# CHAPTER - 2

# A SINGLE PERIOD INVENTORY MODEL WITH DISCRETE DEMAND

## 2.1 INTRODUCTION

The most well-known square-root formula for Economic Order Quantity (EOQ) is such a fundamental result of inventory theory that it appears in almost every textbook on the subject. This fundamental result was developed by Harris (1913). Interestingly, this original work of Harris was lost from sight for many years until it was rediscovered by Erlenkotter (1989). This classical EOQ model assumes that demand is continuous and arising at a constant rate. Because of the simplicity of the EOQ, it is often used, as an approximation, in practice even when demand is stochastic or also when demand is discrete. Even the context for which Harris developed this formula was that of manufacturing parts such as connectors, or studs which are essentially manufactured in discrete units. However, in numerical illustrations, Harris himself uses the fractional answer obtained by the square-root formula after rounding it to nearest feasible quantity. We investigate in this work, whether a model specifically developed for discrete items is more beneficial as compared to this approach of rounding the answers. At the time of this work, we are not aware of any such investigation reported in the literature.

We consider a continuous review inventory system and assume zero lead time. Katy,Juliy & Udayabhanu (2012), and other authors referenced by them including , Cetinkaya & Lee (2000), Lee & Rosenblatt (1986), Thompstone & Silver (1975), assume negligible lead time. The assumption of negligible lead time is practically equivalent to zero load time, and is treated that way in above papers.

## 2.2 NOTATIONS

Notations used in this Chapter are as given below.

$T$: Cycle length

$C_0$: Ordering cost per order.

$C_h$ : Holding cost per unit per unit time.

$C$ : Cost of inventory per unit

$n$ : Initial inventory level, - a positive integer

$D$ : Demand rate per unit time - a positive integer

## 2.3 THE PROPOSED MODEL

Following are the basic assumptions of the proposed model.

1. Demand is discrete at a constant rate of $D$ units per unit time.

2. Supply is instantaneous.

3. Reorder is placed as soon as inventory level reaches zero.

4. Holding cost is incurred for the period during which the inventory item is in the stock.

It is further assumed that the holding cost is not computed for a fraction of unit time. Thus, for example if the demand rate is 20 units per day, then for the 20 units that are sold during the day, the holding cost is incurred for the full day. In our opinion, this is a very realistic assumption.

**Determination of Optimum Inventory Level**

In the proposed inventory model, the demand is discrete, whereas in classical EOQ model the demand is assumed to be continuous and constant over the period. This assumption of the later model results in a linearly decreasing inventory level (as shown in figure 2.3(a)), until the reorder is placed. However, when the demand is discrete, the inventory levels decrease as shown in figure 2.3(b).

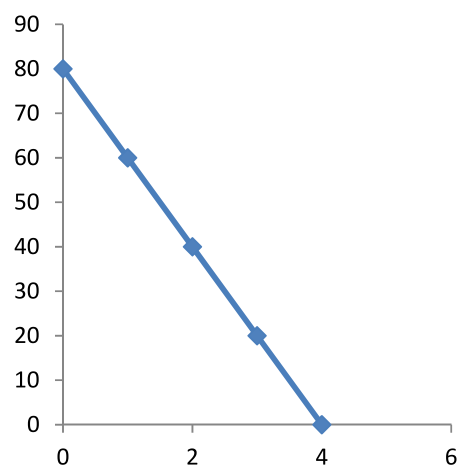


Figure 2.3(a)

Inventory Level v/s. Time for

Continuous Demand

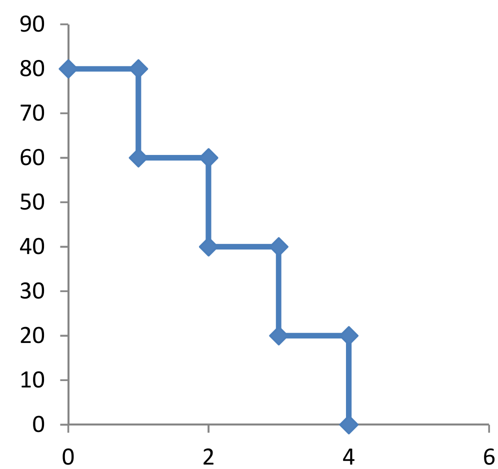


Figure2.3(b)[1]

Inventory Level v/s. Time for

Discrete Demand

The total cost of maintaining inventory for one cycle is

Total cost = Ordering cost + Inventory cost + Holding cost

Ordering cost $= C_0$ ... (2.3.1)

