X)\right)$. Also $a(X)>0$, $C_i(X), i=$ 1, 2 and 3 are nontrivial real-valued statistics, $g(\underline{\theta})$ and $h_i(\underline{\theta})$ are at least twice differentiable functions of θ_i, i = 1, 2 and 3. Here $g(\underline{\theta})=\int_{x>1}(a(x))^{(1-C_1(x)-C_2(x))}\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{C_i(x)}dx$. The density in (6.4.1) so obtained is defined with respect to a measure $\mu(x)$ which is the sum of Lebesgue measure over $(1,\infty)$ a well-known form of a three parameter exponential family with natural parameters $(\eta_1,\eta_2,\eta_3)=\left[\log\left(\frac{\theta\, p_1}{1-p_1-p_2}\right), \log\left(\frac{\theta\, p_2}{1-p_1-p_2}\right), \log\left(e^{-\frac{1}{\theta}}\right)\right]$ generated by underlying indexing parameters $\underline{\theta}=(p_1,p_2,\theta)$. Hence $C(X)=\left(C_1(X),C_2(X),C_3(X)\right)=\left(I_1(X),\ I_2(X),\left(X^\beta-1\right)\left(1-I_1(X)-I_2(X)\right)\right)$ is jointly complete sufficient for $\underline{\theta}=(p_1,p_2,\theta)$. The distributional properties of $C(X)=\left(C_1(X),C_2(X),C_3(X)\right)$ are presented in Muralidharan and Pratima (2017). With this set up we now propose some uniformly minimum variance unbiased estimators for parameters and some parametric function of the model (6.4.1) in various subsections below.

6.4.1. UMVU Estimation of parameters

For the Type II censored sample discussed in the previous section, consider the following transformation

$$Y_1 = (n - r_1 - r_2)\left(X_{(1)}^{\beta} - 1\right),$$

and

$$Y_i = (n - r_1 - r_2 - i + 1)\left[\left(X_{(i)}^{\beta} - 1\right) - \left(X_{(i-1)}^{\beta} - 1\right)\right];\ i = 2, 3, \ldots, n - r_1 - r_2 - c^* \tag{6.4.2}$$

It can be seen that

$$\sum_{i=1}^{n-r_1-r_2-c^*} Y_i = \sum_{i=1}^{n-r_1-r_2-c^*} X_{(i)}^{\beta} + c^* X_{(n-r_1-r_2-c^*)}^{\beta} - (n - r_1 - r_2)$$

and

$$|J| = \frac{c^*!}{(n-r_1-r_2)!\beta^{(n-r_1-r_2-c^*)} \prod_{i=1}^{n-r_1-r_2-c^*} x_{(i)}^{\beta-1}} \tag{6.4.3}$$

Using (6.4.2) and (6.4.3),

$$h\left(\underline{y};\ \underline{\theta}\right) = {p_1}^{r_1}{p_2}^{r_2}\ (1 - p_1 - p_2)^{(n-r_1-r_2)}\ \frac{e^{-\frac{\sum_{i=1}^{n-r_1-r_2-c^*} y_i}{\theta}}}{\theta^{(n-r_1-r_2-c^*)}} \tag{6.4.4}$$

$$= \frac{\left(\frac{\theta\, p_1}{1-p_1-p_2}\right)^{z_1}\left(\frac{\theta\, p_2}{1-p_1-p_2}\right)^{z_2}\left(e^{-\frac{1}{\theta}}\right)^{z_3}(1-p_1-p_2)^{c^*}}{\left(\frac{\theta}{1-p_1-p_2}\right)^{n-c^*}} \tag{6.4.5}$$

where

$$Z_1 = \sum_{i=1}^{n} C_1(X_i) = \sum_{i=1}^{n-c^*} I_1(Y_i) = r_1$$

$$Z_2 = \sum_{i=1}^{n} C_2(X_i) = \sum_{i=1}^{n-c^*} I_2(Y_i) = r_2$$

and

$$Z_3 = \sum_{i=1}^{n} C_3(X_i) = \sum_{i=1}^{n-r_1-r_2-c^*} Y_i$$

Hence by Neyman Factorization theorem $Z = (Z_1, Z_2, Z_3)$ is jointly sufficient for $\underline{\theta} = (p_1, p_2, \theta)$. Also,

$$h\left(\underline{y}; \underline{\theta}\right) = \frac{n!\ {p_1}^{r_1}{p_2}^{r_2}(1-p_1-p_2)^{(n-r_1-r_2)}}{r_1!\ \ r_2!\ (n-r_1-r_2)!} \frac{1}{\frac{n!}{r_1!\ \ r_2!\ (n-r_1-r_2)!}} \frac{e^{-\frac{\sum_{i=1}^{n-r_1-r_2-c^*} y_i}{\theta}}}{\theta^{(n-r_1-r_2-c^*)}}$$

$$= \mathrm{P}(Z_1 = r_1, Z_2 = r_2)\ \ h(\underline{y}; \theta | Z_1 = r_1, Z_2 = r_2)$$

Here distribution of (Z_1, Z_2) is trinomial and is a complete family of distribution and

$$h\left(\underline{y}; \theta \middle| Z_1 = r_1, Z_2 = r_2\right) = \frac{r_1!\ \ r_2!\ (n-r_1-r_2)!}{n!} \frac{e^{-\frac{\sum_{i=1}^{n-r_1-r_2-c^*} y_i}{\theta}}}{\theta^{(n-r_1-r_2-c^*)}}$$

which belongs to the one-parameter exponential family. Hence $Z_3|Z_1, Z_2$ is complete sufficient for θ and also a member of the exponential family. The distribution of $Z_3|Z_1, Z_2$ is Gamma with parameter $(n - r_1 - r_2 - c^*, \theta)$ with pdf

$$h(z_3;\theta|n-r_1-r_2-c^*) = \frac{z_3{}^{(n-r_1-r_2-c^*-1)}}{\Gamma n-r_1-r_2-c^*}\frac{e^{-\frac{z_3}{\theta}}}{\theta^{n-r_1-r_2-c^*}}, z_3>0;\ \theta>0$$

which depends only on θ and is also a complete family of distribution. Therefore, $Z = (Z_1,\ Z_2,\ Z_3)$ is complete sufficient for $\underline{\theta} = (p_1, p_2, \theta)$. The Joint distribution of $Z = (Z_1, Z_2, Z_3)$ is

$$h_Z\left(z;\ \underline{\theta}\right) = \frac{n!}{r_1!\ \ r_2!\ (n-r_1-r_2)!} p_1{}^{r_1} p_2{}^{r_2}(1-p_1-p_2)^{(n-r_1-r_2)},$$

$$\frac{z_3{}^{(n-r_1-r_2-c^*)}}{\Gamma n-r_1-r_2-c^*}\frac{e^{-\frac{z_3}{\theta}}}{\theta^{n-r_1-r_2-c^*}}\quad 0\le r_1, r_2 \le n-c^*;\ z_3>0;\ 0\le p_1,p_2\le 1;\ \theta>0$$

$$= B(z_1,z_2,z_3,c^*,n)\ \frac{\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{z_i}}{g\left(\underline{\theta}\right)^{n-c^*}}(1-p_1-p_2)^{c^*} \tag{6.4.6}$$

where,

$$B(z_1,z_2,z_3,c^*,n) = \begin{cases} \frac{n!}{r_1!\ r_2!\,(n-r_1-r_2)!}\frac{z_3{}^{(n-r_1-r_2-c^*-1)}}{\Gamma n-r_1-r_2-c^*}, & z_3>0;\ r_1+r_2-1<n-c^* \\ 1, & z_3=0;\ r_1=0\ \ or\ r_2=0 \end{cases} \tag{6.4.7}$$

$z_i \in T(n-c^*) \subseteq \mathbb{R}, \underline{\theta} \in \Omega$. Here $z = (z_1,z_2,z_3,c^*,n)$ and $B(z_1,z_2,z_3,c^*,n)$ are such that

$$\frac{g\left(\underline{\theta}\right)^{n-c^*}}{(1-p_1-p_2)^{c^*}} = \int_{z_1\in T(n-c^*)}\int_{z_2\in T(n-c^*)}\int_{z_3\in T(n-c^*)} B(z_1,z_2,z_3,c^*,n)\prod_{i=1}^{3}\left(h_i(\ \underline{\theta}\ )\right)^{z_i} dz_1\,dz_2\,dz_3$$

Referring to the equation (1.5.24), we get $\underline{\mu}$ and Σ from the equation (1.5.31) as:

$$\underline{\mu} = \begin{bmatrix} E\left(C_1(X)\right) \\ E\left(C_2(X)\right) \\ E\left(C_3(X)\right)\end{bmatrix} = \begin{bmatrix} E\left(I_1(X)\right) \\ E\left(I_2(X)\right) \\ E\left(\left(X^\beta-1\right)\left(1-I_1(X)-I_2(X)\right)\right)\end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \theta(1-p_1-p_2)\end{bmatrix} \tag{6.4.8}$$

$$\Sigma = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & -\theta\, p_1(1-p_1-p_2) \\ -p_1p_2 & p_2(1-p_2) & -\theta\, p_2(1-p_1-p_2) \\ -\theta\, p_1(1-p_1-p_2) & -\theta\, p_2(1-p_1-p_2) & \theta^2(1-(p_1+p_2)^2) \end{bmatrix} \quad (6.4.9)$$

