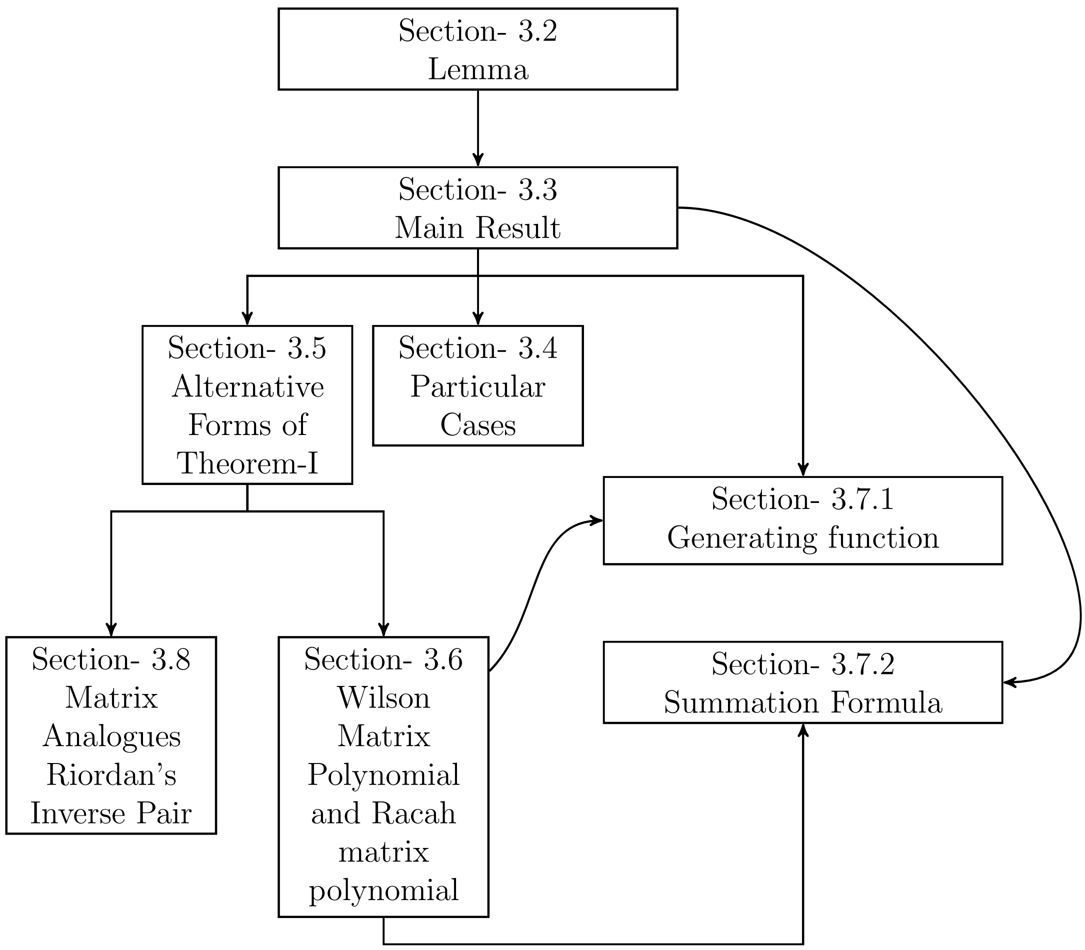
Chapter 3

A General Matrix Inverse Series Relation and Associated Polynomials-II



3.1 Introduction

This chapter incorporates the matrix analogue of the generalized Humbert polynomial [38, Eq.5.11, p.707]:

$$P_n(m,x,\mu,s,c) = \sum_{k=0}^{[n/m]} \binom{s-n+mk}{k}\binom{s}{n-mk}\mu^k \times c^{s-n+mk-k}(-mx)^{n-mk} \tag{3.1.1}$$

and its properties namely inverse relation, generating function relations, summation formulas and matrix version of some Riordan's pairs.
This polynomial occurs as the coefficient of t^n in a series expansion of $(c - mxt + \mu t^m)^s$ as follows.

$$(c - mxt + \mu t^m)^s = \sum_{n=0}^{\infty} P_n(m,x,\mu,s,c). \tag{3.1.2}$$

In his work, H. W. Gould also obtained the inverse series of this polynomial in the form [38, Eq.5.12, p.707]:

$$\frac{(-m)^n}{(c)^{n-s}}\binom{s}{n}x^n = \sum_{k=0}^{[n/m]} \binom{s-n+k}{k}\frac{(-\mu^k)(s-n+mk)}{c^k(s-n+k)} \times P_{n-mk}(m,x,\mu,s,c). \tag{3.1.3}$$

As long ago as in 1722, Liouville discussed a paradox arising from the theories due to Galileo and Huygens related to isochronal property of the cycloid curve [74]. Liouville obtained the power series expansion of $(p^2 - 2qx - x^2)^{-1/2}$ in powers of x. It was shown by Nielsen [74] that the Legendre polynomial $P_n(x)$ and the coefficients of $f_n(p,q)$ of this expansion are connected with

$$f_n(p,q) = i^{-n}p^{-n-1}P_n(iq/p), i^2 = -1.$$

Here the function $(1-2qt-t^2)^{-1/2}$ occurs by substituting $p=1$ and replacing x by t. This function arises in the potential theory involving $P_n(x)$.

As a mater of fact, a Newtonian potential function in potential theory may be given by [63]

$$U = \int\int_V\int \frac{\lambda}{r}\, dV,$$

where $r=(1-2qt-t^2)^{-1/2}$. It is to be noted that the potential function U is associated with Laplace Equation [109]:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

The Humbert polynomials occured in his study (see [46], [47],[48]) of more generalized potential problems associated with the extended Laplace equation [109]:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - 3\frac{\partial^2 U}{\partial x \partial y \partial z} = 0.$$

In a recent years, many researcher worked in the direction of providing matrix analogue to the existing (scalar) hypergeometric function.

The matrix analogue of (3.1.1) has been introduced and studied by some eminent researchers in particular, Ayman Shehata [96], Levet Kargin and Veli Kurt [62] M. A. Pathan and Maged Gumaan Bin-Saad [76] and H. M. Srivastava, Waseem Khan and Hiba Haroon [105]. The gerelized Humbert matrix polynomial, with $a=1, y=1$ in [76] is given by

$$\begin{aligned} P_n^A(m,x,\eta,c) &= \sum_{k=0}^{[n/m]} \eta^k \frac{c^{A-(n-(m-1)k)I}}{(n-mk)!\ k!} \Gamma^{-1}(A+(1-n+mk-k)I)\Gamma(A+I) \\ &\quad \times(-mx)^{n-mk}. \end{aligned} \tag{3.1.4}$$

3.2 Lemma

As a main result of this chapter, we derive the general inversion pair in matrix form. Using this, the inverse matrix series of the polynomial will be deduced. The proof of general inversion pair uses a particular inverse matrix pair which is stated and prove below as

Lemma 3.2.1. *If $A \in C^{p\times p}$ is a positive stable matrix, then*

$$\boldsymbol{f}_j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma(A-(nr-mrk+k-j)I)\ \boldsymbol{g}_k \tag{3.2.1}$$

if and only if

$$\boldsymbol{g}_j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A+(1-nr+mrj-j+k)I) \times(A-(nr-mrk)I)\ \boldsymbol{f}_k. \tag{3.2.2}$$

Proof. We first prove that the series in (3.2.2) $\Rightarrow$ the series (3.2.1). For that we assume that the series in (3.2.2) holds true. If $\mathbf{F}_j$ stands for the right hand side of (3.2.1) then on substituting the series for $\mathbf{g}_k$ from (3.2.2), we get

$$\begin{aligned}\mathbf{F}_j &= \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma(A-(nr-mrk+k-j)I)\mathbf{g}_k\\ &= \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma(A-(nr-mrk+k-j)I)\sum_{i=0}^{k}(-1)^i\binom{k}{i}\\ &\quad\times\Gamma^{-1}(A+(1-nr+mrk-k+i)I)\ (A-nrI+mriI)\ \mathbf{f}_i.\end{aligned}$$

