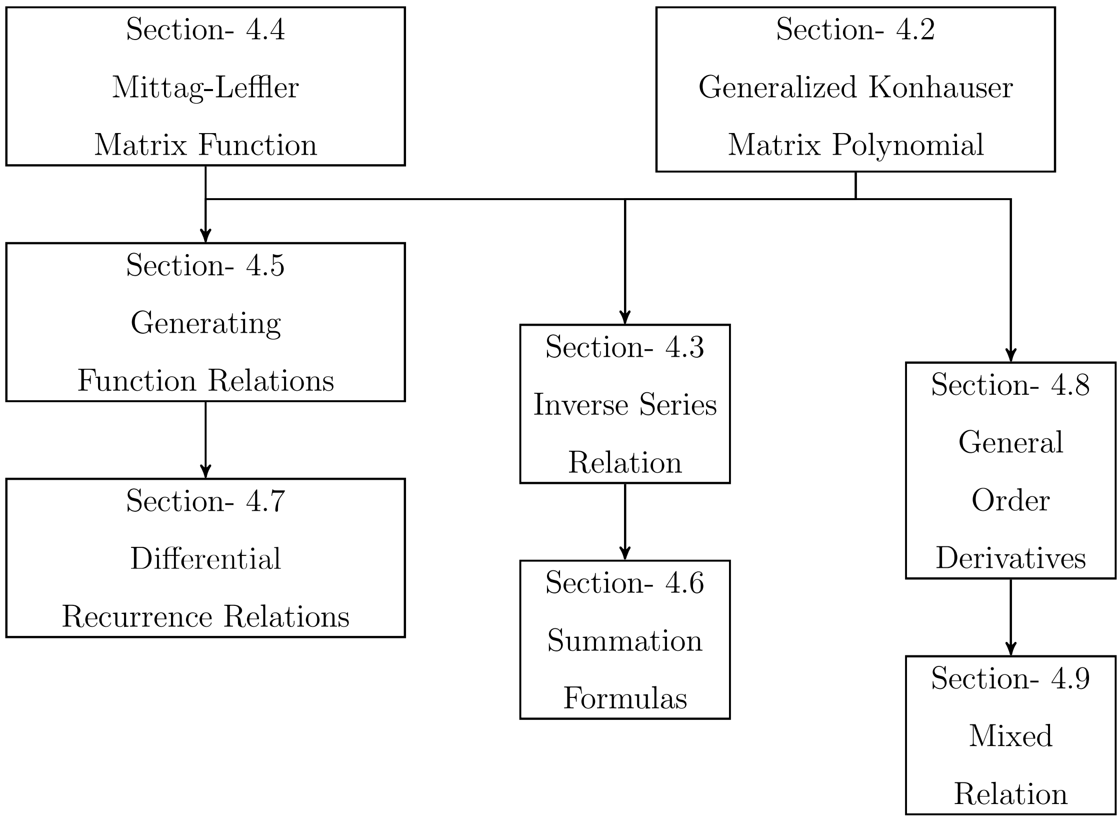
Chapter 4

Generalized Konhauser Matrix Polynomial and its Properties



4.1 Introduction

It is well known that the Konhauser polynomial:

$$Z_m^{\alpha}(x;r) = \frac{\Gamma(rm+\alpha+1)}{\Gamma(m+1)} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{x^{rn}}{\Gamma(rn+\alpha+1)}, \quad (\Re(\alpha) > -1)$$

is the biorthogonal polynomial for the distribution function of the Laguerre polynomial which was introduced by J. D. E. Konhauser [66, Eq.(5), p. 304]. This polynomial was generalized in the form [78, Eq.(5), p.640]:

$$L^{(\alpha,\beta)}_{[\frac{m}{q}]}(z) = \frac{\Gamma(\alpha m+\beta+1)}{m!} \sum_{n=0}^{[\frac{m}{q}]} \frac{(-m)_{qn}}{\Gamma(\alpha n+\beta+1)} \frac{z^n}{n!}, \tag{4.1.1}$$

where $\alpha, \beta \in \mathbb{C}, m, q \in \mathbb{N}, \Re(\beta) > -1$ and $[\frac{m}{q}]$ denotes the integral part of $\frac{m}{q}$. The objective here is to provide a matrix extension to this polynomial and derive certain properties.