Therefore, uniformly minimum variance unbiased estimators for the parameters and parametric function of the model (6.4.1) are given below:

$$E(Z_1) = E\left(\sum_{j=1}^{n} C_1(x_j)\right) = \sum_{j=1}^{n-c^*} E\left(I_1(y_j)\right) = (n-c^*)\, p_1,$$

$$E(Z_2) = E\left(\sum_{j=1}^{n} C_2(x_j)\right) = \sum_{j=1}^{n-c^*} E\left(I_2(y_j)\right) = (n-c^*)\, p_2,$$

and

$$E(Z_3) = E\left(\sum_{j=1}^{n} C_3(x_j)\right) = \sum_{i=1}^{n-r_1-r_2-c^*} E(Y_i) = (n-c^*)\, \theta(1-p_1-p_2),$$

which in turn give UMVUE's of p_1, p_2 and θ respectively as

$$\hat{p}_{1UMVUE} = \frac{Z_1}{n-c^*} = \frac{r_1}{n-c^*} \quad (6.4.10)$$

$$\hat{p}_{2UMVUE} = \frac{Z_2}{n-c^*} = \frac{r_2}{n-c^*} \quad (6.4.11)$$

and

$$\hat{\theta}_{UMVUE} = \frac{Z_3}{(n-c^*)(1-\hat{p}_{1UMVUE}-\hat{p}_{2UMVUE})} \quad (6.4.12)$$

6.4.2. UMVU Estimation of parametric functions

Let $X_1, X_2, \ldots, X_{n-c^*}$ be Type II censored random sample from (6.4.1), then there exists an UMVUE of $\Phi(\underline{\theta})$ if and only if $\Phi(\underline{\theta})\left[g(\underline{\theta})\right]^{n-c^*}$ can be expressed in the form

$$\frac{\Phi(\underline{\theta})[g(\underline{\theta})]^{n-c^*}}{(1-p_1-p_2)^{c^*}} = \int_{z_1\in T(n-c^*)}\int_{z_2\in T(n-c^*)}\int_{z_3\in T(n-c^*)} \alpha(z_1,z_2,z_3,c^*,n)\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{z_i} dz_1\,dz_2\,dz_3$$

Thus, the UMVUE of a function $\Phi(\underline{\theta})$ of $\underline{\theta}$ in $h(\underline{\theta}|x)$ is given by

$$\psi(Z_1,Z_2,Z_3,c^*,n) = \frac{\alpha(Z_1,Z_2,Z_3,c^*,n)}{B(Z_1,Z_2,Z_3,c^*,n)},\ B(Z_1,Z_2,Z_3,c^*,n) \neq 0.$$

The following results are now obvious.

Result 6.2.1. The UMVUE of $\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{k_i} = \left(\frac{\theta}{1-p_1-p_2}\right)^{k_1+k_2} {p_1}^{k_1}\, {p_2}^{k_2} e^{-\frac{k_3}{\theta}}$ is given by

$$H_{k_1,k_2,k_3}(z_1,z_2,z_3,c^*,n) = \frac{B(z_1-k_1,\ z_2-k_2,\ z_3-k_3,\ c^*,n)}{B(z_1,z_2,z_3,c^*,n)}$$

$$= \frac{(r_1)_{k_1}\,(r_2)_{k_2}\,\left(1-\frac{k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)}\,(z_3-k_3)^{k_1+k_2}}{[n-r_1-r_2+1]_{k_1+k_2}[n-r_1-r_2-c^*]_{k_1+k_2}},$$

$k_1 \le r_1; k_2 \le r_2; k_3 \le z_3; k_1+k_2 \le n- r_1-r_2-c^*;\ r_1+r_2-1 < n-c^*;$ $(r)_k = \frac{r!}{(r-k)!}$ and $[r]_k = \frac{\Gamma r+k}{\Gamma r}$.

Corollary 6.2.1. If $k_1 \neq 0,\ k_2 = 0$ and $k_3 = 0$, then UMVUE of $\left(h_1(\underline{\theta})\right)^{k_1} = \left(\frac{\theta\, p_1}{1-p_1-p_2}\right)^{k_1}$ is given by

$$H_{k_1}(z_1,z_2,z_3,c^*,n) = \frac{B(z_1-k_1,z_2,z_3,c^*,n)}{B(z_1,z_2,z_3,c^*,n)}$$

$$= \frac{(r_1)_{k_1}\ {z_3}^{k_1}}{[n-r_1-r_2+1]_{k_1}[n-r_1-r_2-c^*]_{k_1}},$$

$$k_1 \le r_1;\ k_1 \le n-r_1-r_2-c^*;\ \ r_1+r_2-1 < n-c^*.$$

Corollary 6.2.2. If $k_1 = 0, k_2 \neq 0$ and $k_3 = 0$, then UMVUE of $\left(h_2(\underline{\theta})\right)^{k_2} = \left(\frac{\theta\, p_2}{1-p_1-p_2}\right)^{k_2}$ is given by

$$H_{k_2}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2 - k_2, z_3, c^*, n)}{B(z_1, z_2, z_3, c^*, n)}$$

$$= \frac{(r_2)_{k_2}\; {z_3}^{k_2}}{[n-r_1-r_2+1]_{k_2}[n-r_1-r_2-c^*]_{k_2}},$$

$$k_2 \le r_2;\; k_2 \le n - r_1 - r_2 - c^*;\; r_1 + r_2 - 1 < n - c^*.$$

Corollary 6.2.3. If $k_1 = 0, k_2 = 0$ and $k_3 \neq 0$, then UMVUE of $\left(h_3(\underline{\theta})\right)^{k_3} = e^{-\frac{k_3}{\theta}}$ is given by

$$H_{k_3}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2, z_3 - k_3, c^*, n)}{B(z_1, z_2, z_3, c^*, n)}$$

$$= \left(1 - \frac{k_3}{z_3}\right)^{n- r_1 - r_2 - c^* - 1},\; k_3 \le z_3;\; r_1 + r_2 - 1 < n - c^*.$$

Result 6.2.2. *The UMVUE of the variance of* $H_{k_1,k_2,k_3}(Z_1, Z_2, Z_3, c^*, n)$, *is given by*

$$\widehat{var}\left[H_{k_1,k_2,k_3}(z_1, z_2, z_3, c^*, n)\right]$$

$$= H^2_{k_1,k_2,k_3}(z_1, z_2, z_3, c^*, n) - H_{2k_1,2k_2,2k_3}(z_1, z_2, z_3, c^*, n)$$

$$= \left[\frac{(r_1)_{k_1}(r_2)_{k_2}\;\left(1 - \frac{k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)}(z_3 - k_3)^{k_1+k_2}}{[n - r_1 - r_2 + 1]_{k_1+k_2}[n - r_1 - r_2 - c^*]_{k_1+k_2}}\right]^2$$

$$- \frac{(r_1)_{2k_1}(r_2)_{2k_2}\;\left(1 - \frac{2k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)}(z_3 - 2k_3)^{2(k_1+k_2)}}{[n - r_1 - r_2 + 1]_{2(k_1+k_2)}[n - r_1 - r_2 - c^*]_{2(k_1+k_2)}}$$

$$2k_1 \le r_1; 2k_2 \le r_2; 2k_3 \le z_3; 2(k_1 + k_2) \le n - r_1 - r_2 - c^*; r_1 + r_2 - 1 < n - c^*.$$

Corollary 6.2.4. The UMVUE of the variance of $H_{k_1}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_1}(z_1, z_2, z_3, c^*, n)\right] = H_{k_1}^2(z_1, z_2, z_3, c^*, n) - H_{2k_1}(z_1, z_2, z_3, c^*, n)$$

$$= \left[\frac{(r_1)_{k_1}\ \ {z_3}^{k_1}}{[n-r_1-r_2+1]_{k_1}[n-r_1-r_2-c^*]_{k_1}}\right]^2 - \frac{(r_1)_{2k_1}\ \ {z_3}^{2k_1}}{[n-r_1-r_2+1]_{2k_1}[n-r_1-r_2-c^*]_{2k_1}},$$

$$2k_1 \le r_1;\ 2k_1 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*.$$

Corollary 6.2.5. The UMVUE of the variance of $H_{k_2}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_2}(z_1, z_2, z_3, c^*, n)\right] = H_{k_2}^2(z_1, z_2, z_3, c^*, n) - H_{2k_2}(z_1, z_2, z_3, c^*, n)$$

$$= \left[\frac{(r_2)_{k_2}\ \ {z_3}^{k_2}}{[n-r_1-r_2+1]_{k_2}\ [n-r_1-r_2-c^*]_{k_2}}\right]^2 - \frac{(r_2)_{2k_2}\ \ {z_3}^{2k_2}}{[n-r_1-r_2+1]_{2k_2}\ [n-r_1-r_2-c^*]_{2k_2}}$$

$$2k_2 \le r_2;\ 2k_2 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*.$$

Corollary6.2.6. The UMVUE of the variance of $H_{k_3}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_3}(z_1, z_2, z_3, c^*, n)\right] = H_{k_3}^2(z_1, z_2, z_3, c^*, n) - H_{2k_3}(z_1, z_2, z_3, c^*, n)$$

$$= \left(1 - \frac{k_3}{z_3}\right)^{2(n-r_1-r_2-c^*-1)} - \left(1 - \frac{2k_3}{z_3}\right)^{n-r_1-r_2-c^*-1},$$

$$2k_3 \le z_3;\ r_1 + r_2 - 1 < n - c^*.$$

Result 6.2.3. *The UMVUE of* $\left[g(\underline{\theta})\right]^k = \left(\frac{\theta}{1-p_1-p_2}\right)^k, k \neq 0$ *as per the model given in (6.4.1) is*

$$G_k(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2, z_3, c^*, n+k)}{B(z_1, z_2, z_3, c^*, n)}$$

$$= \frac{[n+1]_k \quad {z_3}^k}{[n-r_1-r_2+1]_k \quad [n-r_1-r_2-c^*]_k},$$

$$k \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$$

Result 6.2.4. *The UMVUE of the variance of* $G_k(Z_1, Z_2, Z_3, c^*, n)$ *is given by*