Using the double series relation (1.3.27), this becomes

$$\begin{aligned}\mathbf{F}_j &= \sum_{i=0}^{j}\sum_{k=i}^{j}(-1)^k\binom{j}{k}\binom{k}{i}\Gamma(A-(nr-mrk+k-j)I)\\ &\quad\times\Gamma^{-1}(A+(1-nr+mrk-k)I)\ (A-nrI+mriI)\mathbf{f}_i\\ &= \sum_{i=0}^{j}\sum_{k=0}^{j-i}(-1)^k\binom{j}{k+i}\binom{k+i}{i}\Gamma(A-(nr-mr(k+i)+(k+i)-j)I)\end{aligned}$$

$$\begin{aligned}
& \times\, \Gamma^{-1}(A + (1 - nr + mr(k+i) - k)I)\ (A - nrI + mriI)\ \mathbf{f}_i \\
= & \sum_{i=0}^{j} \binom{j}{i} \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k}\ \Gamma(A - nrI + mr(k+i)I - (k+i)I + jI) \\
& \times\Gamma^{-1}(A + (1 - nr + (k+i)(mr-1) + i)I)\ (A - nrI + mriI)\ \mathbf{f}_i \\
= & \mathbf{f}_j + \sum_{i=0}^{j-1} \binom{j}{i} \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k} \Gamma(A - (nr - mr(k+i) + (k+i) - j)I) \\
& \times\Gamma^{-1}(A + (1 - nr + mrk + mri - k)I)\ (A - nrI + mriI)\ \mathbf{f}_i. \qquad (3.2.3)
\end{aligned}$$

It can be seen that the product of the matrix gamma function and the inverse matrix gamma function simplifies to a matrix polynomial in k, that is,

$$\begin{aligned}
& \Gamma(A - (nr - mr(k+i) + (k+i) - j)I)\Gamma^{-1}(A + (1 - nr + mrk + mri - k)I) \\
& \quad = \sum_{s=0}^{j-i-1} \overline{\alpha}_s\ k^s,
\end{aligned}$$

where the coefficient matrices $\overline{\alpha}_s = \overline{\alpha}_s(A, m, n, r, i)$.

For illustration, let us take $j = 5; i = 2$ and denote $A - nrI - mriI$ by B and $(mr-1)I$ by N then we have

$$\begin{aligned}
\Gamma(B + kN + 3I)\Gamma^{-1}(B + kN + I) &= (B + kNI + 2I)(B + kNI + I) \\
&= k^2N^2 + k(2NB + 3N) + (B^2 + 3B + 2I) \\
&= \sum_{r=0}^{2} \overline{s}_r\ k^r,
\end{aligned}$$

say, where $\overline{s}_2 = N^2 \neq O$. Hence from (3.2.3), we have

$$\mathbf{F}_j = \mathbf{f}_j + \sum_{i=0}^{j-1} \binom{j}{i} \left[\sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k} \sum_{s=0}^{j-i-1} \overline{\alpha}_s\ k^s \right]\ (A - (nr - mri)I)\ \mathbf{f}_i.$$

Here, the two inner series on the right hand side are $(j-i)^{th}$ difference of the polynomial of degree $j - i - 1$, hence in view of Lemma 1.5.1, $\mathbf{F}_j = \mathbf{f}_j$.

The converse part is however complicated. But in the view of the uniqueness of the inverse of a matrix, the converse part can be proved as follows.

In fact, the diagonal elements $\Gamma^{-1}(A - nrI + mrNI - NI + jI)$ of the block matrix corresponding to the series (3.2.1) and those given by $\Gamma(A - nrI + mrNI - NI + jI)$

of the block matrix corresponding to the series (3.2.2) are non singular matrices for every matrix $A \neq nrI - mrNI + NI - jI, j = 0, 1, 2, \ldots,$ hence the inverse of each block matrix is unique. Since (3.2.2)$\Rightarrow$ (3.2.1), it follows that (3.2.1) $\Leftrightarrow$ (3.2.2). $\square$

3.3 Main result

We establish in this section the general matrix inverse series relation:

$$\mathbf{u}(a) = \sum_{k=0}^{M} \overline{\mu}_{n,k}\ \mathbf{v}(a+bk); \quad \mathbf{v}(a) = \sum_{k=0}^{M} \overline{\sigma}_{n,k}\ \mathbf{u}(a+bk),$$

where M is a non negative integer or infinity depending upon whether b is a negative integer or a positive integer. In particular, let a be a non negative integer n. If b is a negative integer $-m$, $m \in N$, then $M = [n/m]$ and if b is a positive integer then $M = \infty$.

As a main result, we derive the inverse matrix series relations in

Theorem 3.3.1. *Let $A - sI$, $s \in \{0\} \cup \boldsymbol{N}$, be a positive stable matrix in $\boldsymbol{C}^{p\times p}$ then*

$$\boldsymbol{U}(n) = \sum_{k=0}^{M} \frac{\eta^k \Gamma^{-1}(A + (1 - nr - brk - k)I)}{k!}\ \boldsymbol{V}(n+bk) \tag{3.3.1}$$

if and only if

$$\boldsymbol{V}(n) = \sum_{k=0}^{M} (-\eta)^k\ \Gamma(A - (nr - k)I)\ \frac{A - (nr + brk)I}{k!}\ \boldsymbol{U}(n+bk). \tag{3.3.2}$$

Proof. We shall let n to be a non negative integer. The theorem will be proved first by taking $M < \infty$ and secondly, for $M = \infty$. For $M < \infty$ the parameter b must take negative integer values. Hence we choose $b = -m, m \in \mathbf{N}$, in which case $M = [n/m]$.

Now, if $\mathbf{U}$ denotes the right hand member of the series (3.3.1), then on substituting the series from (3.3.2) for $\mathbf{V}(n-mk)$, we get

$$\begin{aligned}
\mathbf{U} &= \sum_{mk=0}^{n} \frac{\eta^k\, \Gamma^{-1}(A+(1-nr+mrk-k)I)}{k!} V(n-mk) \\
&= \sum_{mk=0}^{n} \frac{\eta^k\, \Gamma^{-1}(A+(1-nr+mrk-k)I)}{k!} \\
&\quad \times \sum_{mj=0}^{n-mk} \frac{(-\eta)^j\, \Gamma(A-(nr-mrk-j)I)\, (A-(nr-mrk-mrj)I)}{j!} \\
&\quad \times \mathbf{U}(n-mk-mj).
\end{aligned}$$

From the double series relation (1.3.27), we further have

$$\begin{aligned}
\mathbf{U} &= \sum_{mj=0}^{n} \frac{(-\eta)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k}\, \Gamma^{-1}(A-(nr-mrk+k-1)I) \\
&\quad \times \Gamma(A-(nr-mrk+k-j)I)\, (A-(nr-mrj)I) \\
&\quad \times \mathbf{U}(n-mj). \qquad (3.3.3)
\end{aligned}$$

Here the product of inner gamma matrix function and gamma matrix function results in a polynomial of degree $j-1$ in k, that is,

$$\begin{aligned}
&\Gamma^{-1}(A-(nr-mrk+k-1)I)\Gamma(A-(nr-mrk+k-j)I) \\
&= \prod_{i=1}^{j-1}(A-(nr-mrk+k+i)I) \\
&= \sum_{s=0}^{j-1} \bar{\zeta}_s\, k^s,
\end{aligned}$$

where $\bar{\zeta}_0 = \prod_{i=1}^{j-1}(A-nrI-iI)$, $\bar{\zeta}_{j-1} = (mr-1)^{j-1}I$ and for $0 < s < j-1$,

$$\bar{\zeta}_s = \sum_{\substack{u_1,u_2,\ldots u_s=1 \\ u_1\neq u_2\neq\cdots\neq u_s}}^{j-1} \left\{ \prod_{i=1}^{s}(A-nrI+u_iI) \right\}.$$

Therefore from (3.3.3), we get

$$\mathbf{U} = \mathbf{U}(n) + \sum_{mj=1}^{n} \frac{(-\eta)^j}{j!} \left[\sum_{k=0}^{j} (-1)^k \binom{j}{k} \sum_{s=0}^{j-1} \overline{\zeta}_s \; k^s \right] (A - (nr - mrj)I)\mathbf{U}(n - mj).$$

In view of Lemma 1.5.1, the second term on the right hand side will be a null matrix for all $j \geq 1$; consequently $\mathbf{U} = \mathbf{U}(n)$. With this, the proof of the first part is completed.