4.2 Generalized Konhauser Matrix Polynomial

We propose the extension in matrix form of (4.1.1) as follows.

Definition 4.2.1. *For a matrix A in $C^{p\times p}$,*

$$Z^{(A,\lambda)}_{m^*}(x^k;r) = \frac{\Gamma(A+rmI+I)}{m!} \sum_{n=0}^{\lfloor m/s \rfloor} (-mI)_{sn} \Gamma^{-1}(A+rnI+I) \frac{(\lambda x^k)^n}{n!}, \tag{4.2.1}$$

where $r, \lambda, \mu \in \mathbb{C}$; $k \in \mathbb{R}_{>0}$, $s \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\Re(\lambda) > 0$, $\Re(\mu) > -1$ for all eigen values $\mu \in \sigma(A)$ and the floor function $\lfloor u \rfloor = floor\ u$, represents the greatest integer $\leq u$.

It may be seen that when $r = k \in \mathbb{N}$ and $s = 1$, then this polynomial reduces to

$$Z^{(A,\lambda)}_{m}(x;k) = \Gamma(kmI+A+I) \sum_{n=0}^{m} \frac{(-1)^n (\lambda x)^{nk}}{(m-n)!n!} \Gamma^{-1}(knI+A+I)$$

which was studied by Varma, Çekim, and Taşdelen [108]. Further if $k = 1$, then this reduces to the Laguerre matrix polynomial [52]:

$$L_m^{(A,\lambda)}(x) = \sum_{n=0}^{m} \frac{(-1)^n}{n!(m-n)!} \, (A+I)_m \, [(A+I)_n]^{-1} \, (\lambda x)^n.$$

For the polynomial (4.2.1), we derive the differential equation, inverse series relation, the generating function relations, mixed relation etc.
If $B_j + nI$ are invertible for all $n = 0, 1, 2, \ldots,$ then the generalized hypergeometric matrix function [94, Eq. (2.2), p. 608]:

$$\begin{aligned} & {}_pF_q(A_1, A_2, \ldots, A_p; B_1, B_2, \ldots, B_q; z) \\ = & \sum_{k=0}^{\infty} (A_1)_k (A_2)_k \cdots (A_p)_k \, [(B_1)_k]^{-1} [(B_2)_k]^{-1} \cdots [(B_q)_k]^{-1} \frac{z^k}{k!} \end{aligned} \tag{4.2.2}$$

satisfies the matrix differential equation [94, Eq. (2.10), p. 610]:

$$\left[\theta \prod_{j=1}^{q} (\theta I + B_j - I) - z \prod_{i=1}^{p} (\theta I + A_i) \right] \, {}_pF_q(z) = O, \tag{4.2.3}$$

where $\theta = zd/dz$ and O is the zero matrix of order same as the order of the matrices $A_i' s$ and $B_j' s$. Here, if we express the polynomial (4.2.1) in ${}_pF_q$ form then the equation (4.2.3) will readily yield the differential equation corresponding to the polynomial (4.2.1). In fact, for $r,\ s \in \mathbb{N}$ the polynomial is expressible in the desired form as below.

$$\begin{aligned} Z_{m^*}^{(A,\lambda)}(x^k; r) &= \frac{\Gamma(A + rmI + I)}{m!} \Gamma^{-1}(A+I) \sum_{n=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sn} (A+I)_{rn}^{-1} \, (\lambda x^k)^n}{n!} \\ &= \frac{\Gamma(A + rmI + I)}{m!} \Gamma^{-1}(A+I) \sum_{n=0}^{\lfloor m/s \rfloor} \left\{ \prod_{i=1}^{s} \left(\frac{-m+i-1}{s} I \right)_n \right\} \\ & \quad \times \left\{ \prod_{j=1}^{r} \left(\frac{A + jI}{r} \right)_n^{-1} \right\} \frac{1}{n!} \left(\frac{\lambda x^k s^s}{r^r} \right)^n . \end{aligned}$$

