CHAPTER - 5

A DISCRETE INVENTORY MODEL WHEN SALE PRICE DECREASES WITH TIME

5.1 INTRODUCTION

For many products, it is a common phenomenon that after an initial period of popularity, demand of product decreases. As described in this section, many authors have developed inventory models where demand of product varies with time. It is a common practice among sellers to reduce the selling price of product to deal with the decreasing demands. End of season sale of items such as fashion goods, and reducing prices of electronic gazettes over time are excellent examples of this practice. We can, therefore, categorize these types of items as value deteriorating items. Products that are kept lying on the racks for very long, deteriorate in their appearance and may also deteriorate in quality due to dusting and/ or exposure to environment. This also increases holding costs. For such reasons, as soon as the product cycle in terms of the requirement/ demand completes, sellers would like to do away with that product, although at a lower price, which may or may not cover the cost but certainly conserve on the holding cost. Outdated products due to changed fashion or taste of buyers also command lower prices in the market place. This practice of 'stock clearance sale' also helps to ensure smooth flow of money in the business system.

In the present chapter we incorporate this phenomenon in the proposed model and obtain optimum inventory levels. Since the product is required to be sold at a lower price (sometimes significantly lower), the model can be regarded as a model for value deteriorating items.

Whitin (1952) appears to be first who recognized and incorporated this phenomenon while proposing an inventory model for fashion goods. He assumed that items on hand at the end of the period are liquidated at a loss (i.e. value deteriorated). He assumed that the cost of liquidating an unsold item dominates the set up cost and holding cost and hence did not consider these costs in his analysis.

Although, we are not directly modeling demand as a function of time, the model proposed in this chapter can also be viewed from the point of view of time dependent demand. By “time dependent demand”, we mean demand that varies with time. Several researchers have considered time dependent demand patterns, as discussed in section 1.1.2, for inventory models. The age-on-shelf of inventory may make a negative impact on demand because of the loss of confidence of consumers on the quality of such products. As discussed by Goyal and Giri (2001), and Ruxain et al. (2010) in their review, most of the continuous-time inventory models have been developed assuming either linearly varying demand or exponentially varying demand patterns. Resh et al. (1976) and Donaldson (1977) have also considered an inventory model with a linear trend in demand. Goyal (1986), Goswami and Chaudhuri (1991), Chung and Ting(1993), Dave and Patel (1981), and Dutta and Pal (1992) are some more authors who also considered models with time proportional (i.e. linearly varying with time) demand. Hariga and Benkherouf (1994), and Wee (1995) considered exponentially time varying demand. Giri and Chaudhuri (1997), Papachristos and Skouri (2000), Chu and Chen (2002), Khanra and Chandhuri (2003), Yang (2005), Dye et al. (2006) also considered time dependent demand.

Mishra (2013) proposed a deterministic model with demand rate and holding cost as liner function of time. Tripathi (2013) developed an inventory model for time varying demand and constant demand. Sing, et al (2013) and Mishra et al. (2013) also proposed model with time dependent demand. Jagadeeswari and Chenniappan (2014) developed Inventory Model for Deteriorating Items with Time – Quadratic Demand. S.Kumar and Rajput (2015) developed fuzzy inventory model for time dependent demand.

Hill (1995) proposed a "ramp-type" time dependent demand as described in section 1.1.2. It is a combination of linearly varying demand and exponentially varying demands over the entire time horizon. Mandal and Pal (1998), Deng et al (2007), Skouri et al. (2007) also consider the ramp type demand in the study of inventory for the deteriorating items. Manna and Chaudhuri (2003) also considered ramp type demand pattern that is generally followed by new brands of consumer goods coming in the market. Panda et al. (2008) discussed an inventory model for a seasonal product with constant deterioration rate and a ramp-type time dependent demand. Tend et al. (2011) also consider ramp type demand rate in their work. Karmakar and Dutta Chaudhury (2014) also consider ramp type demand, constant deterioration rate and holding cost as a linearly increasing function of time. Jagadeeswari and Chenniappan (2014) considered deterioration rate is constant and quadratic time dependent demand.

Generally, in market, it is observed that a price reduction results in an increased demand. The retailers exploit this phenomenon by offering price discounts to their customers to address the problem of decreasing demands.

This policy is frequently used in supermarkets, malls etc. Wee and Yu (1997) considered the effects of the temporary discount sale when the items deteriorate exponentially with time.

In the work referenced above, it appears that Whitin (1952) is the only author who considered items with discrete demand. Many authors considered time varying demand but no one seems to have considered price discount with time in order to maintain demand.