In order to prove the converse part, let us take

$$\mathbf{V} = \sum_{mk=0}^{n} \frac{(-\eta)^k}{k!} \, \Gamma(A - (nr - k)I) \; (A - (nr - mrk)I)\mathbf{U}(n - mk),$$

then substituting the series (3.3.1) for $\mathbf{U}(n - mk)$ and using the double sum (1.3.27), we arrive at

$$\begin{aligned} \mathbf{V} \quad &= \sum_{mj=0}^{n} \frac{\eta^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \\ &\quad \times \Gamma(A - (nr - k)I)(A - (nr - mrk)I)\mathbf{V}(n - mj). \end{aligned} \tag{3.3.4}$$

We claim that the inner series in (3.3.4) is equal to $\delta_{jo}I$. Here, denoting the inner series in (3.3.4) by $\mathbf{g}_j$, and replacing $\Gamma(A-(nr-k)I)$ by $\mathbf{f}_k$ then in view of (1.3.5) and (1.3.6), we get

$$\begin{aligned} \mathbf{g}_j \quad &= \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I) \\ &\quad \times (A - (nr - mrk)I) \, \mathbf{f}_k. \end{aligned} \tag{3.3.5}$$

The inverse series of this occurs from Lemma 3.2.1, in the form:

$$\mathbf{f}_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma(A - (nr - mrk + k - j)I) \; \mathbf{g}_k. \tag{3.3.6}$$

If we put

$$\mathbf{g}_s = \binom{0}{s} I$$

in the inverse series (3.3.6), then $\mathbf{f}_J = \Gamma(A - (nr - j)I)$ (is recovered), and using the same substitution in (3.3.5), we find the matrix series orthogonality relation:

$$\binom{0}{j} I = \delta_{jo} I \;=\; \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Gamma^{-1}(A + (1 - nr + mrj - j + k)I)$$
$$\times\, \Gamma(A - (nr - k)I)(A - (nr - mrk)I).$$

Thus (3.3.4) becomes, in the notation of the kronecker delta,

$$\begin{aligned}\mathbf{V} &= \mathbf{V}(n) + \sum_{mj=1}^{n} \frac{\eta^j}{j!}\, \mathbf{V}(n - mj)\ \delta_{jo} \\ &= V(n).\end{aligned}$$

This completes the proof of the converse part and hence the proof of the theorem when $M = [n/m]$.

We now prove the theorem for the case $M = \infty$ which runs almost parallel to the above proof. We assume that for the sequences $\{\mathbf{U}(n)\}$ and $\{\mathbf{V}(n)\}$ are such that $\|\mathbf{U}(n)\| < \infty$ and $\|\mathbf{V}(n)\| < \infty$. In order to prove the first part, we denote the right hand side of (3.3.1) by $\mathbf{R}$ and substitute the series for $\mathbf{V}(n - mk)$ from (3.3.2), to get

$$\begin{aligned}\mathbf{R} &= \sum_{k=0}^{\infty} \frac{\eta^k\, \Gamma^{-1}(A + (1 - nr - brk - k)I)}{k!} \mathbf{V}(n + bk). \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\eta^{k+j}\, \Gamma^{-1}(A + (1 - nr - brk - k)I)}{k!\ j!} \\ &\quad \times \Gamma(A - (nr + brk - j)I)\ (A - (nr + brk + brj)I)\ \mathbf{U}(n + bk + bj).\end{aligned}$$

This, with the help of (1.3.30) takes the form:

$$\begin{aligned}\mathbf{R} &= \sum_{j=0}^{\infty}(-1)^j\eta^j\sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A+(1-nr-brk-k)I)\\ &\quad\times\Gamma(A-(nr+brk-j+k)I)\ \frac{(A-(nr+brj)I)}{j!}\ \mathbf{U}(n+bj)\\ &= \mathbf{U}(n)+\sum_{j=1}^{\infty}(-1)^j\eta^j\sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma^{-1}(A+(1-nr-brk-k)I)\\ &\quad\times\Gamma(A-(nr+brk-j+k)I)\ \frac{(A-(nr+brj)I)}{j!}\ \mathbf{U}(n+bj).\end{aligned} \tag{3.3.7}$$

Since the inner series in this last expression is resembling with the inner series occurring in (3.3.3), it follows that the expression (3.3.7) yields the relation $\mathbf{R}=\mathbf{U}(n)$. Conversely, let us put

$$\sum_{k=0}^{\infty}\frac{(-\eta)^k}{k!}\Gamma(A-(nr-k)I)\ (A-(nr+brk)I)\mathbf{U}(n+bk)=\mathbf{S}.$$

Then on making use of the series (3.3.1) and (1.3.30) in turn, we find that

$$\begin{aligned}\mathbf{S} &= \sum_{j=0}^{\infty}\frac{(-\eta)^j}{j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\ \Gamma^{-1}(A+(1-nr-brj-j+k)I)\\ &\quad\times\Gamma(A-(nr-k)I)(A-(nr+brk)I)\ \mathbf{V}(n+bj).\end{aligned} \tag{3.3.8}$$

Again, that the inner series in (3.3.7) is of the similar form as that of (3.3.4). Thus following the same arguments, we find the following orthogonal relation implied by the inner series in (3.3.7).

$$\begin{aligned}\binom{0}{j}I &= \sum_{k=0}^{j}(-1)^k\binom{j}{k}\Gamma^{-1}(A+(1-nr-brj-j+k)I)\\ &\quad\times\Gamma(A-(nr-k)I)(A-(nr+brk)I).\end{aligned} \tag{3.3.9}$$

Using this orthogonal relation, we get from (3.3.8),

$$\mathbf{S} = \mathbf{V}(n). \tag{3.3.10}$$

The proof of the second part is completed and hence the proof of the theorem. □

3.4 Particular cases

The general inversion pair of Theorem 3.3.1 contains several particular polynomials as its special cases; amongst these, the worth mentioning is the matrix analogue of the generalized Humbert polynomials which is denoted here by $P_n^A(m, x, \eta, c)$.

- Generalized Humbert matrix polynomial and its inverse series:

  The Generalized Humbert matrix polynomial occurs with the help of the substitutions $r = 1, b = -m, m \in \mathbf{N}$, $\mathbf{V}(n - mk) = (-mx)^{n-mk} c^{A-(n-mk)I}/(n - mk)!$ and replacing η by $\eta\, c^{-I}$, then $\mathbf{U}(n) = P_n^A(m, x, \eta, c)\Gamma^{-1}(A + I)$ yields the polynomial (3.1.4):

$$P_n^A(m, x, \eta, c) = \sum_{k=0}^{[n/m]} \eta^k \frac{c^{A-(n-(m-1)k)I}}{(n - mk)!\ k!} \Gamma^{-1}(A + (1 - n + mk - k)I) \times\Gamma(A + I)(-mx)^{n-mk}.$$

  Its inverse series occurs in the form:

$$\frac{(-mx)^n}{n!} I = \sum_{k=0}^{[n/m]} (-\eta)^k \frac{(A - nI + mkI)(A - nI + kI)^{-1} c^{nI-kI-A}}{k!} \times\Gamma(A + (1 + k - n)I)\ P_{n-mk}^A(m, x, \eta, c).$$

  The alternative form (3.1.4), that is

$$P_n^A(m, x, \eta, c) = \sum_{k=0}^{[n/m]} (-\eta)^k\ c^{-(A+(n-mk-k)I)}\ \frac{(-A)_{n-mk+k}}{(n - mk)!\ k!}$$

$$\times (mx)^{n-mk} \tag{3.4.1}$$

is obtainable from the formula (1.3.20). The inverse series

$$\frac{(mx)^n}{n!} I = \sum_{k=0}^{[n/m]} \eta^k \frac{(A - nI + mkI)(A - nI + kI)^{-1} c^{nI-kI-A}}{k!}$$

$$\times (-A)^{-1}_{n-k} \; P^A_{n-mk}(m, x, \eta, c) \tag{3.4.2}$$

follows similarly.