Hence, in (4.2.2), setting
$p = s,\ q = r,\ A_i = (-m+i-1)I/s,\ B_j = (A+jI)/r,\ z = \lambda s^s x^k / r^r$, the equation immediately leads us to the differential equation for (4.2.1) of order

$max.\{r+1,\ s\}$. It is stated in

Theorem 4.2.1. *If $r, s \in \mathbb{N}$ and the operator Θ is defined by $\Theta f(x) = \frac{x}{k}\frac{d}{dx}f(x)$ then $U = Z_{m^*}^{(A,\lambda)}(x^k; r)$ satisfies the equation*

$$\Bigg[\Bigg\{\Theta\prod_{j=1}^{r}\left(\Theta\ I + \frac{A+jI}{r} - I\right)\Bigg\} - \left(\frac{s^s}{r^r}\right)\ \lambda\ x^k\ \Bigg\{\prod_{i=1}^{s}\left(\Theta\ I + \frac{-m+i-1}{s}I\right)\Bigg\}\Bigg]U = O.$$

4.3 Inverse Series Relation

For deriving the inverse series of the matrix polynomial (4.2.1), the following lemma will be used.

Lemma 4.3.1. *If $\{P_n\}$ and $\{Q_n\}$ are finite sequences of matrices in $C^{p\times p}$, then*

$$Q_n = \sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ P_j \ \Leftrightarrow\ P_n = \sum_{j=0}^{n}\frac{(-nI)_j}{j!}\ Q_j.$$

Proof. Let us denote the right hand side of second series by the matrix T_n, then substituting the series for Q_k we get

$$\begin{aligned} T_n &= \sum_{k=0}^{n}\frac{(-nI)_k}{k!}\ Q_k \\ &= \sum_{k=0}^{n}\frac{(-1)^k n!}{k!\ (n-k)!}\ I\sum_{j=0}^{k}\frac{(-kI)_j}{j!}\ P_j \\ &= \sum_{k=0}^{n}\frac{(-1)^k n!}{k!\ (n-k)!}I\ \sum_{j=0}^{k}\frac{(-1)^j\ k!}{j!\ (k-j)!}\ P_j. \end{aligned}$$

Using the double series relation (1.3.31), we further get

$$\begin{aligned} T_n &= \sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}(-1)^k\ \binom{n-j}{k}\ P_j \\ &= P_n + \sum_{j=0}^{n-1}\ \binom{n}{j}\ \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\ P_j. \end{aligned}$$

Thus, $T_n = P_n$ and hence, first series implies the second series. The converse part follows by just interchanging P_r and Q_r in this proof. Hence it is omitted for the sake of brevity. □

Using this lemma, we now establish the inverse series relation in

Theorem 4.3.1. *For a matrix* $A \in C^{p\times p},\ r,\lambda \in \mathbb{C},\ s \in \mathbb{N},\ m \in \mathbb{N}\cup\{0\}$,

$$Z_{m^*}^{(A,\lambda)}(x^k;r) = \frac{\Gamma(A+rmI+I)}{m!}\sum_{j=0}^{\lfloor m/s\rfloor}(-mI)_{sj}\Gamma^{-1}(A+rjI+I)\frac{(\lambda x^k)^j}{j!} \quad (4.3.1)$$

if and only if

$$\frac{(\lambda x^k)^m}{m!}I = \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{(A,\lambda)}(x^k;r), \quad (4.3.2)$$

and for $m \neq sl,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r) = O, \quad (4.3.3)$$

where O *is the zero matrix of order* p.