In the present chapter we propose an inventory model for the inventory items having unit demand, where seller takes the approach described above to deal with reducing demand rate. We further assume that the price discount is offered in such a way that the demand rate remain unchanged after the discount (i.e. demand rate is maintained).

5.2 NOTATIONS

Following notations are used in this paper.

n : Lot size. (Initial inventory level)

T : Cycle length (time to reorder)

T_0 : Time after which items are sold at a reduced price.

C_1: Holding cost per unit per unit time before time T_0.

C_2 : Holding cost per unit per unit time after time T_0.

C : Cost of inventory per unit.

C_0: Ordering cost per order.

θ : Mean customer inter arrival time.(Equivalently, Customer arrival rate = $\frac{1}{\theta}$)

5.3 THE PROPOSED MODEL

Following are the basic assumptions of the model.

1. Customer arrival process is Poisson with arrival rate of $\frac{1}{\theta}$ customers per unit time. It is further assumed that each customer has a demand of one unit.
2. Supply/ replenishment is instantaneous.
3. Reorder is placed as soon as inventory level reaches zero.
4. Holding cost is incurred only for the period during which the items are in stock.

Suppose inventory is maintained for a discrete item. The stock level is raised to n units in the beginning. Items are sold at a regular unit price p_1 up to time T_0. If there are any unsold units left at time T_0, they are sold at a reduce price p_2. Since holding cost includes the cost of tied up capital, reduction in the selling price leads to an increased holding cost. In the following development, instead of directly using sale prices p_1 and p_2, we use the resultant holding costs C_1 and C_2 .

Determination of Optimum Inventory Level

Initially at time $t_0 = 0$, let the inventory level be raised to n units. Suppose i^{th}-customer arrives at time t_i, so that at time $t_1 + \cdots + t_i$, inventory level reduces to $(n - i)$ units for i = 1, 2, ….

Let x be the number of units sold up to a specified time T_0. In practice, two cases may arise $x = n$ or $x < n$ .If $x < n$, then remaining $n - x$ units are sold at a reduce price. Here x should be viewed as a realized value of r.v. X.

In the proposed model, number of unit sold is same as the number of customers arrived. As customer arrival process is Poisson and since number of units sold cannot exceed the initial inventory level n, it is clear that X follows truncated Poisson distribution, truncated above at n.

At time $T = t_1 + t_2 + \ldots + t_n$, inventory level becomes zero. At this time reorder is made and the next cycle starts.

Clearly, the cycle length T is a random variable.

As the customer arrival process is Poisson, the inter-arrival times t_i are identically and independently distributed (iid) exponential random variables, with common mean θ.

That is,

$t_i \sim Exp(\theta) , i = 1,2, \ldots , n.$

Then the expected value of Cycle length $T = \Sigma t_i$ is given by

$E(T) = n\theta$

In order to obtain the optimal order quantity, we need to minimize the expected total inventory cost per unit time with respect to the initial inventory level n. The total inventory cost for one cycle is given by

Total cost = Ordering cost + Inventory cost + Holding cost

Ordering cost = C_0 ... (5.3.1)

Cost of inventory = nC ... (5.3.2)

The holding cost that incur before time T_0 is

$$HC_1 = \begin{cases} C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + (n-x)(T_0 - T_x)) ; & x < n \\ C_1(nt_1 + (n-1)t_2 + \cdots + 1t_n) & ; x = n \end{cases}$$

The holding cost that incur after time T_0, is

$$HC_2 = \begin{cases} C_2\big((n-x)(T_{x+1} - T_0) + (n-(x+1))t_{x+2} + \cdots + t_n\big) & ; x < n \\ 0 ; & ; x = n \end{cases}$$

Here T_x and T_{x+1} are the arrival times of x^{th} and $(x+1)^{\text{th}}$ customers respectively, i.e. $T_x = \sum_{i=1}^{x} t_i$ and $T_{x+1} = T_x + t_{x+1}$

Hence we have

Total holding cost = HC_1+HC_2

$$= \begin{cases} C_1(nt_1 + (n-1)t_2 + \cdots + (n-(x-1))t_x + (n-x)(T_0 - T_x)) + \\ \quad C_2\big((n-x)(T_{x+1} - T_0) + (n-(x+1))t_{x+2} + \cdots + t_n\big) ; & x < n \\ C_1(nt_1 + (n-1)t_2 + \cdots + \big(n-(x-1)\big)t_n & ; x = n \end{cases} \quad \text{... (5.3.3)}$$

Total cost for one cycle is sum of ordering cost, inventory cost and holding cost.