- Humbert matrix polynomial and its inverse series:

The further reducibility of this polynomial is subject to the substitutions $c = \eta = 1$ in (3.4.1) and (3.4.2). We find the inverse pair of Humbert matrix polynomial (1.4.10):

$$\left.\begin{aligned} \Pi^A_{n,m}(x) &= \sum_{k=0}^{[n/m]} (-1)^k \frac{(-A)_{n-mk+k}}{(n-mk)!\; k!} \; (mx)^{n-mk} \\ &\Leftrightarrow \\ \frac{(mx)^n}{n!} I &= \sum_{k=0}^{[n/m]} \frac{(A - nI + mkI)(A - nI + kI)^{-1}}{k!} \\ &\quad \times (-A)^{-1}_{n-k} \; \Pi^A_{n-mk,m}(x). \end{aligned}\right\} \tag{3.4.3}$$

In fact, this polynomial constitutes the class $\{\Pi^A_{n,m}(x); n = 0, 1, 2, \ldots\}$ of polynomials which includes several well known polynomials as well as *not so well known* polynomials. The following are amongst them.

- Pincherle matrix polynomial and its inverse series:

For $m = 3$, and $A = -\dfrac{1}{2} I$ the pair (3.4.3) provides the Pincherle matrix polynomial (1.4.11) and its inverse series relation given by

$$\left.\begin{aligned} \mathcal{P}_n(x) &= \sum_{k=0}^{[n/3]} (-1)^k \frac{((1/2)I)_{n-2k}}{(n-3k)!\ k!}\ (3x)^{n-3k} \\ &\Leftrightarrow \\ \frac{(3x)^n}{n!} I &= \sum_{k=0}^{[n/3]} \frac{((1/2)I + (n-3k)I)\ \ ((1/2)I + (n-k)I)^{-1}}{k!} \\ &\quad \times ((1/2)I)_{n-k}^{-1}\ \mathcal{P}_{n-3k}(x). \end{aligned}\right\}$$

- Kinney matrix polynomial and its inverse series:

If $A = \dfrac{1}{m} I$, then (3.4.3) reduces to the Kinney matrix polynomial (1.4.12) and its inverse series relation:

$$\left.\begin{aligned} K_n^A(m,x) &= \sum_{k=0}^{[n/m]} (-1)^k \frac{((-1/m)I)_{n-mk+k}}{(n-mk)!\ k!}\ (mx)^{n-mk} \\ &\Leftrightarrow \\ \frac{(-mx)^n}{n!} I &= \sum_{k=0}^{[n/m]} \frac{\left(-\frac{1}{m}I - (n-mk)I\right)\ \left(-\frac{1}{m}I - (n-k)I\right)^{-1}}{k!} \\ &\quad \times ((-1/m)I)_{n-k}^{-1}\ K_{n-mk}^A(m,x). \end{aligned}\right\}$$

- Gegenbauer matrix polynomial and its inverse series:

  In (3.4.3), taking $m = 2$, we get inverse relation of (1.4.5) in the form [7, p.104 (15)]:

$$\left.\begin{aligned}
&C_n^A(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(-A)_{n-k}}{(n-2k)!\ k!}\ (2x)^{n-2k} \\
&\Leftrightarrow \\
&\frac{(2x)^n}{n!} I = \sum_{k=0}^{[n/2]} \frac{(A+(n-2k)I)(A+(n-k)I)^{-1}}{k!} \\
&\qquad\qquad \times(-A)_{n-k}^{-1}\ C_{n-2k}^A(x).
\end{aligned}\right\}$$

3.5 Alternative Forms of Theorem -I

In this section, several alternative forms of Theorem 3.3.1 are deduced which will be used in the next section for illustrating the various particular cases namely, the matrix version of (i) the Wilson polynomial [65], (ii) the Racah polynomial [65] and (iii) Riordan's inverse pairs [82].

We begin with the Theorem 3.3.1 in which we put $\eta = 1$ and replace $\mathbf{V}(n)$ by $\Gamma(A - nrI + I)\mathbf{V}(n)$ to get

- Inverse matrix series relation- 1.

$$\left.\begin{aligned}
&\mathbf{U}(n) = \sum_{k=0}^{M} \frac{\Gamma(A-(nr+brk-1)I)}{k!} \\
&\qquad \times\Gamma^{-1}(A+(1-nr-brk-k)I)\ \mathbf{V}(n+bk) \\
&\Leftrightarrow \\
&\mathbf{V}(n) = \sum_{k=0}^{M} (-1)^k \frac{(A-(nr+brk)I)}{k!}\ \Gamma(A-(nr-k)I) \\
&\qquad \times\Gamma^{-1}(A-(nr-1)I)\mathbf{U}(n+bk).
\end{aligned}\right\} \tag{3.5.1}$$

From this pair, we obtain some more alternative inverse pairs. First, on multiplying both the relation in (3.5.1) by $A-nrI$ and putting $(A-nrI)\mathbf{U}(n) = \mathbf{U}_1(n)$ and $(A-nrI)\mathbf{V}(n) = \mathbf{V}_1(n)$, we get

- Inverse matrix series relation- 2.

$$\left.\begin{aligned} \mathbf{U}_1(n) &= \sum_{k=0}^{M} \frac{\Gamma(A-(nr+brk)I)}{k!}\,\Gamma^{-1}(A-(nr+brk+k-1)I) \\ &\quad \times(A-nrI)\ \mathbf{V}_1(n+bk) \\ &\Leftrightarrow \\ \mathbf{V}_1(n) &= \sum_{k=0}^{M}(-1)^k\frac{\Gamma(A-(nr-k)I)}{k!}\Gamma^{-1}(A-nrI)\mathbf{U}_1(n+bk). \end{aligned}\right\} \tag{3.5.2}$$

Next, on replacing A and $A+I$ and r by $-r$, (3.5.2) changes to

- Inverse matrix series relation- 3.

$$\left.\begin{aligned} U_1(n) &= \sum_{k=0}^{M} \frac{\Gamma(A+(nr+brk+1)I)}{k!}\Gamma^{-1}(A+(nr+brk-k+2)I) \\ &\quad \times(A+nrI+I)V_1(n+bk) \\ &\Leftrightarrow \\ V_1(n) &= \sum_{k=0}^{M}(-1)^k\frac{\Gamma(A+(nr+k+1)I)}{k!}\ \Gamma^{-1}(A+nrI+I) \\ &\quad \times U_1(n+bk). \end{aligned}\right\} \tag{3.5.3}$$

Using the formula (1.3.20) in (3.5.1), we obtain the pair:

- Inverse matrix series relation- 4.

$$\left.\begin{aligned}
&\mathbf{U}_2(n) = \sum_{k=0}^{M} \frac{(-1)^k}{k!}(-A + nrI + brkI)_k \; \mathbf{V}_2(n+bk) \\
&\Leftrightarrow \\
&\mathbf{V}_2(n) = \sum_{k=0}^{M} \frac{(A - (nr + brk)I)}{k!} \; (A - nrI + kI)^{-1} \\
&\qquad\qquad \times \; (-A + nrI)_{-k}^{-1} \, U_2(n+bk).
\end{aligned}\right\} \tag{3.5.4}$$

Also, applying the formula (1.3.20) in (3.5.2) and then replacing $-A$ by A, we find

- Inverse matrix series relation- 5.

$$\left.\begin{aligned}
&\mathbf{U}_2(n) = \sum_{k=0}^{M} \frac{(-1)^k}{k!}(A + nrI)(A + nrI + brkI + kI)^{-1} \\
&\qquad\qquad \times (A + (nr + brk + 1)I)_k \; \mathbf{V}_2(n+bk) \\
&\Leftrightarrow \\
&\mathbf{V}_2(n) = \sum_{k=0}^{M} \frac{(A + nrI + I)_{-k}^{-1}}{k!} \; \mathbf{U}_2(n+bk).
\end{aligned}\right\} \tag{3.5.5}$$

3.6 Wilson Matrix Polynomials and Racah Matrix Polynomials

It is well known that the polynomials which are orthogonal as well as having hypergeometric representation in ${}_4F_3[*]$ form are the Wilson polynomial and the Racah polynomial [65]. It is interesting to see that both these polynomials assume matrix extension by means of our main Theorem that is, Theorem 3.3.1.