$$\widehat{var}[G_k(z_1, z_2, z_3, n)] = G_k^2(z_1, z_2, z_3, c^*, n) - G_{2k}(z_1, z_2, z_3, c^*, n)$$

$$= \left[\frac{[n+1]_k \quad {z_3}^k}{[n-r_1-r_2+1]_k \quad [n-r_1-r_2-c^*]_k}\right]^2 - \frac{[n+1]_{2k} \quad {z_3}^{2k}}{[n-r_1-r_2+1]_{2k} \quad [n-r_1-r_2-c^*]_{2k}},$$

$$2k \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*.$$

Result 6.2.5. *For fixed x, the UMVUE of the density is given by*

$$\phi_x(z_1, z_2, z_3, c^*, n) = a(x)\, \frac{B(z_1 - C_1(x),\ z_2 - C_2(x),\ z_3 - C_3(x), c^*, n-1)}{B(z_1, z_2, z_3, c^*, n)}$$

$$= \left(\beta x^{\beta-1}\right) \frac{(r_1)_{I_1(x)}\ (r_2)_{I_2(x)}\ (n - r_1 - r_2)_{(1-I_1(x)-I_2(x))}\ (n - r_1 - r_2 - c^* - 1)_{(1-I_1(x)-I_2(x))}}{n\ [\ z_3 - (x^\beta - 1)(1 - I_1(x) - I_2(x))]^{(1-I_1(x)-I_2(x))}}$$

$$\left(1 - \frac{(x^\beta - 1)(1 - I_1(x) - I_2(x))}{z_3}\right)^{(n-r_1-r_2-c^*-1)}, z_3 > \left(x^\beta - 1\right);\ r_1 + r_2 - 1 < n - c^*.$$

Result 6 2.6. *The UMVUE of the variance of* $\phi_x(Z_1, Z_2, Z_3, c^*, n)$ *is given by*

$$\widehat{var}[\phi_x(z_1, z_2, z_3, c^*, n)] = \phi_x^2(z_1, z_2, z_3, c^*, n)$$

$$-\phi_x(z_1, z_2, z_3, c^*, n)\ \phi_x(z_1 - C_1(x),\ z_2 - C_2(x),\ z_3 - C_3(x), c^*, n-1)$$

$$= \phi_x^2(z_1, z_2, z_3, c^*, n) - \left(\beta x^{\beta-1}\right)^2$$

$$\left(\frac{(r_1)_{2I_1(x)}\ (r_2)_{2I_2(x)}\ (n - r_1 - r_2)_{2(1-I_1(x)-I_2(x))}\ (n - r_1 - r_2 - c^* - 1)_{2(1-I_1(x)-I_2(x))}}{n(n-1)[\ z_3 - 2(x^\beta - 1)(1 - I_1(x) - I_2(x)]^{2(1-I_1(x)-I_2(x))}}\right)$$

$$\left(1 - \frac{2(x^\beta - 1)(1 - I_1(x) - I_2(x))}{z_3}\right)^{(n-r_1-r_2-c^*-1)}, z_3 > 2\left(x^\beta - 1\right);\ \ r_1 + r_2 - 1 < n - c^*.$$

Result 6.2.7. *For a fixed* $z = (z_1, z_2, z_3, c^*, n)$, *the UMVUE of the survival function* $S(t) = P(X > t),\ t \geq 0$ *is obtained as*

$$\hat{S}(t) = \left(\frac{(r_1)_{I_1(t)}(r_2)_{I_2(t)}(n - r_1 - r_2)_{(1-I_1(t)-I_2(t))}(n - r_1 - r_2 - c^* - 1)_{(1-I_1(t)-I_2(t))}}{n\left[(n - r_1 - r_2 - c^*) - \left(1 - I_1(t) - I_2(t)\right)\right]}\right)$$

$$\left(Z_3 - (t^\beta - 1)\left(1 - I_1(t) - I_2(t)\right)\right)^{\left(I_1(t)+I_2(t)\right)^{(n-r_1-r_2-c^*-1)}}$$

$$\left(1 - \frac{(t^\beta - 1)\left(1 - I_1(t) - I_2(t)\right)}{Z_3}\right), Z_3 > (t^\beta - 1);\ r_1 + r_2 - 1 < n - c^*$$

Result 6.2.8. *For the fixed* $z = (z_1, z_2, z_3, c^*, n)$, *the UMVUE of the* $var\left(\hat{S}(t)\right)$, *is obtained as*

$$\widehat{var}\left(\hat{S}(t)\right) = \left[\hat{S}(t)\right]^2 - \frac{1}{n(n-1)}\left(1 - \frac{2(t^\beta - 1)\left(1 - I_1(t) - I_2(t)\right)}{z_3}\right)^{(n-r1-r2-c^*-1)}$$

$$\left(\frac{(r_1)_{2I_1(t)}(r_2)_{2I_2(t)}(n - r_1 - r_2)_{2\left(1-I_1(t)-I_2(t)\right)}}{\left[(n - r_1 - r_2 - c^*) - 2\left(1 - I_1(t) - I_2(t)\right)\right]}\right)$$

$$\left(\frac{(n - r_1 - r_2 - c - 1^*)_{2\left(1-I_1(t)-I_2(t)\right)}}{\left[(n - r_1 - r_2 - c^* + 1) - 2\left(1 - I_1(t) - I_2(t)\right)\right]}\right)$$

$$\left(z_3 - 2(t^\beta - 1)(1 - I_1(t) - I_2(t)\right)^{2\left(I_1(t)+I_2(t)\right)},$$

$$Z_3 > 2(t^\beta - 1);\ r_1 + r_2 - 1 < n - c^*$$

In all the above cases for β=1, the results reduce to the case of Type II censoring samples of the exponential distribution. Further for β=1 and c^*=0 the results reduce to the case of exponential distribution (see Muralidharan and Pratima, 2017), and hence not repeated again.

6.5. Least Squares and Weighted Least Squares Estimation

In this section, we provide the regression based estimators of the unknown parameters keeping β known and equal to 1. Suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from a distribution function $H(X)$ and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denotes the order statistics of the observed sample. Here,

$$H\left(X_{(i)};\, p_1, p_2, \theta\right) = \begin{cases} 0, & X_{(i)} < 0 \\ p_1, & 0 \le X_{(i)} < 1 \\ p_1 + p_2, & X_{(i)} = 1 \\ 1 - (1 - p_1 - p_2)e^{-\left(\frac{x-1}{\theta}\right)}, & X_{(i)} > 1 \end{cases} \quad (6.5.1)$$

Then using equation (1.5.3), the Least Squares Estimator (LSE) is obtained by minimizing the function

$$\sum_{i=1}^{n}\left[{p_1}^{I_1(x_{(i)})}(p_1 + p_2)^{I_2(x_{(i)})}\left(1 - (1 - p_1 - p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)^{\left(1 - I_1(x_{(i)}) - I_2(x_{(i)})\right)} - \frac{i}{n+1}\right]^2 \quad (6.5.2)$$

with respect to unknown parameters. The least squares estimator of $\underline{\theta} = (p_1, p_2, \theta)$ is $\underline{\hat{\theta}}_{LSE} = (\hat{p}_{1LSE}, \hat{p}_{2LSE}, \hat{\theta}_{LSE})$, say, are obtained by solving the estimating equations (6.5.3), (6.5.4) and (6.5.5) as given below, simultaneously.

$$\sum_{i=1}^{n}\left(H\left(x_{(i)}\right) - \frac{i}{n+1}\right)H\left(x_{(i)}\right)\left\{\frac{I_1\left(x_{(i)}\right)}{p_1} + \frac{\left[1 - I_1\left(x_{(i)}\right) - I_2\left(x_{(i)}\right)\right]e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1 - (1 - p_1 - p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\} = 0 \quad (6.5.3)$$

$$\sum_{i=1}^{n}\left(H\left(x_{(i)}\right)-\frac{i}{n+1}\right)H\left(x_{(i)}\right)\left\{\frac{I_2\left(x_{(i)}\right)}{p_1+p_2}+\frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right]e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\}=0 \tag{6.5.4}$$

and

$$\sum_{i=1}^{n}\left(H\left(x_{(i)}\right)-\frac{i}{n+1}\right)H\left(x_{(i)}\right)\left\{\frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right](1-p_1-p_2)\left(\frac{x_{(i)}-1}{\theta^2}\right)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\}=0 \tag{6.5.5}$$

In a similar argument, using the equation (1.5.4) we obtain the weighted least squares estimator by minimizing

$$\sum_{i=1}^{n}w_i\left[{p_1}^{I_1\left(x_{(i)}\right)}(p_1+p_2)^{I_2\left(x_{(i)}\right)}\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)^{\left(1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right)}-\frac{i}{n+1}\right]^2 \tag{6.5.6}$$

with respect to unknown parameters. Hence, the weighted least squares estimator of $\underline{\theta}=(p_1,p_2,\theta)$ is $\hat{\underline{\theta}}_{WLSE}=(\hat{p}_{1\,WLSE},\hat{p}_{2\,WLSE},\hat{\theta}_{WLSE})$, say, are obtained by solving the estimating equations (6.5.7), (6.5.8) and (6.5.9) are given below:

$$\sum_{i=1}^{n}w_i\left(H\left(x_{(i)}\right)-\frac{i}{n+1}\right)H\left(x_{(i)}\right)\left\{\frac{I_1\left(x_{(i)}\right)}{p_1}+\frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right]e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\}=0 \tag{6.5.7}$$

$$\sum_{i=1}^{n}w_i\left(H\left(x_{(i)}\right)-\frac{i}{n+1}\right)H\left(x_{(i)}\right)\left\{\frac{I_2\left(x_{(i)}\right)}{p_1+p_2}+\frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right]e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\}=0 \tag{6.5.8}$$

and

$$\sum_{i=1}^{n} w_i \left(H\left(x_{(i)}\right) - \frac{i}{n+1}\right) H\left(x_{(i)}\right) \left\{ \frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right](1-p_1-p_2)\left(\frac{x_{(i)}-1}{\theta^2}\right)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)} \right\} = 0$$