From (5.3.1) - (5.3.3),

The total inventory cost per unit time, with initial inventory level n, is

$TC(n) =$

$$\begin{cases} \frac{1}{T}\left(\begin{array}{c} C_0 + \ nc + C_1\begin{pmatrix} nt_1 + (n-1)t_2 + (n-2)t_3 + \cdots + \\ \big(n-(x-1)\big)t_x + (n-x)(T_0 - T_x) \end{pmatrix} \\ +C_2\Big(\big(n-x\big)(T_{x+1} - T_0) + \big(n-(x+1)\big)t_{x+2} + \cdots + \ t_n\Big) \end{array} \right) & ; x < n \\ \frac{1}{T}\ (\ C_0 + \ nc + C_1(nt_1 + (n-1)t_2 \ + (n-2)t_3 + \cdots \ + \ t_n)\ ) & ; x = n \end{cases}$$

… (5.3.4)

Since $TC(n)$ is a random variable, we minimize $E(TC(n))$ with respect to n for obtaining the optimum inventory level.

The case, $n = 1$ requires a special attention.

For $n = 1, T = t_1 \sim Exp(\theta)$. Hence

$$E\left(\frac{1}{T}\right) \ = \ \frac{1}{\theta}\int_0^\infty e^{-t/\theta}t^{-1}dt$$

This is a Gamma integral with shape parameter α = 0. It may be recalled that the Gamma integral converges if and only if α > 0. Thus, it follows that

$E\left(\frac{1}{T}\right)$ diverges to ∞.

As a result, for $n = 1$, expected total cost per unit time is infinite. So optimum inventory level cannot be 1.

$For\ n > 1, E\big(TC(n)\big)$ is computed as $E\big(TC(n)\big) = E(E(TC(n)|X = x))$

$$E(TC(n)|X = x)$$

$$= \begin{cases} (C_0 + nc)E\left(\frac{1}{T}\right) + C_1(n\,E\left(\frac{t_1}{T}\right) + (n-1)E\left(\frac{t_2}{T}\right) + \cdots + \big(n - (x-1)\big)\ E\left(\frac{t_x}{T}\right) + \\ (n-x)E\left(\frac{T_0 - T_x}{T}\right)) + C_2((n-x)E\left(\frac{T_{x+1} - T_0}{T}\right) + \big(n - (x+1)\big)\,E\left(\frac{t_{x+2}}{T}\right) + \cdots + \\ 2\,E\left(\frac{t_{n-1}}{T}\right) + E\left(\frac{t_n}{T}\right)) & ; x < n \\ (C_0 + nc)E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\big(n + (n-1) + (n-2) + \cdots + (1)\big) & ; x = n \end{cases}$$

$$E(TC(n)|X = x)$$

$$= \begin{cases} (C_0 + nc)\,E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\big(n + (n-1) + (n-2) + \cdots + (n - (x-1)\big) + \\ (n-x)C_1\left(E\left(\frac{T_0}{T}\right) - E\left(\frac{T_x}{T}\right)\right) + (n-x)C_2\left(E\left(\frac{T_{x+1}}{T}\right) - E\left(\frac{T_0}{T}\right)\right) + \\ C_2\,E\left(\frac{t_i}{T}\right)((n-(x+1)) + \cdots + 2 + 1) & ; x < n \\ (C_0 + nc)E\left(\frac{1}{T}\right) + C_1\,E\left(\frac{t_i}{T}\right)\big(n + (n-1) + (n-2) + \cdots + (1)\big) & ; x = n \end{cases}$$

… (5.3.5)

As per the model assumptions, we have for n>1,

$$E\left(\frac{1}{T}\right) = \frac{1}{\theta(n-1)}\ ,\ \ E\left(\frac{t_i}{T}\right) = \frac{1}{n}\ , E\left(\frac{T_0}{T}\right) = \frac{T_0}{\theta(n-1)}$$

$$E\left(\frac{T_x}{T}\right) = \frac{x}{n}\ , E\left(\frac{T_{x+1}}{T}\right) = \frac{x+1}{n}$$