Cost of inventory $= nC$ ... (2.3.2)

*[1] This figure shows inventory level over time when demand is discrete having rate of 20 units per unit time, and initial inventory level is 80.*

Total holding cost = $nC_h + \ (n - D)\ C_h\ + (n - 2D)C_h\ + \ \dots \dots\ +$

$$(n - (T - 1)D)C_h)$$

$$= C_h(n + (n - D) + (n - 2D) + \cdots + D)$$

Thus,

$$Total\ holding\ cost = C_h T \frac{(n + D)}{2} \qquad \dots (2.3.3)$$

Equations (2.3.1) – (2.3.3) imply that total cost for one cycle of length $T$ is

$$C_0 + nC + \ C_h T \frac{(n + D)}{2}$$

The total cost per unit time, with initial inventory level $n$, is

$$TC(n) = \frac{C_0 + nC + \ C_h T \frac{(n + D)}{2}}{T}$$

By substituting $T = \frac{n}{D}$, we get for n = 1, 2, …

$$TC(n) = \frac{C_0 D}{n} + \ DC + C_h\ \frac{(n + D)}{2}$$

$$= \frac{C_0 D}{n} + \ DC + n\frac{C_h}{2} + D\frac{C_h}{2} \qquad \dots (2.3.4)$$

An optimal value of $n$ is the one that minimizes $TC(n)$.

Note that

$$TC(n + 1) - TC(n) = \frac{C_h}{2} - \frac{C_0 D}{n(n + 1)}$$

is an increasing function for all $n$. Thus, the total cost function $TC(n)$ is a convex function of $n$. This further implies that the cost function has unique minima.

This unique optimal solution is the integer value of $n$ that satisfies

$TC(n) \leq TC(n-1)$ as well as

$TC(n) \leq TC(n+1)$

Here, we take $TC(0) = \infty$.

Equivalently, optimal value of $n$ is the smallest integer value of $n$ that satisfies

$TC(n) \leq TC(n+1)$.

From equation 2.3.4, we have

$$TC(n+1) = \frac{C_0 D}{n+1} + DC + \frac{(n+1)C_h}{2} + D\frac{C_h}{2}$$

$TC(n) \leq TC(n+1)$

$$\Rightarrow \frac{C_0 D}{n} + n\frac{C_h}{2} + D\frac{C_h}{2} + DC \leq \frac{C_0 D}{n+1} + \frac{(n+1)C_h}{2} + D\frac{C_h}{2} + DC$$

$$\Rightarrow \frac{C_0 D}{n} - \frac{C_0 D}{n+1} \leq \frac{C_h}{2}$$

$$\Rightarrow \frac{C_0 D}{n(n+1)} \leq \frac{C_h}{2}$$

$$\Rightarrow \frac{2C_0 D}{C_h} \leq n(n+1) \qquad \text{... (2.3.5)}$$

Therefore, for obtaining optimal value $n^*$, we solve

$$n(n+1) = \frac{2DC_0}{C_h}$$

$$\Rightarrow n^2 + n - \frac{2DC_0}{C_h} = 0$$

Comparing above equation with general quadratic equation $ax^2 + bx + c = 0$, we have

$$a = 1, b = 1, c = -\frac{2DC_0}{C_h}$$

$$\Rightarrow \Delta = b^2 - 4ac = 1 - 4\left(-\frac{2DC_0}{C_h}\right) = 1 + \frac{8DC_0}{C_h}$$

$$\Rightarrow n = \frac{-b + \sqrt{\Delta}}{2a}$$

as $n = \frac{-b - \sqrt{\Delta}}{2a}$ is not feasible, being a negative value..

$$\therefore n = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 + \sqrt{1 + \frac{8DC_0}{C_h}}}{2}$$

Thus the optimal solution $n^*$is the smallest integer greater than or equal to the above solution.

Therefore,

$$n^* = \left\lceil \frac{-1 + \sqrt{1 + \frac{8DC_o}{C_h}}}{2} \right\rceil \qquad \ldots (2.3.6)$$

Since $n^* = D * T$, the cycle length $T$ is given by

$$T = \frac{n^*}{D}$$

$$\therefore T = \frac{1}{D} \left\lceil \frac{-1 + \sqrt{1 + \frac{8DC_o}{C_h}}}{2} \right\rceil \qquad \ldots (2.3.7)$$

Next we compare above formula (2.3.6) with classical EOQ formula

$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

Writing $z = \frac{2DC_0}{C_h}$, we can write Q* = $\sqrt{z}$, and

the optimal solution at (2.3.6) can be expressed as

$$n^* = \left\lceil -0.5 + \sqrt{\frac{1}{4} + \frac{4 * z}{4}} \right\rceil$$

$$= \left\lceil -0.5 + \sqrt{.25 + z} \right\rceil$$

For all values of z the difference between $\sqrt{z}$ and $\sqrt{0.25+z}$ is practically negligible. So the difference between Q* and $n$* is approximately 0.5.