(6.5.9)

6.6. Percentile Estimation

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from (6.5.1) and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of the observed sample. If P_i denotes some estimate of $h(x)$ given in (6.2.2) with $\beta = 1$, then the percentile estimate of unknown parameters can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\log P_i - \log H\left(x_{(i)}; p_1, p_2, \theta\right)\right]^2 \qquad (6.6.1)$$

with respect to unknown parameters. Considering $P_i = \frac{i}{(n+1)}$, the percentile estimator of $\underline{\theta} = (p_1, p_2, \theta)$ is $\hat{\underline{\theta}}_{PE} = (\hat{p}_{1PE}, \hat{p}_{2PE}, \hat{\theta}_{PE})$, say, can be obtained by solving estimating equations (6.6.2), (6.6.3) and (6.6.4) simultaneously.

$$\sum_{i=1}^{n}\left(\log\left(\frac{i}{n+1}\right) - \log H\left(x_{(i)}\right)\right)\left\{\frac{I_1\left(x_{(i)}\right)}{p_1} + \frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right] e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\} = 0$$

(6.6.2)

$$\sum_{i=1}^{n}\left(\log\left(\frac{i}{n+1}\right) - \log H\left(x_{(i)}\right)\right)\left\{\frac{I_2\left(x_{(i)}\right)}{p_2} + \frac{\left[1-I_1\left(x_{(i)}\right)-I_2\left(x_{(i)}\right)\right] e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\} = 0$$

(6.6.3)

and

$$\sum_{i=1}^{n}\left(\log\left(\frac{i}{n+1}\right)-\log H\left(x_{(i)}\right)\right)\left\{\frac{[1-I_1(x_{(i)})-I_2(x_{(i)})](1-p_1-p_2)\left(\frac{x_{(i)}-1}{\theta^2}\right)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}}{\left(1-(1-p_1-p_2)e^{-\left(\frac{x_{(i)}-1}{\theta}\right)}\right)}\right\}=0 \tag{6.6.4}$$

Note: All the above equations are nonlinear and hence, we use some nonlinear method of solving equations. Specifically, we used the *optim function* in R.

6.7. Simulation study

Here, we conduct simulation experiments to check the performance of estimators under different Type II censored sampling plans. We present the estimates of the parameters and parametric functions for various choices of inliers (r_1, r_2) based on the simulation of 10000 Type II censored samples. We have taken three sample size n (= 25, 50 and 100) and for each sample size we considered two Type II censoring schemes so that experiment contains 80% (c^* = 20%), and 60% (c^* = 40%), failure information with different choices of r_1 and r_2 . For all models, a value of $\beta = 1.2$ and $\theta = 1.6$ is considered for the Weibull distribution. Table 6.1 presents the ML estimates of parameters when c^* = 20% *and* c^* = 40% along with their standard error (SE) of the estimate (shown in bracket). It is observed that as the number of inliers increases the standard error of the ML estimate of model parameters increase for all values of n. Similarly, as the censoring proportion c^* increases then also the SE increases for every combination of (n, r_1, r_2).

The summary of UMVU estimates of parametric functions of the model for some selected value of parameters of the simulated data for two censoring schemes with c^* = 20% *and* c^* = 40% are shown in Table 6.2. The entry in brackets is the UMVUE of the standard error of the estimates of parametric functions. It is observed that the estimates of the parametric function decrease as n and c^* increases and increases as the number of inliers increases, whereas the UMVUE of the standard error of estimates of parametric functions increase as c^* increase and decrease as n increase for every combination of (n, r_1, r_2).

The summary of UMVU estimates of pdf and survival function of the model of the simulated data for two censoring schemes, c^* = 20% *and* c^* = 40% is shown in Table 6.3. The entry in brackets is the UMVUE of the standard error of estimates. It is observed that as the number of inliers increases the estimate of pdf decreases. The survival function decreases at various time points and decreases as c^* and n increases for every combination of (n, r_1, r_2). Note that the standard error of both estimate of pdf as well as survival function increases as $c^*$is increases and decreases as n increases for every combination of (n, r_1, r_2). Also, note that the magnitude of standard error is negligible everywhere.

It is interesting to note that the parameter (β, θ) is orthogonal to the parameters (p_1, p_2) for the likelihood estimation, whereas, this is not the case with UMVU estimation. Further, the UMVUE of the standard error of θ is less than the standard error of MLE of θ. However, when we deal with degeneracy of distributions, we recommend the use of MLE's for estimating individual parameters and UMVUE for parametric functions.

Table 6.4 presents the MLE, LSE, WLSE, and PE of parameters along with their standard error of the estimate (shown in bracket) correspond to β=1 for complete sample. Note that the estimate of p_1, p_2 and θ is comparable in all cases. It is seen that the standard error is very small for every combination of (n, r_1, r_2). It is also found that, the LSE is close to ML estimators, whereas, the WLSE and PE underestimate the parameters slightly as the number of inliers increases.

Table 6.1. *Summary of estimates of parameters when c^* = 20% and c^* = 40% (entries below)*

N	(r_1, r_2)	$\hat{p}_{1MLE}$	$\hat{p}_{2MLE}$	$\hat{\beta}_{MLE}$	$\hat{\theta}_{MLE}$
25	(0,0)	0 (-) 0 (-)	0 (-) 0 (-)	1.205795 (0.000256) 1.209836 (0.000301)	1.601745 (0.000275) 1.602248 (0.000314)
	(1,1)	0.039572 (0.029695) 0.039916 (0.030035)	0.040288 (0.030063) 0.040036 (0.030081)	1.208307 (0.000263) 1.208498 (0.000319)	1.598631 (0.000271) 1.599690 (0.000326)
	(1,2)	0.040116 (0.030073) 0.040072 (0.030097)	0.079764 (0.048230) 0.080552 (0.048514)	1.208261 (0.000272) 1.208752 (0.000347)	1.602626 (0.000269) 1.598922 (0.000331)
	(2,2)	0.079044 (0.048080) 0.080016 (0.048099)	0.079516 (0.048207) 0.080072 (0.048435)	1.201995 (0.000279) 1.202006 (0.000339)	1.599496 (0.000289) 1.600298 (0.000340)
	(3,1)	0.120080 (0.060953) 0.119762 (0.068440)	0.039872 (0.029873) 0.039942 (0.029998)	1.203478 (0.000287) 1.208569 (0.000350)	1.601933 (0.000292) 1.605283 (0.000329)
	(4,0)	0.159160 (0.070014) 0.159940 (0.070154)	0 (-) 0 (-)	1.209912 (0.000282) 1.205256 (0.000336)	1.599581 (0.000294) 1.601992 (0.000343)
50	(0,0)	0 (-) 0 (-)	0 (-) 0 (-)	1.201370 (0.000169) 1.208842 (0.000236)	1.601533 (0.000165) 1.604078 (0.000225)
	(1,2)	0.019811 (0.015109) 0.019824 (0.015118)	0.039474 (0.024540) 0.039893 (0.024758)	1.202494 (0.000175) 1.208250 (0.000222)	1.601720 (0.000165) 1.600727 (0.000211)
	(3,3)	0.059704 (0.031453) 0.060490 (0.031701)	0.060186 (0.031626) 0.060874 (0.031835)	1.204491 (0.000177) 1.205496 (0.000232)	1.610106 (0.000181) 1.594701 (0.001836)
	(4,4)	0.079732 (0.036763) 0.079600 (0.036772)	0.079516 (0.036637) 0.080466 (0.036860)	1.202917 (0.000185) 1.206270 (0.002246)	1.606965 (0.000184) 1.604335 (0.000226)
	(6,2)	0.119500 (0.044706) 0.120840 (0.044899)	0.040032 (0.024804) 0.040020 (0.024810)	1.201680 (0.000188) 1.201323 (0.000237)	1.606397 (0.000181) 1.595685 (0.000231)
	(8,0)	0.159716 (0.050799) 0.160372 (0.050877)	0 (-) 0 (-)	1.205469 (0.000183) 1.206884 (0.000244)	1.604492 (0.000182) 1.601742 (0.000230)
100	(0,0)	0 (-) 0 (-)	0 (-) 0 (-)	1.203593 (0.000106) 1.202053 (0.000131)	1.605647 (0.000109) 1.599184 (0.000153)
	(2,2)	0.019939 (0.012551) 0.020673 (0.060844)	0.019988 (0.012518) 0.019991 (0.012555)	1.202847 (0.000108) 1.204905 (0.000144)	1.606117 (0.000111) 1.601124 (0.000174)
	(2,4)	0.020121 (0.012624) 0.019962 (0.012840)	0.040106 (0.018843) 0.040238 (0.018847)	1.203082 (0.000109) 1.201515 (0.000150)	1.609934 (0.000110) 1.599123 (0.000202)
	(4,4)	0.039961 (0.018804) 0.040011 (0.018811)	0.040083 (0.018809) 0.399999 (0.018810)	1.205306 (0.000116) 1.209189 (0.000153)	1.600626 (0.000113) 1.601221 (0.000215)
	(6,4)	0.060006 (0.023154) 0.059923 (0.023158)	0.040157 (0.018835) 0.040262 (0.018863)	1.203875 (0.000116) 1.210815 (0.000158)	1.602492 (0.000112) 1.598820 (0.000217)
	(8,8)	0.080197 (0.026607) 0.079832 (0.026608)	0.079877 (0.026613) 0.080071 (0.026649)	1.200144 (0.000116) 1.203002 (0.000142)	1.605126 (0.000118) 1.598969 (0.000205)
	(0,16)	0 (-) 0 (-)	0.160393 (0.036347) 0.160210 (0.036494)	1.200512 (0.000120) 1.204799 (0.000137)	1.603861 (0.000122) 1.600120 (0.000170)