Also, we have,

$$n + (n-1) + (n-2) + \cdots + (n - (x-1)) = \frac{x}{2}\ (n + n - (x-1))$$

$$(n - (x+1)) + \cdots + 2 + 1) = \frac{n-(x+1)}{2}\ (n - (x+1) + 1)$$

Therefore,

$E(TC(n)|X = x)$

$$= \begin{cases} \dfrac{C_0 + nc}{\theta(n-1)} + C_1\dfrac{x(2n-x+1)}{2n} + (n-x)C_1\left(\dfrac{T_0}{\theta(n-1)} - \dfrac{x}{n}\right) + \\ \quad (n-x)C_2\left(\dfrac{x+1}{n} - \dfrac{T_0}{\theta(n-1)}\right) + C_2\dfrac{(n-x-1)(n-x)}{2n} & ; x < n \\ \dfrac{C_0 + nc}{\theta(n-1)} + C_1\dfrac{n+1}{2} & ; x = n \end{cases}$$

$$= \begin{cases} \dfrac{C_0 + nc}{\theta(n-1)} + C_1 x - C_1\dfrac{x^2}{2n} + C_1\dfrac{x}{2n} + (n-x)C_1\dfrac{T_0}{\theta(n-1)} - (n-x)C_1\dfrac{x}{n} + \\ \quad (n-x)C_2\dfrac{x+1}{n} - (n-x)C_2\dfrac{T_0}{\theta(n-1)} + C_2\dfrac{n^2 + x^2 - 2nx - n + x}{2n} & ; x < n \\ \dfrac{C_0 + nc}{\theta(n-1)} + C_1\dfrac{n+1}{2} & ; x = n \end{cases}$$

$E(TC(n)|X = x)$

$$= \begin{cases} \dfrac{C_0 + nc}{\theta(n-1)} + \dfrac{n}{n-1}\dfrac{T_0}{\theta}(C_1 - C_2) + \dfrac{x}{n-1}\dfrac{T_0}{\theta}(C_2 - C_1) + \dfrac{n+1}{2}C_2 + \dfrac{x}{2n}(C_1 - C_2) \\ \quad + \dfrac{(C_1 - C_2)}{2n}x^2 & ; x < n \\ \dfrac{C_0 + nc}{\theta(n-1)} + C_1\dfrac{n+1}{2}; & ; x = n \end{cases}$$

writing $\lambda = \frac{T_0}{\theta}$, and $k_n = \sum_{x=0}^{n} \frac{\lambda^x}{x!}$

$$E\big(TC(n)\big)$$

$$= \sum_{x=0}^{n-1}\left(\frac{C_0+nc}{\theta(n-1)} - \frac{n}{n-1}\lambda(C_2-C_1) + \frac{x}{n-1}\,\lambda(C_2-C_1) + \frac{n+1}{2}\,C_2\right.$$

$$\left. - \frac{x}{2n}\,(C_2-C_1) - \frac{(C_2-C_1)}{2n}x^2\right)\frac{\lambda^x/x!}{k_n}$$

$$+\left(\frac{C_0+\,nc}{\theta(n-1)}\right.$$

$$\left.+\,C_1\frac{n+1}{2}\right)\frac{\lambda^n/n!}{k_n} \qquad \ldots\ (5.3.6)$$

Add and subtract n^{th} term in 2nd, 3rd, 5th and 6th term in above expression (5.3.6),

$$E\big(TC(n)\big) = \sum_{x=0}^{n-1}\frac{C_0+nc}{\theta(n-1)}\,\frac{\lambda^x/x!}{k_n} + \frac{C_0+\,nc}{\theta(n-1)}\,\frac{\lambda^n/n!}{k_n}$$

$$-\sum_{x=0}^{n-1}\frac{n}{n-1}\lambda(C_2-C_1)\,\frac{\lambda^x/x!}{k_n} - \frac{n}{n-1}\,\lambda(C_2-C_1)\frac{\lambda^n/n!}{k_n}$$

$$+\frac{n}{n-1}\lambda(C_2-C_1)\frac{\lambda^n/n!}{k_n}$$

$$+\sum_{x=0}^{n-1}\frac{x}{n-1}\,\lambda(C_2-C_1)\frac{\lambda^x/x!}{k_n} \qquad +\sum_{x=0}^{n-1}\frac{n+1}{2}\,C_2\,\frac{\lambda^x/x!}{k_n}$$

$$+\frac{n+1}{2}\,C_2\,\frac{\lambda^n/n!}{k_n} - \frac{n+1}{2}\,C_2\,\frac{\lambda^n/n!}{k_n} + \frac{n+1}{2}\,C_1\,\frac{\lambda^n/n!}{k_n}$$