So, we now deduce the required form as follows.

In Theorem 3.3.1, putting $b = -1, \eta = 1, r = 2$ and then reversing the series and also assuming that $A - NI$ is invertible for all $N \geq 0$, we obtain

$$\begin{aligned}\mathbf{U}(n) &= \sum_{k=0}^{n} \frac{\Gamma^{-1}(A - nI - kI + I)}{(n-k)!}\, \mathbf{V}(k)\\ &\Leftrightarrow\\ \mathbf{V}(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(A - 2kI)(A - nI - kI)^{-1}\Gamma(A - nI - kI + I)}{(n-k)!}\, \mathbf{U}(k).\end{aligned}$$

Replacing $\mathbf{U}(n)$ by $\mathbf{U}(n)\Gamma^{-1}(A + I)$, we find

$$\begin{aligned}\mathbf{U}(n) &= \sum_{k=0}^{n} \frac{\Gamma^{-1}(A - nI - kI + I)\Gamma(A + I)}{(n-k)!}\, \mathbf{V}(k)\\ &\Leftrightarrow\\ \mathbf{V}(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(A - 2kI)(A - nI - kI)^{-1}\Gamma(A - (n + k - 1)I)}{(n-k)!}\\ &\quad \times\Gamma^{-1}(A + I)\, \mathbf{U}(k).\end{aligned}$$

In view of formula (1.3.20), this pair may be written in the form:

$$\begin{aligned}\mathbf{U}(n) &= \sum_{k=0}^{n} \frac{(-A)_{n+k}}{(n-k)!}\, \mathbf{V}(k)\\ &\Leftrightarrow\\ \mathbf{V}(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(A - 2kI)(A - nI - kI)^{-1}(-A)_{n+k}^{-1}}{(n-k)!}\, \mathbf{U}(k).\end{aligned}$$

If the second series is re-written in slightly different form, it becomes

$$\begin{aligned}\mathbf{U}(n) &= \sum_{k=0}^{n} \frac{(-A)_{n+k}}{(n-k)!}\, \mathbf{V}(k)\\ &\Leftrightarrow\\ \mathbf{V}(n) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}(-A + 2kI)(-A)_{n+k+1}^{-1}}{(n-k)!}\, \mathbf{U}(k).\end{aligned}$$

Finally, using the formulas

$$(A)_{m+n} = (A)_m(A + mI)_n$$

and

$$(-1)^k(n!)I/(n-k)! = (-nI)_k,$$

we get the pair:

$$\left.\begin{aligned}
\mathbf{U}(n) &= \frac{(-A)_n}{n!}\sum_{k=0}^{n}(-1)^k(-nI)_k(-A+nI)_k\ \mathbf{V}(k)\\
&\Leftrightarrow\\
\mathbf{V}(n) &= \frac{(-A)_n^{-1}}{n!}\sum_{k=0}^{n}(-1)^k(-nI)_k(2kI-A)(-A+nI)_{k+1}^{-1}\ \mathbf{U}(k).
\end{aligned}\right\}\tag{3.6.1}$$

This inverse pair is capable of providing us the matrix polynomials of Wilson as well as that of Racah. In fact, the Wilson matrix polynomial together with the inverse series relation is obtained from the pair (3.6.1) if A is replaced by $A+B+C+D-I$ and

$$\mathbf{V}(n) = \frac{(-1)^n(A+ixI)_n(A-ixI)_n}{n!}\,(A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}.$$

With this choice, (3.6.1) particularizes to

$$\begin{aligned}
&P_n(x^2)(A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}\\
&\quad= \sum_{k=0}^{n}\frac{(-nI)_k}{k!}(A+B+C+D+nI-I)_k(A+ixI)_k(A-ixI)_k\\
&\qquad\times(A+B)_k^{-1}(A+C)_k^{-1}(A+D)_k^{-1}.
\end{aligned}$$

The inverse series occurs from the second series of (3.6.1) in the form:

$$\begin{aligned}
&\frac{(A+ixI)_n(A-ixI)_n}{n!}(A+B)_n^{-1}(A+C)_n^{-1}(A+D)_n^{-1}\\
&\quad= \sum_{k=0}^{n}(-1)^{n-k}\frac{(-nI)_k}{n!}(A+B+C+D+2kI-I)\\
&\qquad\times(A+B+C+D+kI-I)_{n+1}^{-1}(A+B)_k^{-1}(A+C)_k^{-1}(A+D)_k^{-1}P_k(x^2).
\end{aligned}$$

Similarly, replacing $-A$ by $A+B+I$ and putting

$$\mathbf{V}(n) = \frac{(-xI)_n(xI+D+E+I)_n}{n!}(A+I)_n^{-1}(B+E+I)_n^{-1}(D+I)_n^{-1}$$

in (3.6.1), we obtain the following pair of inverse series relation of the Racah matrix polynomials.

$$R_n(x(xI+D+E+I);A,B,D,E)$$

$$= \sum_{k=0}^{n} \frac{(-nI)_k}{k!}(A+B+nI+I)_k(-x)_k(xI+D+E+I)_k(A+I)_k^{-1}$$

$$\times(B+E+I)_k^{-1}(D+I)_k^{-1};$$

$$\frac{(-xI)_n(xI+D+E+I)_n}{k!}\,[(A+I)_n]^{-1}(B+E+I)_n^{-1}[(D+I)_n]^{-1}$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\frac{(-nI)_k}{n!}(A+B+2kI+I)\ (A+B+kI+I)_{n+1}^{-1}$$

$$\times R_k(x(xI+D+E+I);A,B,D,E).$$

Since the classical polynomials of Wilson and that of Racah contain a number of polynomials, namely the polynomilas of Hahn, dual Hahn, continuous Hahn, continuous dual Hahn, Meixner - Pollaczek, Meixner, Krawtchouk, Jacobi, Charlier, Bessel, Laguerre, Hermite etc. (see [5, p. 46] and [65, p. 183] for complete reducibility chart), it will be interesting to examine the reducibility of their matrix analogues from these two matrix polynomials together with their inverse series relations.