Table 6.2. *Summary of estimates of parametric functions when c^* = 20% and c^* = 40% (entries below)*

n	(r_1, r_2)	UMVUE of parametric function (SE)				
		$\prod_{i=1}^{3}(h_i(\underline{\theta}))^{k_i} = \left(\frac{\theta}{1-p_1-p_2}\right)^2 p_1p_2e$ $k_1 = 1, k_2 = 1, k_3 = 1$	$h_1(\underline{\theta}) = \frac{\theta\, p_1}{1-p_1-p_2}$, $k_1 = 1, k_2 = 0, k_3 = 0$	$h_2(\underline{\theta}) = \frac{\theta\, p_2}{1-p_1-p_2}$, $k_1 = 0, k_2 = 1, k_3 = 0$	$h_3(\underline{\theta}) = e^{-\frac{1}{\theta}}$, $k_1 = 0, k_2 = 0, k_3 = 1$	$g(\underline{\theta}) = \frac{\theta}{1-p_1-p_2}$, $k = 1$
25	(0,0)	- -	- -	- -	0.535219 (0.076313) 0.535193 (0.088703)	1.599834 (0.348958) 1.599586 (0.399896)
	(1,1)	0.006407 (0.014850) 0.006393 (0.015040)	0.108566 (0.115736) 0.108486 (0.117269)	0.108697 (0.115614) 0.108516 (0.117330)	0.534882 (0.080836) 0.534725 (0.096153)	1.737678 (0.411728) 1.737030 (0.476936)
	(1,2)	0.010170 (0.018571) 0.010151 (0.019034)	0.113721 (0.121481) 0.113698 (0.123503)	0.166262 (0.083411) 0.166021 (0.100705)	0.535132 (0.083411) 0.534848 (0.100705)	1.817839 (0.449398) 1.816337 (0.524389)
	(2,2)	0.016313 (0.023849) 0.016299 (0.024568)	0.174131 (0.141266) 0.174108 (0.144901)	0.174268 (0.141178) 0.174248 (0.144800)	0.535392 (0.086237) 0.535368 (0.105973)	1.905500 (0.491903) 1.905477 (1.580060)
	(3,1)	0.015201 (0.024004) 0.015179 (0.024794)	0.238650 (0.158660) 0.238609 (0.163749)	0.118803 (0.127424) 0.118697 (0.129761)	0.535256 (0.086255) 0.535104 (0.106009)	1.904615 (0.491636) 1.904271 (0.579582)
	(4,0)	0.166094 (0.107249) 0.164770 (0.115510)	0.310019 (0.179581) 0.308102 (0.087282)	- -	0.535745 (0.086243) 0.534980 (0.105935)	1.908010 (0.491104) 1.903112 (0.580933)
50	(0,0)	- -	- -	- -	0.535384 (0.053423) 0.535358 (0.061888)	1.600606 (0.249946) 1.600436 (0.287477)
	(1,2)	0.002241 (0.003734) 0.002238 (0.003790)	0.053411 (0.054404) 0.053387 (0.054774)	0.078159 (0.059851) 0.078110 (0.060393)	0.535325 (0.055625) 0.535194 (0.065431)	1.702129 (0.282676) 1.701490 (0.327506)
	(3,3)	0.006998 (0.006990) 0.006982 (0.007172)	0.114647 (0.071682) 0.114538 (0.072728)	0.114365 (0.071519) 0.114284 (0.072569)	0.535409 (0.058157) 0.535226 (0.069718)	1.819091 (0.321971) 1.817863 (0.376902)
	(4,4)	0.012780 (0.010773) 0.012770 (0.011189)	0.154527 (0.084075) 0.154500 (0.085919)	0.154541 (0.084082) 0.154515 (0.085927)	0.535168 (0.060042) 0.535076 (0.073057)	1.903940 (0.351836) 1.903597 (0.415803)
	(6,2)	0.010702 (0.010581) 0.010695 (0.010961)	0.228726 (0.104270) 0.228714 (0.107446)	0.087413 (0.067344) 0.087405 (0.068280)	0.535443 (0.060022) 0.535368 (0.073033)	1.905024 (0.351994) 1.904734 (0.416001)
	(8,0)	0.163138 (0.075244) 0.133125 (0.080821)	0.304788 (0.125131) 0.304767 (0.129894)	- -	0.535226 (0.060044) 0.535202 (0.073063)	1.904465 (0.351984) 1.904336 (0.416053)
100	(0,0)	- -	- -	- -	0.535100 (0.037598) 0.535075 (0.043486)	1.599195 (0.177688) 1.599182 (0.271242)
	(2,2)	0.000788 (0.000994) 0.000788 (0.001004)	0.038403 (0.028632) 0.038400 (0.028765)	0.038304 (0.028639) 0.038313 (0.028785)	0.535416 (0.038576) 0.535479 (0.055033)	1.667502 (0.367596) 1.667543 (0.223531)
	(2,4)	0.001453 (0.001454) 0.001453 (0.001476)	0.039181 (0.029287) 0.039187 (0.029440)	0.069210 (0.035966) 0.069207 (0.036268)	0.535266 (0.039115) 0.535252 (0.045898)	1.702163 (0.201199) 1.702070 (0.233629)
	(4,4)	0.002686 (0.002091) 0.002684 (0.002133)	0.070816 (0.036811) 0.070795 (0.037133)	0.070804 (0.036817) 0.070766 (0.037135)	0.535460 (0.039662) 0.535276 (0.046799)	1.739915 (0.210024) 1.739379 (0.244574)
	(6,4)	0.004123 (0.002929) 0.004121 (0.003004)	0.106764 (0.045706) 0.106710 (0.046319)	0.072206 (0.037627) 0.072197 (0.037982)	0.535377 (0.040234) 0.535229 (0.047754)	1.778049 (0.219171) 1.777290 (0.256070)
	(8,8)	0.012404 (0.007286) 0.012373 (0.007614)	0.152188 (0.058008) 0.152080 (0.059251)	0.152277 (0.058464) 0.152174 (0.059293)	0.535267 (0.042133) 0.535022 (0.051043)	1.904089 (0.250417) 1.903051 (0.296232)
	(0,16)	0.163082 (0.054285) 0.163042 (0.058402)	- -	0.304724 (0.089887) 0.304679 (0.093316)	0.535140 (0.042140) 0.535078 (0.051047)	1.904201 (0.250479) 1.903844 (0.296438)

Table 6.3. *Summary of estimates of pdf for c^* = 20% and c^* = 40% (entries below)*