$$-\sum_{x=0}^{n-1}\frac{x}{2n}\,(C_2-C_1)\frac{\lambda^x/x!}{k_n} - \frac{n}{2n}\,(C_2-C_1)\frac{\frac{\lambda^n}{n!}}{k_n} + \frac{n}{2n}\,(C_2-C_1)\frac{\frac{\lambda^n}{n!}}{k_n}$$

$$-\sum_{x=0}^{n-1}\frac{x^2}{2n}\,(C_2-C_1)\frac{\lambda^x/x!}{k_n} - \frac{n^2}{2n}\,(C_2-C_1)\frac{\frac{\lambda^n}{n!}}{k_n} + \frac{n^2}{2n}\,(C_2-C_1)\frac{\lambda^n/n!}{k_n}$$

$$\Rightarrow E\big(TC(n)\big) = \frac{C_0 + nc}{\theta(n-1)} - \frac{n}{n-1}\lambda(C_2 - C_1) + \sum_{x=0}^{n} \frac{x}{n-1}\,\lambda(C_2 - C_1)\frac{\lambda^x/x!}{k_n}$$

$$+\frac{n+1}{2}\,C_2 - \frac{n+1}{2}(C_2 - C_1)\frac{\frac{\lambda^n}{n!}}{k_n} + \frac{n+1}{2}\,(C_2 - C_1)\frac{\frac{\lambda^n}{n!}}{k_n}$$

$$-\sum_{x=0}^{n} \frac{x}{2n}\,(C_2 - C_1)\frac{\lambda^x/x!}{k_n} - \sum_{x=0}^{n} \frac{x^2}{2n}\,(C_2 - C_1)\frac{\lambda^x/x!}{k_n}$$

Thus,

$$E\big(TC(n)\big) = \frac{C_0 + nc}{\theta(n-1)} - \frac{n}{n-1}\lambda\,(C_2 - C_1) + \frac{n+1}{2}\,C_2$$

$$+\frac{\lambda}{n-1}\,(C_2 - C_1)\sum_{x=0}^{n}\frac{\frac{x\,\lambda^x}{x!}}{k_n} - \frac{(C_2 - C_1)}{2n}\sum_{x=0}^{n}\frac{\frac{x\,\lambda^x}{x!}}{k_n}$$

$$-\frac{(C_2 - C_1)}{2n}\sum_{x=0}^{n}\frac{\frac{x^2\,\lambda^x}{x!}}{k_n} \qquad \ldots(5.3.7)$$

$$E\big(TC(n)\big) = \frac{C_0 + nc}{\theta(n-1)} - \frac{n}{n-1}\lambda\,(C_2 - C_1) + \frac{n+1}{2}\,C_2 + \frac{\lambda}{n-1}\,(C_2 - C_1)E(X)$$

$$-\frac{(C_2 - C_1)}{2n}\big(E(X^2) + E(X)\big)$$

Where

$$E(X) = \sum_{x=0}^{n}\frac{x\,\lambda^x/x!}{k_n} \quad and \quad E(X^2) = \sum_{x=0}^{n}\frac{x^2\,\lambda^x/x!}{k_n}$$

It may be noted here that the expected total cost does not directly depend on C_1 and C_2, it rather depends on the difference (C_2-C_1). This difference is precisely the cost of liquidating unsold items, the cost considered by Whitin (1952). Thus our result is in agreement with Whitin's assumption that holding cost is not an important variable for determining optimal ordering quantity for value deteriorating items. However, since our model is different from that of Whitin, the overall cost function obviously differs.

As moments of truncated above Poisson distribution are not available in a closed form, obtaining the formula of E(TC(n)) and hence that of the optimal value of n in a closed form is difficult. We, therefore, present an algorithm for the calculation of optimal value of n and the associated minimum cost. An implementation of the same in C++ is presented in the appendix. The algorithm is described in the next section.

5.4 ALGORITHM

Step 1: Enter the value of C_0, C_1, C_2, C, θ, T_0.

Step 2: Compute $\lambda = \frac{T_0}{\theta}$.