We now show the particularization of the Racah matrix polynomial

- Dual Hahn matrix polynomialand its inverse series relation:

  If $B = \beta I, E = \delta I, A+I = -NI, N \in \mathbb{N}$, are substituted in (3.6.2). Further, if $\beta \to \infty$, then

$R_n(x(xI+D+\delta I+I);N,\beta I,D,\delta I)\to D_n(x(xI+D+\delta I+I);N,\beta I,D,\delta I)$, thus

$$D_n(x(xI+D+\delta I+I);N,\beta I,D,\delta I) = \sum_{k=0}^{n}\frac{(-nI)_k}{k!}(-x)_k(xI+\delta I+D+I)_k(-NI)_k^{-1}(D+I)_k^{-1} \tag{3.6.2}$$

$\Leftrightarrow$

$$\frac{(-xI)_n(xI+D+\delta I+I)_n}{n!}[(-NI)_n]^{-1}[(D+I)_n]^{-1} = \sum_{k=0}^{n}\frac{(-nI)_k}{n!}\,D_k(x(xI+D+\delta I+I);N,\beta I,D,\delta I).$$

- Hahn matrix polynomial and its inverse series relation:

  If $B=\beta I, E=\delta I$, $D+I=-NI$ substituted in (3.6.2), we get with $\delta\to\infty$, $R_n(x(xI+\delta I-NI);A,\beta I,N,\delta I)\to Q_n(x,A,\beta I,,\delta I)$, then

$$Q_n(x,A,\beta I,\delta I) = \sum_{k=0}^{n}\frac{(-nI)_k}{k!}(-xI)_k(nI+A+\beta I+I)_k(-NI)_k^{-1}(A+I)_k^{-1} \tag{3.6.3}$$

$\Leftrightarrow$

$$\frac{(-xI)_n}{n!}[(-NI)_n]^{-1}[(A+I)_n]^{-1} = \sum_{k=0}^{n}\frac{(-nI)_k}{k!}(I+A+\beta I+2kI)(kI+A+\beta I+I)_{n+1}^{-1}\,Q_k(x,A,\beta I,,\delta I).$$

- Jacobi matrix polynomial and its inverse series relation:

  To find the Jacobi polynomials from the Hahn polynomials (3.6.3), we take $x\to Nx$ and $N\to\infty$, then we get

$$P_n^{(A,B)}(x)=\sum_{k=0}^{n}\frac{(-nI)_k}{n!\;k!}(nI+A+B+I)_k(A+I)_n(A+I)_k^{-1}\left(\frac{1-x}{2}\right)^k \tag{3.6.4}$$

$\Leftrightarrow$

$$\frac{(A+I)_n^{-1}}{n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{n}(-1)^k\frac{(-nI)_k}{k!}(A+B+2kI+I) \times(A+B+I)_{n+k+1}^{-1}(A+I)_k^{-1}P_k^{(A,B)}(x).$$

- Krawtchouk matrix polynomial and its inverse series relation:

  If $D = ptI$, $\delta I = (1-p)tI$ and taking $t \to \infty$, in (3.6.2) we get

$$\begin{aligned} K_n(x, pI, NI) &= \sum_{k=0}^{n}\frac{(-nI)_k(-NI)_k}{k!}(-xI)_k \\ &\Leftrightarrow \\ (-NI)_n^{-1}\frac{(-xI)_n}{n!} &= \sum_{k=0}^{n}\frac{(-nI)_k}{k!}K_n(x, pI, NI)\ . \end{aligned} \tag{3.6.5}$$

- Meixner matrix polynomial and its inverse series relation:

  If $D = (\beta-1)I$, $\delta I = N(1-c)c^{-1}I$ and taking $N \to \infty$, in (3.6.2) we get

$$\begin{aligned} M_n(x, \beta I, cI) &= \sum_{k=0}^{n}\frac{(-nI)_k(\beta I)_k^{-1}}{k!}(-xI)_k \\ &\Leftrightarrow \\ (\beta I)_n^{-1}\frac{(-xI)_n}{n!} &= \sum_{k=0}^{n}\frac{(-nI)_k}{k!}M_n(x, \beta I, cI)\ . \end{aligned} \tag{3.6.6}$$

- *Extended* Legendre matrix polynomials and its inverse series relation:

  On the other hand, the *extended* Legendre matrix polynomials $P_n(x, C)$ is a special case of (2.5.4) when $A = O, B = O$.

$$P_n(x) = \sum_{k=0}^{n}\frac{(-nI)_k}{k!\ k!}(nI+I)_k\left(\frac{1-x}{2}\right)^k \tag{3.6.7}$$

$$\Leftrightarrow$$

$$\frac{1}{n!n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{n}(-1)^k\frac{(-nI)_k}{k!k!}(2kI+I)(I)_{n+k+1}^{-1}P_k(x).$$

3.7 Application

In this section, the matrix generating functions will be derived from the first series of the inverse pair (3.6.1); whereas from the second series, certain matrix summation formulas will be obtained.

3.7.1 Generating Function Relations

Theorem 3.7.1. *For a positive stable matrix C in $\mathbb{A}^{p\times p}$ and $|t|<1$, the following generating function relation holds.*

$$\sum_{n=0}^{\infty} U(n)\ t^n = (1-t)^{-A} \sum_{k=0}^{\infty}(1-t)^{-2kI}(-A)_{2k}\, V(k)\ t^k.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} U(n)\ t^n &= \sum_{n=0}^{\infty}\frac{(-A)_n}{n!}\sum_{k=0}^{n}(-1)^k(-nI)_k(-A+nI)_k V(k)\ t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-A)_n(-A+nI)_k}{(n-k)!}\ V(k)\ t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-A)_{n+k}(-A+(n+k)I)_k}{n!}\ V(k)\ t^{n+k} \\
&= \sum_{k=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(-A+2kI)_n}{n!}\ t^n\right](-A)_{2k}\ V(k)\ t^k \\
&= \sum_{k=0}^{\infty}(1-t)^{A-2kI}(-A)_{2k}\ V(k)\ t^k \\
&= (1-t)^{A}\sum_{k=0}^{\infty}(1-t)^{-2kI}(-A)_{2k}\ V(k)\ t^k.
\end{aligned}$$

□

Theorem 3.7.2. *For the invertible matrices $C+nI,\ n=0,1,\ldots$ and $|t|<1$,*

$$\sum_{n=0}^{\infty} F(n)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\ (C)_{rk}\ {}_1F_1\left(C+rkI;C+skI;t\right)\left[(C)_{sk}\right]^{-1} G(k)(-t)^{sk}.$$

Proof. From the first series of (2.6.1), we have the left hand side

$$\sum_{n=0}^{\infty} F(n)\frac{t^n}{n!} \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}(-nI)_{sk}(C+nI)_{lk}G(k)\frac{t^n}{n!}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-1)^{sk}\ (C)_{n+lk}}{(n-sk)!}[(C)_n]^{-1}G(k)t^n\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{sk}(C)_{n+rk}}{n!}[(C)_{n+sk}]^{-1}\ G(k)t^{n+sk}\\
&= \sum_{k=0}^{\infty}(C)_{rk}\left(\sum_{n=0}^{\infty}\frac{(C+rkI)_n}{n!}[(C+skI)_n]^{-1}\ t^n\right)\\
&\quad \times[(C)_{sk}]^{-1}G(k)(-t)^{sk}\\
&= \sum_{k=0}^{\infty}\ (C)_{rk}\ {}_1F_1\left(C+rkI;C+skI;t\right)[(C)_{sk}]^{-1}\ G(k)(-t)^{sk}.
\end{aligned}$$

□

Theorem 3.7.3.

$$\begin{aligned}
&\sum_{n=0}^{\infty}[(A_1)_n]^{-1}[(A_2)_n]^{-1}[(A_3)_n]^{-1}P_n(x^2)\frac{t^n}{n!}\\
&\quad = \sum_{n=0}^{\infty}\frac{t^n}{n!}\ {}_4F_4(\prec 2;R+nI\succ,z_1I,z_2I;R+nI,A_1,A_2,A_3;-t)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}[(A_1)_n]^{-1}[(A_2)_n]^{-1}[(A_3)_n]^{-1}P_n(x^2)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}(-1)^k\frac{(R+nI)_k}{(n-k)!k!}(z_1I)_k(z_2I)_k[(A_1)_n]^{-1}[(A_2)_n]^{-1}[(A_3)_n]^{-1}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\Gamma(R+nI+2kI)\Gamma^{-1}(R+nI+kI)(z_1I)_k(z_2I)_k[(A_1)_n]^{-1}[(A_2)_n]^{-1}\\
&\quad\times[(A_3)_n]^{-1}\frac{(-t)^k}{k!}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}(R+nI)_{2k}(R+nI)_k^{-1}(z_1I)_k(z_2I)_k[(A_1)_n]^{-1}[(A_2)_n]^{-1}[(A_3)_n]^{-1}\frac{(-t)^k}{k!}\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\ {}_4F_4(\prec 2;R+nI\succ,z_1I,z_2I;R+nI,A_1,A_2,A_3;-t),
\end{aligned}$$

for all t. □

On the other hand, if $A+B+I=Q, Q+nI=Q_n, D+E+I=B_1, A+I=B_2, B+E+I=B_3$, and $D+I=B_4$, then the generating function relation holds.