n	(r_1, r_2)	UMVUE of pdf (SE) $x=2$	UMVUE of pdf (SE) $x=3$	UMVUE of pdf (SE) $x=4$	UMVUE of survival function (SE) $x=2$	UMVUE of survival function (SE) $x=3$	UMVUE of survival function (SE) $x=4$
25	(0,0)	0.382928 (0.022636) 0.382978 (0.028767)	0.168836 (0.028119) 0.168836 (0.034012)	0.068261 (0.025519) 0.068235 (0.029273)	0.444424 (0.081892) 0.444386 (0.095037)	0.180696 (0.067992) 0.180633 (0.077883)	0.068975 (0.038279) 0.068930 (0.042920)
	(1,1)	0.352313 (0.026877) 0.352322 (0.031704)	0.155186 (0.028477) 0.155128 (0.033990)	0.062692 (0.024952) 0.062647 (0.029579)	0.408521 (0.083177) 0.408369 (0.097375)	0.165983 (0.066334) 0.165868 (0.077004)	0.063332 (0.036819) 0.063271 (0.041894)
	(1,2)	0.336894 (0.029967) 0.336923 (0.033600)	0.148527 (0.028849) 0.148449 (0.036412)	0.060038 (0.024676) 0.060024 (0.050570)	0.390923 (0.083594) 0.390663 (0.098539)	0.158908 (0.065538) 0.158700 (0.077356)	0.060633 (0.036130) 0.058392 (0.050717)
	(2,2)	0.321594 (0.032702) 0.321613 (0.035654)	0.141856 (0.029212) 0.141795 (0.035577)	0.057375 (0.024378) 0.055839 (0.038458)	0.373397 (0.083880) 0.373382 (0.099778)	0.151929 (0.064678) 0.151852 (0.076931)	0.058795 (0.035413) 0.057969 (0.045324)
	(3,1)	0.321664 (0.032662) 0.321702 (0.035619)	0.141828 (0.029252) 0.141818 (0.035351)	0.057319 (0.024461) 0.057282 (0.035910)	0.376268 (0.083892) 0.373188 (0.099772)	0.151722 (0.064661) 0.151704 (0.076788)	0.057886 (0.035387) 0.058079 (0.044537)
	(4,0)	0.321462 (0.032694) 0.321654 (0.035743)	0.142062 (0.029228) 0.141762 (0.035323)	0.058023 (0.024352) 0.057305 (0.036214)	0.373607(0.083861) 0.371316 (0.099776)	0.152066 (0.064612) 0.151681 (0.076828)	0.057918 (0.035354) 0.057884 (0.044381)
50	(0,0)	0.382888 (0.013239) 0.382904 (0.016388)	0.168900 (0.019118) 0.168893 (0.022228)	0.068323 (0.018119) 0.068309 (0.020911)	0.444605 (0.057454) 0.444573 (0.066514)	0.180854 (0.048558) 0.180817 (0.055911)	0.069066 (0.028185) 0.069041 (0.032142)
	(1,2)	0.360030 (0.016952) 0.360057 (0.018567)	0.158782 (0.019644) 0.158735 (0.022806)	0.064219 (0.017851) 0.064177 (0.020835)	0.417997 (0.058150) 0.417869 (0.067690)	0.170013 (0.047789) 0.169906 (0.055618)	0.064931 (0.027542) 0.064871 (0.031700)
	(3,3)	0.336914 (0.020816) 0.336975 (0.022121)	0.148628 (0.020066) 0.148584 (0.032457)	0.060118 (0.017575) 0.060060 (0.020811)	0.391225 (0.058622) 0.391054 (0.068827)	0.159134 (0.046981) 0.158964 (0.055436)	0.060766 (0.026880) 0.060651 (0.031303)
	(4,4)	0.321709 (0.022756) 0.321720 (0.022756)	0.141832 (0.020298) 0.141796 (0.023881)	0.057333 (0.017372) 0.057313 (0.020789)	0.373331 (0.058807) 0.373251 (0.069574)	0.151770 (0.046381) 0.151721 (0.055337)	0.057924 (0.026387) 0.057904 (0.031039)
	(6,2)	0.321727 (0.022764) 0.321747 (0.024122)	0.141940 (0.020280) 0.141911 (0.023868)	0.057414 (0.017381) 0.057386 (0.020805)	0.373609 (0.058810) 0.373531 (0.069579)	0.151970 (0.046417) 0.151903 (0.055383)	0.058026 (0.026425) 0.057990 (0.031080)
	(8,0)	0.321662 (0.022766) 0.321690 (0.024133)	0.141834 (0.020294) 0.141818 (0.023876)	0.057336 (0.017370) 0.057335 (0.020802)	0.373307 (0.058807) 0.373303 (0.069584)	0.151776 (0.046383) 0.151754 (0.055370)	0.057935 (0.026392) 0.057906 (0.031058)
100	(0,0)	0.382962 (0.008299) 0.382949 (0.009991)	0.168801 (0.013483) 0.168785 (0.015583)	0.068220 (0.012803) 0.068215 (0.014786)	0.444295 (0.040467) 0.444272 (0.046789)	0.180576 (0.034447) 0.180571 (0.039724)	0.068896 (0.020265) 0.068902 (0.023261)
	(2,2)	0.367596 (0.010807) 0.367600 (0.011750)	0.162189 (0.013693) 0.162166 (0.015848)	0.065621 (0.012682) 0.065620 (0.014761)	0.426950 (0.040801) 0.426922 (0.034128)	0.173707 (0.034129) 0.173703 (0.039653)	0.066351 (0.020022) 0.066349 (0.023160)
	(2.4)	0.359950 (0.011965) 0.359995 (0.012844)	0.158731 (0.013810) 0.158724 (0.016006)	0.064186 (0.012618) 0.064180 (0.014746)	0.417828 (0.040949) 0.417807 (0.047585)	0.169908 (0.033934) 0.169891 (0.039586)	0.064867 (0.019858) 0.064856 (0.023062)
	(4,4)	0.352256 (0.012988) 0.352260 (0.013847)	0.155394 (0.013897) 0.155344 (0.016137)	0.062859 (0.012555) 0.062824 (0.014734)	0.409052 (0.040801) 0.408931 (0.047836)	0.166394 (0.034129) 0.166308 (0.039546)	0.063546 (0.020022) 0.063504 (0.023000)
	(6,4)	0.344663 (0.013870) 0.344699 (0.014712)	0.152026 (0.013984) 0.151985 (0.016276)	0.061490 (0.012489) 0.061446 (0.014732)	0.400187 (0.041179) 0.400053 (0.048088)	0.162773 (0.033574) 0.162650 (0.039511)	0.062158 (0.019577) 0.062081 (0.022929)
	(8,8)	0.321682 (0.016013) 0.321712 (0.016869)	0.141870 (0.014192) 0.141841 (0.014734)	0.057378 (0.012283) 0.057344 (0.014734)	0.373464 (0.041394) 0.373360 (0.048833)	0.151893 (0.032989) 0.151797 (0.039469)	0.058003 (0.019123) 0.057941 (0.022768)
	(0,16)	0.321659 (0.016012) 0.321674 (0.016870)	0.141795 (0.014201) 0.141779 (0.016668)	0.057317 (0.012282) 0.057299 (0.014733)	0.373235 (0.041395) 0.373178 (0.048836)	0.151722 (0.032974) 0.151671 (0.039457)	0.057903 (0.019104) 0.057870 (0.022752)

Table 6.4. *Summary of estimates of parameters for model*

(r_1, r_2)	Parameter	MLE	LSE	WLSE	PE
(05,10)	$\hat{p}_1$	0.0506 (0.0010)	0.0419 (0.0222)	0.0287 (0.0004)	0.0269 (0.0034)
	$\hat{p}_2$	0.0989 (0.0012)	0.1075 (0.0198)	0.1064 (0.0005)	0.1053 (0.0053)
	$\hat{\theta}$	1.5991 (0.0075)	1.6164 (0.0736)	1.5761 (0.0026)	1.5345 (0.0996)
(05,20)	$\hat{p}_1$	0.0505 (0.0073)	0.0722 (0.0204)	0.0392 (0.0004)	0.0281 (0.0011)
	$\hat{p}_2$	0.1984 (0.0013)	0.1622 (0.0149)	0.1505 (0.0005)	0.1506 (0.0023)
	$\hat{\theta}$	1.5950 (0.0061)	1.5776 (0.0841)	1.4661 (0.0027)	1.3035 (0.0322)
(10,05)	$\hat{p}_1$	0.0906 (0.0027)	0.0475 (0.0198)	0.0337 (0.0004)	0.0435 (0.0010)
	$\hat{p}_2$	0.0494 (0.0019)	0.1091 (0.0222)	0.1185 (0.0007)	0.1180 (0.0031)
	$\hat{\theta}$	1.5966 (0.0154)	1.6352 (0.0749)	1.6145 (0.0028)	1.6695 (0.0360)
(10,15)	$\hat{p}_1$	0.1009 (0.0297)	0.0640 (0.0176)	0.0403 (0.0004)	0.0457 (0.0011)
	$\hat{p}_2$	0.1503 (0.0354)	0.1825 (0.0159)	0.1837 (0.0006)	0.1790 (0.0031)
	$\hat{\theta}$	1.6128 (0.1866)	1.6255 (0.0860)	1.5601 (0.0030)	1.5135 (0.0407)
(10,40)	$\hat{p}_1$	0.1009 (0.0296)	0.1122 (0.0183)	0.0512 (0.0004)	0.0460 (0.0011)
	$\hat{p}_2$	0.4020 (0.0488)	0.3166 (0.0107)	0.2936 (0.0004)	0.2872 (0.0032)
	$\hat{\theta}$	1.5960 (0.2273)	1.3239 (0.1141)	1.5678 (0.0032)	1.4941 (0.0476)
(20,05)	$\hat{p}_1$	0.1983 (0.0098)	0.0923 (0.0149)	0.0563 (0.0003)	0.0807 (0.0013)
	$\hat{p}_2$	0.0500 (0.0052)	0.1751 (0.0202)	0.2013 (0.0008)	0.1983 (0.0049)
	$\hat{\theta}$	1.5972 (0.0449)	1.6804 (0.0932)	1.6353 (0.0034)	1.7680 (0.0548)
(25,25)	$\hat{p}_1$	0.2491 (0.0429)	0.1280 (0.0128)	0.0733 (0.0003)	0.1000 (0.0015)
	$\hat{p}_2$	0.2493 (0.0431)	0.3733 (0.0127)	0.3825 (0.0006)	0.3709 (0.0050)
	$\hat{\theta}$	1.5975 (0.2274)	1.6590 (0.1531)	1.4782 (0.0044)	1.4913 (0.0906)
(30,40)	$\hat{p}_1$	0.2998 (0.4553)	0.1590 (0.0123)	0.0869 (0.0003)	0.1194 (0.0016)
	$\hat{p}_2$	0.3995 (0.0487)	0.5028 (0.0108)	0.5127 (0.0005)	0.4893 (0.0055)
	$\hat{\theta}$	1.6047 (0.2951)	1.4393 (0.2616)	1.1899 (0.0057)	1.0934 (0.1351)
(35,35)	$\hat{p}_1$	0.3511 (0.0475)	0.1784 (0.0113)	0.0976 (0.0003)	0.1384 (0.0017)
	$\hat{p}_2$	0.3507 (0.0473)	0.5230 (0.0113)	0.5374 (0.0005)	0.5170 (0.0061)
	$\hat{\theta}$	1.6040 (0.2913)	1.7524 (0.3761)	1.3280 (0.0066)	1.3768 (0.1942)
(40,10)	$\hat{p}_1$	0.3988 (0.0487)	0.1873 (0.0106)	0.1041 (0.0003)	0.1550 (0.0017)
	$\hat{p}_2$	0.0997 (0.0295)	0.3807 (0.0169)	0.4195 (0.0008)	0.4150 (0.0079)
	$\hat{\theta}$	1.6001 (0.2271)	1.8927 (0.2367)	1.7313 (0.0058)	1.8976 (0.1814)
(40,30)	$\hat{p}_1$	0.4016 (0.0488)	0.1983 (0.0106)	0.1085 (0.0031)	0.1567 (0.0018)
	$\hat{p}_2$	0.2991 (0.0455)	0.5418 (0.1198)	0.5602 (0.0006)	0.5430 (0.0069)
	$\hat{\theta}$	1.6082 (0.2966)	1.8429 (0.9842)	1.4981 (0.0079)	1.7578 (0.3328)

6.8. Numerical study

In this section, we discuss two examples from Appendix. The first example is based on the dataset A.7 of tumor size of 509 samples from 509 female breast cancer patients with histologically confirmed invasive ductal breast carcinoma (IDC) of cohort 1. In this data set, lesions are absent for six patients, whereas for 22 patients lesions present but non-measurable by CT scan or MRI. The second example is based on dataset A.6

of child's age at death from the woman's questionnaire of NFHS-3 for Gujarat state. There are 15 stillbirths (the death of a baby before or during the birth after 28 weeks of gestation) considered as observation 0, 37 neonatal deaths (the death of a baby within the first 28 days of life) considered as observation 1.