Now, for any positive number x, the relationship round(x) = $\lceil$x - 0.5$\rceil$ is well-known. We, therefore, have

$$n^* = \text{round}(Q^*)$$

***Remark***:

1. In above, for rounding, we have assumed that the tie breaking is towards negative infinity. If instead, the tie breaking is to be towards positive infinity, we have the relationship, round(x) = $\lfloor$x + 0.5$\rfloor$. The corresponding formula for $n^*$ is given by

   $$n^* = \lfloor 0.5 + \sqrt{.25+z} \rfloor$$

   which is obtained as the largest value of n that satisfies the inequality

   $$TC(n) \leq TC(n-1) .$$

   Thus, the identity stated above would still hold true.

2. The equality stated above holds for all values of Q* ≥ 1. For the case of Q* < 1. However, in that case we note that rounding of Q* would lead to an infeasible solution, and in practice we need to use the value 1 for the solution, which again would be same as n*.

## 2.4 AN ILLUSTRATIVE EXAMPLE

A company stocks an item that is consumed at the rate of $\theta = 0.05$ per unit time. The holding cost per unit per unit time is Rs. 500, ordering cost is Rs. 10000 per order. Suppose shortages are not occurring (no shortage is allowed) and the purchasing cost per unit is Rs. 12000. Determine expected total cost and optimal quantity.

This example solved by taking

$C_o$= 10000

$C_h$ =500

$C$ = 12000

a = 1/$\theta$ =1/ 0.05 = 20

Using the classical EOQ formula $Q^*$ = 28.284

And using the formula (2.3.6), $n^*$ = $\lceil 28.03 \rceil$ = 28

and E(TC(28))= 259144

From This example, we can observe that although there is a difference between value of Q* and $n^*$, the relationship n* = ROUND(Q*) prevails.

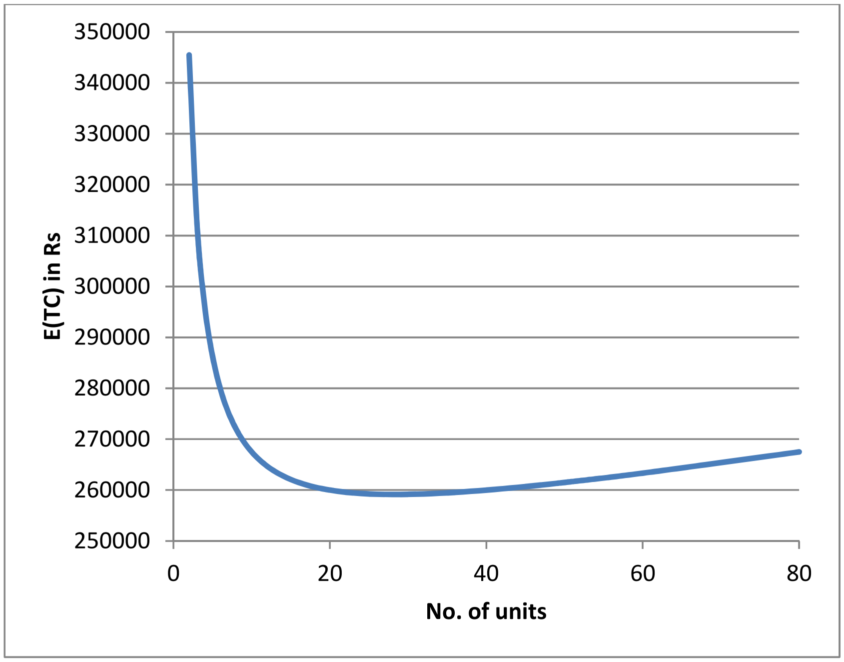


Figure 2.4 Expected total cost as a function of n

The graph describes how expected total cost varies with the initial inventory level. The graph describes the relationship between expected total cost and initial inventory level. As it can be observed, the curve for E(TC(n)) is a convex function of n, with minimum cost achieved at n=28 units.