$$\begin{aligned}
&\sum_{n=0}^{\infty} R_n(x(xI+D+E),A,B,C,D,E)\frac{t^n}{n!}\\
=\;&\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!}(Q+nI)_k(-xI)_k(xI+B_1)_k[(B_2)_k]^{-1}\\
&\times[(B_3)_k]^{-1}[(B_4)_k]^{-1}\\
=\;&\sum_{n=0}^{\infty} \frac{t^n}{n!}\sum_{k=0}^{\infty}\Gamma(Q+nI+2kI)\Gamma^{-1}(Q+nI+kI)(-xI)_k(xI+B_1)_k\\
&\times[(B_2)_k]^{-1}[(B_3)_k]^{-1}[(B_4)_k]^{-1}\frac{(-t)^{sk}}{k!}\\
=\;&\sum_{n=0}^{\infty} \frac{t^n}{n!}\sum_{k=0}^{\infty}(Q+nI)_{2k}(Q+nI)_k^{-1}(-xI)_k(xI+B_1)_k[(B_2)_k]^{-1}\\
&\times[(B_3)_k]^{-1}[(B_4)_k]^{-1}\frac{(-t)^k}{k!}\\
=\;&\sum_{n=0}^{\infty} \frac{t^n}{n!}\;{}_4F_4(\prec 2;Q_n\succ,-xI,xI+B_1;Q_n,B_2,B_3,B_4;-t)
\end{aligned}$$

for all t.

3.7.2 Summation Formulas

The inverse series relation of the generalized Humbert matrix polynomial is given by

$$\begin{aligned}
\frac{(mx)^n}{n!}I \;=\; &\sum_{k=0}^{[n/m]} \frac{\eta^k}{k!}\,(A-nI+mkI)\,(A-nI+kI)^{-1}\,c^{nI-kI-A}\\
&\times(-A)_{n-k}^{-1}\,P_{n-mk}^{A}(m,x,\eta,c). \qquad (3.7.1)
\end{aligned}$$

Taking infinite sum both sides, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(mx)^n}{n!}I &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{\eta^k}{k!}\,(A-nI+mkI)\,(A-nI+kI)^{-1}\,c^{nI-kI-A}\\ &\quad\times(-A)^{-1}_{n-k}\,P^A_{n-mk}(m,x,\eta,c)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{-\eta^k}{k!}\,(A-nI)\,c^{nI+mkI-kI-A}\,(-A)^{-1}_{n+mk-k+1}\\ &\quad\times P^A_n(m,x,\eta,c)\\ &= \sum_{n=0}^{\infty}c^{nI-A}\,(-A)^{-1}_n\,P^A_n(m,x,\eta,c)\sum_{k=0}^{\infty}\frac{\eta^k}{k!}\,c^{mkI-kI}\\ &\quad\times\,(-A+nI+I)^{-1}_{mk-k}.\end{aligned}$$

We thus get for $x\in\mathbb{R}$,

$$\begin{aligned}e^{mx}I &= \sum_{n=0}^{\infty}c^{nI-A}\,(-A)^{-1}_n\,P^A_n(m,x,\eta,c)\sum_{k=0}^{\infty}\frac{\eta^k}{k!}\,c^{mkI-kI}\\ &\quad\times(-A+nI+I)^{-1}_{mk-k}.\end{aligned}\tag{3.7.2}$$

Also from (3.7.1), we have for $|mx|<1$,

$$\begin{aligned}\sum_{n=0}^{\infty}(mx)^nI &= \sum_{n=0}^{\infty}n!\sum_{k=0}^{[n/m]}\frac{\eta^k}{k!}\,(A-nI+mkI)\,(A-nI+kI)^{-1}\,c^{nI-kI-A}\\ &\quad\times(-A)^{-1}_{n-k}\,P^A_{n-mk}(m,x,\eta,c)\\ &= \sum_{n=0}^{\infty}c^{nI-A}\,(-A)^{-1}_n\,P^A_n(m,x,\eta,c)\sum_{k=0}^{\infty}\frac{\eta^k}{k!}\,(n+mk)!\,c^{mkI-kI}\\ &\quad\times(-A+nI+I)^{-1}_{mk-k}.\end{aligned}$$

Hence, we get

$$\frac{c^A}{1-mx}=\sum_{n=0}^{\infty}c^{nI}\,(-A)^{-1}_n\,P^A_n(m,x,\eta,c)\sum_{k=0}^{\infty}\frac{\eta^k}{k!}\,(n{+}mk)!\,c^{mkI-kI}\,(-A{+}nI{+}I)^{-1}_{mk-k}.$$

The summation formulas involving the Wilson matrix polynomial and the Racah matrix polynomial are obtained from their respective inverse series (2.6.2) and (3.6.2). They are stated below.

With $A+B+C+D+I=R$, and applying $\sum t^n$ both sides in (2.6.2), gives

$$
\begin{aligned}
&{}_2F_3\left(A+ixI, A-ixI; A+B, A+C, A+D; t\right)\\
&=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}(-nI)_k(R+2kI)[(R+kI)_{n+1}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\
&\times[(A+D)_k]^{-1}P_k(x^2).
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k}{n!k!}(R+2kI)[(R+kI)_{n+k+1}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\
&\times[(A+D)_k]^{-1}P_k(x^2)t^k
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(R+2kI+I)_n^{-1}}{n!}t^n\right\}[(R)_{2k}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\
&\times[(A+D)_k]^{-1}P_k(x^2)t^k
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=0}^{\infty}\ {}_0F_1[-;R+2kI+I;t][(R)_{2k}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\
&\times[(A+D)_k]^{-1}P_k(x^2)t^k.
\end{aligned}
$$

Similarly, considering $\sum t^n$ both sides in (3.6.2), yields the sum:

$$
\begin{aligned}
&{}_2F_3\left(-xI, xI+D+E+I; A+I, B+E+I, D+I; t\right)\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(sn)!}\sum_{k=0}^{sn}(-snI)_k(A+B+(rk/s)I+I)\ [(A+B+kI+I)_{ln+1}]^{-1}\\
&\times R_{k,l,s}(x(xI+D+E+I);A,B,D,E). \qquad (3.7.3)
\end{aligned}
$$

Here it is noteworthy that for the particular values of x, a number of sums may be obtained. For example, for $x=0$, the sum (3.7.3) simplifies to

$$
\begin{aligned}
& {}_2F_3(A,A;A+B,A+C,A+D;t) \\
= & \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{k=0}^{sn} (-snI)_k (R+(rk/s)I)[(R+kI)_{ln+1}]^{-1} \\
& \times \sum_{j=0}^{\lfloor k/s \rfloor} \frac{(-kI)_{sj}}{j!} (R+kI)_{lj} (A)_j (A)_j [(A+B)_j]^{-1} [(A+C)_j]^{-1} [(A+D)_j]^{-1} \\
= & \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{sn-sj} (-1)^k \binom{sn-sj}{k} \frac{(k+sj)!}{(sn-sj)!k!j!} (R+(r/s)kI+rjI) \\
& \times [(R+kI+sjI)_{ln+1}]^{-1} (R+kI+sjI)_{lj} [(A)_j]^2 [(A+B)_j]^{-1} [(A+C)_j]^{-1} \\
& \times [(A+D)_j]^{-1} t^n \\
= & \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{j=0}^{\infty} \sum_{k=0}^{sn} (-1)^k \binom{sn}{k} \frac{(k+sj)!}{k!j!} (R+(r/s)kI+rjI) \\
& \times (R+kI+sjI)^{-1}_{ln-sj+1} (R+kI+sjI)_{lj} [(A)_j]^2 [(A+B)_j]^{-1} [(A+C)_j]^{-1} \\
& \times [(A+D)_j]^{-1}
\end{aligned}
$$

On the other hand, when $x = 0$, then since $R_{n,l,s}(0; A, B, C, D) = I$, the sum (3.7.3) gets reduced to the elegant form:

$$
I = \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{k=0}^{sn} (-snI)_k (A+B+(rk/s)I+I) \, [(A+B+kI+I)_{ln+1}]^{-1}.
$$

3.8 Matrix analogues of Riordan's Inverse Pairs

In [82, Ch. 2], John Riordan studied and classified a number of inverse series relations into several classes; among them we refer to here the Gould classes, Simpler Legendre classes and the Legendre-Chebyshev classes. These inverse pairs are provided the matrix extension in this section with the aid of the alternative inverse pairs obtained in section 3.5. They are tabulated below.