We fitted the number of distributions to both datasets by using various information criteria. Table 6.5 and Table 6.6 shows the fitted values and estimates for respective datasets. The superscripts that indicate the rank obtained by the distribution according to the selection criteria (the smaller the better). For both datasets the first two competing model includes Gamma and Weibull distribution across the criteria's.

Table 6.5. *Parameter estimates and goodness-of-fit criteria for distributions fitted to the breast tumor size data*

Distribution	MLE (SE)	AIC	BIC	K-S	CVM	AD
Exponential	$\hat{\theta}$=0.0349 (0.0016)	4191.4030^{5}	4195.5790^{5}	0.3236^{5}	13.9263^{5}	70.7563^{4}
Gamma	$\hat{\beta}$=4.3043 (0.2674) $\hat{\theta}$=0.1503 (0.0099)	3815.4970^{1}	3823.8490^{1}	0.1301^{1}	1.6984^{1}	9.5293^{1}
Normal	$\hat{\mu}$=28.6424 (0.8076) $\hat{\theta}$=17.7565 (0.5725)	4136.4550^{4}	4144.8070^{4}	0.2013^{4}	5.6140^{4}	Inf^{6}
Pareto	$\hat{\theta}$=0.3052 (0.0139)	5257.6140^{6}	5261.7900^{6}	0.5316^{6}	36.3271^{6}	1685236^{5}
Rayleigh	$\hat{\theta}$=23.8300 (0.5433)	3953.5270^{3}	3957.7030^{2}	0.1846^{3}	4.0913^{3}	21.8297^{3}
Weibull	$\hat{\beta}$=1.7859 (0.0524) $\hat{\theta}$=32.4110 (0.8802)	3939.0980^{2}	3947.4500^{3}	0.1503^{2}	3.3870^{2}	20.6991^{2}

Table 6.6. *Parameter estimates and goodness-of-fit criteria for distributions fitted to the NFHS-3 data*

Distribution	MLE (SE)	AIC	BIC	K-S	CVM	AD
Exponential	$\hat{\theta}$=0.0086 (0.0009)	1003.4520^{4}	1005.9180^{4}	0.5891^{6}	7.2735^{6}	129.9538^{6}
Gamma	$\hat{\beta}$=0.1628 (0.0181) $\hat{\theta}$=0.0014 (0.0003)	534.6685^{1}	539.6003^{1}	0.2284^{1}	0.6593^{1}	3.5283^{1}
Gumbel	$\hat{\mu}$=37.1592 (11.3940) $\hat{\theta}$=103.7490 (10.3819)	1118.3080^{5}	1123.2390^{5}	0.3553^{5}	2.5353^{4}	13.8217^{4}
Normal	$\hat{\mu}$ =116.2184 (24.9533) $\hat{\theta}$= 232.7506 (17.6449)	1199.1900^{6}	1204.1210^{6}	0.3088^{3}	2.7108^{5}	13.6803^{3}
Pareto	$\hat{\theta}$=0.4212 (0.0452)	739.5819^{3}	742.0479^{3}	0.3445^{4}	1.6969^{3}	31.6350^{5}
Rayleigh	$\hat{\theta}$=183.9041(9.8541)	1974.9590^{7}	1977.4250^{7}	0.6459^{7}	12.1311^{7}	395.7277^{7}
Weibull	$\hat{\beta}$ =0.2468 (0.0221) $\hat{\theta}$=14.1284 (6.4501)	547.0436^{2}	551.9754^{2}	0.2332^{2}	0.7801^{2}	4.9760^{2}

The histogram of the datasets and plots of the fitted densities are displayed in Figure 6.3 and Figure 6.4 respectively. These plots indicate that the Weibull distribution provides satisfactory a good fit to both datasets.

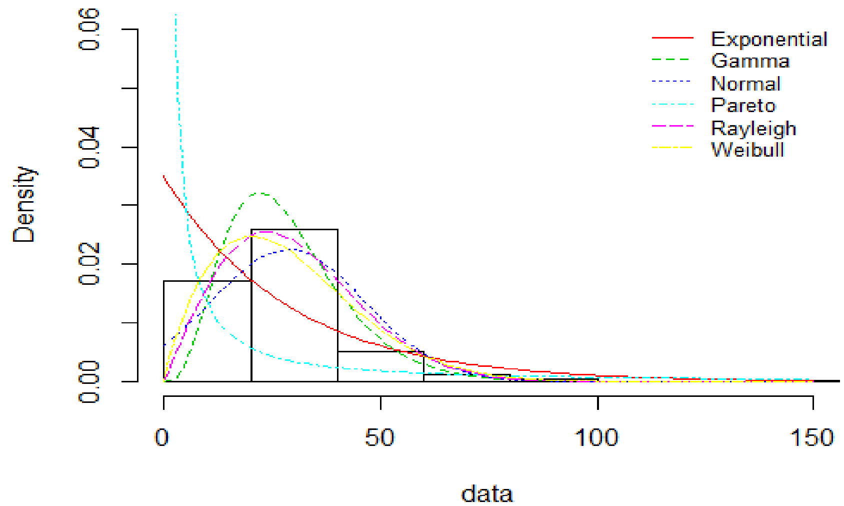

Fig. 6.3. *Histogram and theoretical densities for breast tumor size data*

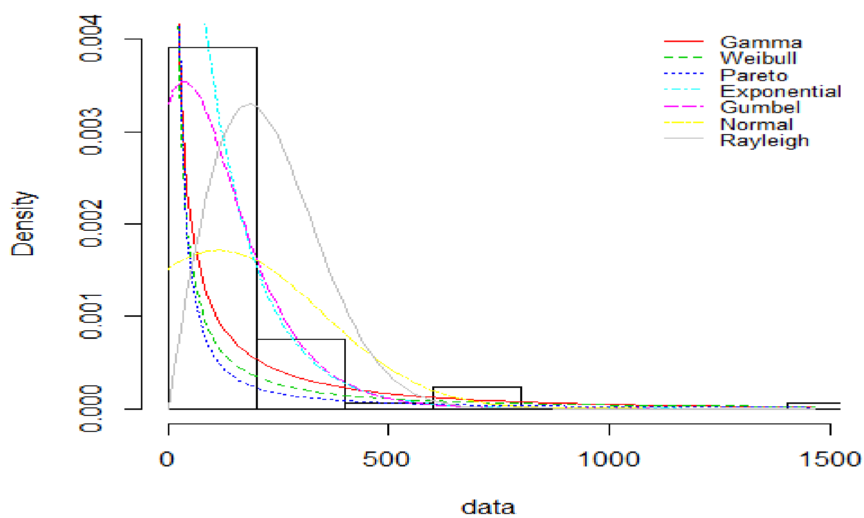

Fig. 6.4. *Histogram and theoretical densities for NFHS-3 data*

The Figure 6.5 and Figure 6.6 shows the plot the pdf $h(x)$ and survival function $S(x)$ of Type II censored Weibull inlier model respectively of censored breast tumor size data with c^*= 6 and censored NFHS-3 data with c^*= 1.

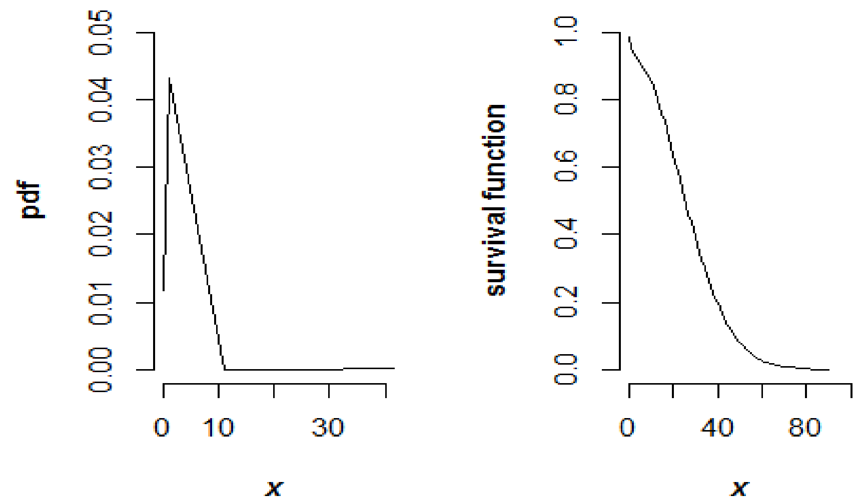

Figure 6.5. *Density function and survival function of Breast tumor size data*

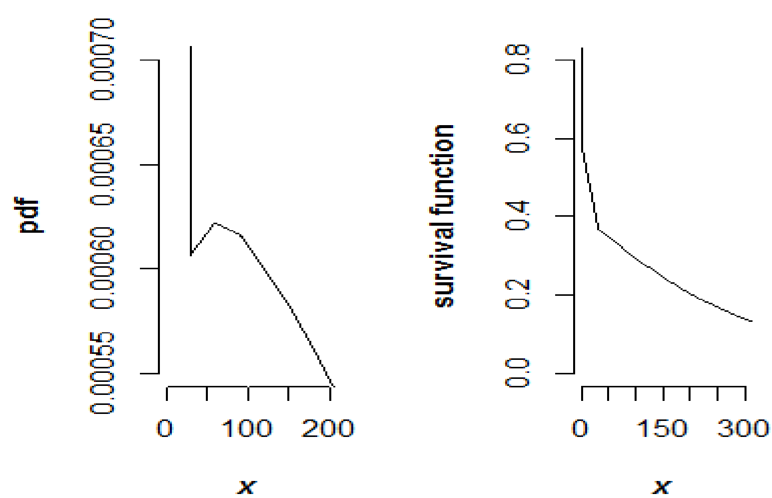

Figure 6.6. *Density function and survival function of NFHS-3 data*

For Type II censored Weibull inlier model, the summary of the ML estimates of parameters along with their standard error (shown in bracket) and 95 % confidence interval considering various censoring schemes on breast tumor size data NFHS-3 data is given in Table 6.7. and the summary of the UMVU estimates of parameters/parametric functions estimate for particular values of parameter k, using two censoring schemes on breast tumor size data and NFHS-3 data is given in Table 6.8.