Step 3: Set $n = 2, E(TC(1))$ = LARGE

Step 4: Compute

$$k_n = \sum_{x=0}^{n} \frac{\lambda^x}{x!}$$

Step 5: Compute

$$e_1 = \frac{C_0 + nC}{\theta(n-1)}, e_2 = \frac{n}{n-1} \lambda (C_1 - C_2), \quad e_3 = \frac{n+1}{2} C_2$$

Step 6: Compute

$$sum1 = \frac{1}{k_n} \sum_{x=0}^{n} x\lambda^x/x! \;\; as\; sum1 = \lambda(1 - \frac{\lambda^n/n!}{k_n})$$

Step 7: Compute

$$e_4 = \frac{\lambda\,(C_1 - C_2\,)}{n-1}\,sum1$$

$$e_5 = \frac{1}{2n}\,(C_1 - C_2)sum1$$

Step 8: Compute sum2 = $\sum_{x=0}^{n} \frac{x^2\,\lambda^x/_{x!}}{k_n}$ as

$$\mathrm{Sum2} \;=\; \mathrm{sum1} + \lambda^2\left(1 - \frac{1}{k_n}\frac{\lambda^{n-1}}{(n-1)!} - \frac{1}{k_n}\frac{\lambda^n}{n!}\right)$$

Compute $e_6 = \frac{C_1 - C_2}{2n}\,sum2$

Step 9: Compute $ETC(n) = \; e_1 + \; e_2 + e_3 - \, e_4 + \, e_5 + \, e_6$

Step 10: If $E\big(TC(n)\big) > E(TC(n-1))$ then out put the value of $n-1$ as the optimal solution and stop.

Else set $n = n + 1$. And go to step 4.

5.5 AN ILLUSTRATIVE EXAMPLE

Suppose that demand for a product is 20 per month and customer arrival process is a Poisson process with each customer having demand of one unit. Holding cost per unit per unit time is Rs. 500 before time T_0. Holding cost per unit per unit time is Rs. 750 after time T_0. Ordering cost is Rs.10000 per order. Suppose shortages are not allowed and the purchasing cost per unit is Rs.12000. Here we determine expected total cost and optimal quantity.

This example is solved with the help of program develop in C++ (see Appendix-E)

$C = 12000$, $C_0 = 10000$, $C_1 = 500$, $C_2 = 750$, $T_0 = 3$, $\theta = 0.05$, $1/\theta = 20$

Using the program developed in Appendix – E, we get the optimal value of

n= 43 with E(TC)= 2,61,251.1 .

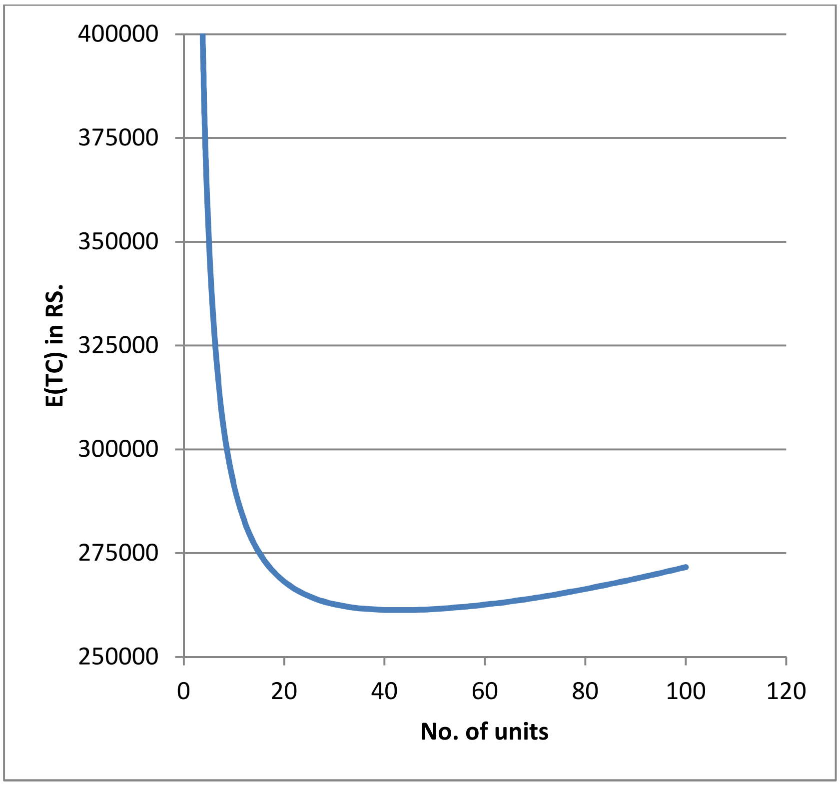

Figure 5.5 Expected total cost as a function of n when Sale Price decreases with Time

The graph describes the relationship between expected total cost and initial inventory level. As it can be observed, the curve for E(TC(n)) is a convex function of n, with minimum cost achieved at n=43 units.