Table-1: Gould matrix classes

$$F(n) = \sum C_{n,k} G(k); \quad G(n) = \sum (-1)^{n-k} D_{n,k} F(k)$$

$$(\; h_{r,s} = qr - s, B = A + I \;)$$

Inv. pair No.	b	r	A	$C_{n,k}$	$D_{n,k}$
(3.5.1)	-1	$1-q$	A	$\frac{\Gamma(B + h_{k,k} I)}{(n-k)!} \times \Gamma^{-1}(B + h_{k,n} I)$	$\frac{(A + (h_{k,k}) I)}{(n-k)!} \times \Gamma^{-1}(B + h_{n,n} I) \times \Gamma(A + (h_{n,k}) I)$
(3.5.3)	-1	$q-1$	A	$\frac{(B + h_{n,n} I)}{(n-k)!} \times \Gamma(B + h_{k,k} I) \times \Gamma^{-1}(B + (h_{k,n}+1) I)$	$\frac{\Gamma(B + h_{n,k} I)}{(n-k)!} \times \Gamma^{-1}(B + h_{n,n} I)$
(3.5.5)	1	$q-1$	A	$\frac{(A + (h_{n,n}) I)}{(k-n)!} \times \Gamma(A + (h_{k,n}) I) \times \Gamma^{-1}(B + h_{k,k} I)$	$\frac{\Gamma(B + h_{n,n} I)}{(k-n)!} \times \Gamma^{-1}(B + h_{n,k} I)$
(3.5.4)	1	$q-1$	$-A-I$	$\frac{\Gamma(B + h_{k,n} I)}{(k-n)!} \times \Gamma^{-1}(B + h_{k,k} I)$	$\frac{(B + h_{k,k} I)}{(k-n)!} \times \Gamma(B + h_{n,n} I) \times \Gamma^{-1}(B + (h_{n,k}+1) I)$

Table-2A: Simpler Legendre matrix classes-I

$$F(n) = \sum C_{n,k}\, G(k); \quad G(n) = \sum (-1)^{n-k}\, D_{n,k}\, F(k)$$

$$(B = A + I)$$

Inv. pair No.	b	r	A	$C_{n,k}$	$D_{n,k}$
(3.5.4)	-1	2	$-A-I$	$\frac{\Gamma(B+nI+kI)}{(n-k)!}$ $\times\Gamma^{-1}(B+2kI)$	$\frac{(B+2kI)}{(n-k)!}$ $\times\Gamma(B+2nI)$ $\times(B+nI+kI)^{-1}$ $\times\Gamma^{-1}(B+nI+kI)$
(3.5.5)	-1	2	A	$\frac{(A+2nI)}{(n-k)!}$ $\times\Gamma(A+nI+kI)$ $\times\Gamma^{-1}(B+2kI)$	$\frac{\Gamma(B+2nI)}{(n-k)!}$ $\times\Gamma^{-1}(B+nI+kI)$
(3.5.3)	1	2	A	$\frac{(B+2nI)}{(k-n)!}$ $\times\Gamma(B+2kI)$ $\times\Gamma^{-1}(A+nI+kI+2I)$	$\frac{\Gamma(B+nI+kI)}{(k-n)!}$ $\times\Gamma^{-1}(B+2nI)$
(3.5.1)	1	-2	A	$\frac{\Gamma(B+2kI)}{(k-n)!}$ $\times\Gamma^{-1}(B+nI+kI)$	$\frac{(A+2kI)}{(k-n)!}$ $\times\Gamma(A+nI+kI)$ $\times\Gamma(A+nI+kI)$ $\times\Gamma^{-1}(B+2nI)$

Table-2B : Simpler Legendre matrix classes-II

$$F(n) = \sum C_{n,k} G(n-2k); \quad G(n) = \sum (-1)^k D_{n,k} F(n-2k)$$

$(B = A + I)$

Inv. pair No.	b	r	A	$C_{n,k}$	$D_{n,k}$
(3.5.5)	-2	2	A	$\frac{(A+2nI)}{k!}$ $\times\Gamma(A+2nI-3kI)$ $\times\Gamma^{-1}(B+2nI-4kI)$	$\frac{\Gamma(B+2nI)}{k!}$ $\times\Gamma^{-1}(B+2nI-kI)$
(3.5.4)	-2	2	$-A-I$	$\frac{\Gamma(B+2nI-3kI)}{k!}$ $\times\Gamma^{-1}(B+2nI-4kI)$	$\frac{(B+2nI-4kI)}{k!}$ $\times(B+2nI-kI)^{-1}$ $\times\Gamma(B+2nI)$ $\times\Gamma^{-1}(B+2nI-kI)$

Table-3 : Legendre-Chebyshev matrix classes

$$F(n) = \sum C_{n,k} G(k); \quad G(n) = \sum (-1)^{n-k} D_{n,k} F(k)$$

$$U = A + cnI, \quad V = A + ckI; \qquad U + I = C, \quad V + I = D$$

Inv. pair No.	b	r	A	$C_{n,k}$	$D_{n,k}$
(3.5.5)	-1	A	c	$\frac{U\Gamma(V+nI-kI)}{(n-k)!}$ $\times\Gamma^{-1}(D)$	$\frac{\Gamma(C)}{(n-k)!}$ $\Gamma^{-1}(U-nI+kI+I)$
(3.5.5)	1	A	c	$\frac{U\Gamma(V+kI-nI)}{(k-n)!}$ $\times\Gamma^{-1}(D)$	$\frac{\Gamma(C)}{(k-n)!}$ $\times\Gamma^{-1}(C-kI+nI)$
(3.5.1)	-1	A	$-c$	$\frac{\Gamma(D)}{(n-k)!}$ $\times\Gamma^{-1}(D+kI-nI)$	$\frac{V\ \Gamma(U+nI-kI)}{(n-k)!}$ $\times\Gamma^{-1}(C)$
(3.5.1)	1	A	$-c$	$\frac{D}{(k-n)!}$ $\times\Gamma^{-1}(D-kI+nI)$	$\frac{V\Gamma(U+kI-nI)}{(k-n)!}$ $\times\Gamma^{-1}(C)$
(3.5.4)	-1	$-A-I$	c	$\frac{\Gamma(D+nI-kI)}{(n-k)!}$ $\times\Gamma^{-1}(D)$	$\frac{D\ \Gamma(C)}{(n-k)!}$ $\times\Gamma^{-1}(C-nI+kI+I)$
(3.5.4)	1	$-A-I$	c	$\frac{\Gamma(D+kI-nI)}{(k-n)!}$ $\times\Gamma^{-1}(D)$	$\frac{D\ \Gamma(C)}{(k-n)!}$ $\times\Gamma^{-1}(C+nI-kI+I)$
(3.5.3)	-1	A	c	$\frac{C\ \Gamma(D)}{(n-k)!}$ $\Gamma^{-1}(D-nI+kI+I)$	$\frac{\Gamma(C+nI-kI)}{(n-k)!}$ $\Gamma^{-1}(C)$
(3.5.3)	1	A	c	$\frac{C\ \Gamma(V+I)}{(k-n)!}$ $\Gamma^{-1}(D+nI-kI+I)$	$\frac{\Gamma(C-nI+kI)}{(k-n)!}$ $\Gamma^{-1}(C)$