Table 6.7. *Summary of MLE estimates of parameters of breast tumor size data and NFHS-3 data*

	Breast tumor size data			NFHS-3 data		
	c^*	MLE (SE)	95% CI	c^*	MLE (SE)	95% CI
p_1	6	0.0118 (0.0048)	(0.0024, 0.0212)	1	0.1724 (0.0405)	(0.0930, 0.2518)
p_2		0.0432 (0.0090)	(0.0256, 0.0609)		0.4253 (0.0530)	(0.3214, 0.5292)
β		2.0042 (0.0124)	(1.9804, 2.0280)		1.1071 (0.0282)	(1.0518, 1.1623)
θ		1034.8440 (47.4654)	(941.8137, 1127.8747)		519.3012 (87.2868)	(346.2623, 629.3402)
p_1	10	0.0118 (0.0048)	(0.0024, 0.0212)	5	0.1724 (0.0405)	(0.0930, 0.2518)
p_2		0.0432 (0.0090)	(0.0256, 0.0609)		0.4253 (0.0530)	(0.3214, 0.5292)
β		2.1462 (0.0124)	(2.1219, 2.1704)		1.1779 (0.0313)	(1.1166, 1.2393)
θ		1653.1010 (76.1264)	(1503.8960, 1802.3060)		729.5832 (123.0263)	(480.6163, 978.5501)

Table 6.8. *Summary of UMVU estimates of parameters/parametric functions of breast tumor size data and NFHS-3 data*

Parameter/Parametric function	UMVUE (SE)			
	Breast tumor size data		NFHS-3 data	
p_1	$c^*= 6$ and $\beta = 2.0042$	0.0119 (0.0048)	$c^*= 1$ and $\beta = 1.1071$	0.1744 (0.0409)
p_2		0.0437 (0.0091)		0.4302 (0.0534)
θ		1034.8440 (46.0699)		519.3012 (44.6062)
$\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{k_i} = \left(\frac{\theta}{1-p_1-p_2}\right)^2 p_1 p_2 e^{-\frac{1}{\theta}}$ $k_1 = 1, k_2 = 1, k_3 = 1$		605.3350 (282.4393)		108937.5000 (56705.5000)
$h_1(\underline{\theta}) = \frac{\theta\, p_1}{1-p_1-p_2}, k_1 = 1, k_2 = 0, k_3 = 0$		12.8819 (5.3135)		216.3755 (74.2763)
$h_2(\underline{\theta}) = \frac{\theta\, p_2}{1-p_1-p_2}, k_1 = 0, k_2 = 1, k_3 = 0$		47.2336 (10.5016)		533.7263 (151.3107)
$h_3(\underline{\theta}) = e^{-\frac{1}{\theta}}, k_1 = 0, k_2 = 0, k_3 = 1$		0.9990 (0.0002)		0.9981 (0.0003)
$g(\underline{\theta}) = \frac{\theta}{1-p_1-p_2}, k = 1$		1094.9600 (51.5321)		1269.4030 (266.5312)
pdf at time x, ϕ_x pdf at time x, ϕ_x		ϕ_{20}=0.0251 (0.0008) ϕ_{40}=0.0155 (0.0004)		ϕ_{100}=0.0010 (1.81e-04) ϕ_{500}=0.0003 (5.26e-05)
Survival function at time $t, \hat{S}_t$ Survival function at time t, $\hat{S}_t$		$\hat{S}_{20}$=0.6399 (0.0133) $\hat{S}_{40}$=0.1968 (0.1808)		$\hat{S}_{100}$=0.2964 (0.0420) $\hat{S}_{500}$=0.0621 (0.0217)
p_1	$c^*= 10$ and $\beta = 2.1462/$	0.0120 (0.0143)	$c^*= 5$ and $\beta = 1.1779$	0.1829 (0.0427)
p_2		0.0441 (0.0092)		0.4512 (0.0550)
θ		1653.1010 (73.8864)		729.5832 (62.2970)
$\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{k_i} = \left(\frac{\theta}{1-p_1-p_2}\right)^2 p_1 p_2 e^{-\frac{1}{\theta}}$ $k_1 = 1, k_2 = 1, k_3 = 1$		1545.2330 (721.0685)		214329.9000(113680.5000)
$h_1(\underline{\theta}) = \frac{\theta\, p_1}{1-p_1-p_2}, k_1 = 1, k_2 = 0, k_3 = 0$		20.5780 (8.4884)		303.9930 (105.8252)
$h_2(\underline{\theta}) = \frac{\theta\, p_2}{1-p_1-p_2}, k_1 = 0, k_2 = 1, k_3 = 0$		75.4527 (16.7785)		749.8494 (217.1478)
$h_3(\underline{\theta}) = e^{-\frac{1}{\theta}}, k_1 = 0, k_2 = 0, k_3 = 1$		0.9994 (2.78e-05)		0.9987 (0.0002)
$g(\underline{\theta}) = \frac{\theta}{1-p_1-p_2}, k = 1$		1749.1310 (85.6294)		1783.4260 (389.5607)
pdf at time x, ϕ_x Pdf at time x, ϕ_x		ϕ_{20}=0.0261 (0.0008) ϕ_{40}=0.0160 (0.0005)		ϕ_{100}=0.0011 (1.97e-04) ϕ_{500}=0.0003 (6.10e-05)
survival function at time t, $\hat{S}_t$ survival function at time t, $\hat{S}_t$		$\hat{S}_{20}$=0.6503 (0.0132) $\hat{S}_{40}$=0.1800 (0.0139)		$\hat{S}_{100}$=0.297768 (0.0425) $\hat{S}_{500}$=0.050627 (0.0204)

From the row data one can observe that the survival probability for patients having tumor sizes in mm 20 and 40 are respectively 0.59 (=301/509) and 0.13 (=65/509) which are almost identical with the table value estimates. Also, the survival probability for days 100 and 500 are respectively 0.24 (=21/87) and 0.057 (=5/87) which are almost identical with the table value estimates.

For exponential inlier model, the summary of the various estimates of parameters along with their standard error of the estimate is shown in bracket and 95 % confidence interval for *breast tumor size data* NFHS-3 data set is given in Table 6.9.

Table 6.9. *Summary of estimates of parameters of breast tumor size data and NFHS-3 data*

Data	Parameter		Estimators			
			MLE	LSE	WLSE	PE
Breast tumor size	p_1		0.0118 (0.0048)	0.0436 (0.0090)	0.0538 (3.6e-05)	0.0053 (3.7e-05)
	p_2		0.0432 (0.0090)	0.0863 (0.0062)	0.0654 (4.5e-05)	0.0274 (4.3e-05)
	θ		27.6424 (1.2604)	20.2256 (0.1224)	27.3326 (0.0021)	45.5702 (0.0019)
	95% CI	p_1	(0.0024, 0.0212)	(0.0260, 0.0612)	(0.0537, 0,0539)	(0.0052, 0.0054)
		p_2	0.0256, 0.06089)	(0.0741,0.0984)	(0.0653, 0.0655)	(0.0273, 0.0275)
		θ	(25.1721, 30.1127)	(19.9857, 20.4655)	(27.3285, 27.3367)	(45.5665, 45.5739)
NFHS-3	p_1		0.1724(0.0405)	0.1310(0.0184)	0.0648(0.0004)	0.0742(0.0015)
	p_2		0.4253(0.0530)	0.4034(0.0122)	0.3894(0.0006)	0.3834(0.0055)
	θ		286.8286(48.6519)	282.9151(43.4851)	200.7400(0.8463)	290.9114(37.7872)
	95% CI	p_1	(0.0903, 0.2518)	(0.0949,0.1671)	(0.0639,0.0657)	(0.0713,0.0771)
		p_2	(0.3214, 0.5292)	(0.3795,0.4273)	(0.3882,0.3906)	(0.0378,0.3942)
		θ	(191.8039,383.8533)	(197.6859,368.1444)	(199.0813,202.3987)	(216.8498,364.9730)

It is observed that the standard error of estimate of all parameters/parametric functions decreases as n increases and increases as censored value c^* and number of inliers increases for every combination of (n, r_1, r_2). It is also inferred that degeneracy actually plays some role in the estimation of parameters/parametric functions. When we deal with degeneracy at zero and one, we recommend the use of MLEs for estimating individual parameters and the UMVUE for parametric functions. As expected, the UMVUE of the variance of θ is less than the variance of MLE. Although the model description is smooth, it poses some challenges in studying the inferential aspects of the model completely.