Proof. We first show that the series (4.3.1) implies both (4.3.2) and (4.3.3). The proof of (4.3.1) implies (4.3.2) runs as follows.
Denoting the right hand side of (4.3.2) by the matrix Ξ_m, and substituting for $Z_{j^*}^{(A,\lambda)}(x^k;r)$ from (4.3.1) and then using the double series relation (1.3.28), we get

$$\begin{aligned}
\Xi_m &= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,\lambda)}(x^k;r)\\
&= \frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}\frac{(-msI)_j}{j!}\sum_{i=0}^{\lfloor j/s\rfloor}(-jI)_{si}\ \Gamma^{-1}(A+riI+I)\ \frac{(\lambda x^k)^i}{i!}\\
&= \sum_{j=0}^{ms}\sum_{i=0}^{\lfloor j/s\rfloor}\frac{\Gamma(A+rmI+I)\ (-1)^{j+si}\ \Gamma^{-1}(A+riI+I)}{(ms-j)!\ (j-si)!\ i!}(\lambda x^k)^i\\
&= \sum_{i=0}^{m}\sum_{j=0}^{ms-si}\frac{\Gamma(A+rmI+I)\ (-1)^{j}\ \Gamma^{-1}(A+riI+I)}{(ms-si-j)!\ j!\ i!}(\lambda x^k)^i
\end{aligned}$$

$$= \frac{(\lambda x^k)^m}{m!} I + \sum_{i=0}^{m-1} \frac{\Gamma(A + rmI + I)\ \Gamma^{-1}(A + riI + I)}{(ms - si)!\ i!} (\lambda x^k)^i$$
$$\times \sum_{j=0}^{ms-si} (-1)^j \binom{ms - si}{j}.$$

Here the inner sum in the second term on the right hand side vanishes, consequently, we arrive at $\Xi_m = \frac{(\lambda x^k)^m}{m!} I$.
To show further that (4.3.1) also implies (4.3.3), let us substitute for $Z_{j^*}^{(A,\lambda)}(x^k; r)$ from (4.3.1) to the left hand side of (4.3.3). Then in view of (1.3.28), we get

$$\sum_{j=0}^{m} (-mI)_j \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,\lambda)}(x^k; r)$$
$$= \sum_{j=0}^{m} \frac{(-1)^j m!}{(m - j)!} I \sum_{i=0}^{\lfloor j/s \rfloor} \frac{(-1)^{si}\ \Gamma^{-1}(A + riI + I)}{(j - si)!\ i!} (\lambda x^k)^i$$
$$= \sum_{i=0}^{\lfloor m/s \rfloor} \frac{m!\ \Gamma^{-1}(A + ri + I)}{(m - si)!\ i!} (\lambda x^k)^i \sum_{j=0}^{m-si} (-1)^j \binom{m - si}{j}$$
$$= O,$$

if $m \neq sl,\ l \in \mathbb{N}$. Thus completing the first part. The proof of the converse part which uses the technique employed in [16], runs as follows.
In order to show that the series (4.3.2) and the condition (4.3.3) together imply the series (4.3.1), we use Lemma 4.3.1 with

$$P_j = j!\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,\lambda)}(x^k; r),$$

and consider one sided relation in the lemma that is, the series on the left hand side implies the series on the right hand side. Then

$$Q_m = \sum_{j=0}^{m} (-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,\lambda)}(x^k; r) \tag{4.3.4}$$
$$\Rightarrow$$
$$Z_{m^*}^{(A,\lambda)}(x^k; r) = \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{m} \frac{(-mI)_j}{j!} Q_j. \tag{4.3.5}$$

Since the condition (4.3.3) holds, $Q_m = 0$ for $m \neq sl,\ l \in \mathbb{N}$, whereas

$$Q_{ms} = \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,\lambda)}(x^k; r).$$

Also the series (4.3.2) holds true, whence it follows that

$$\begin{aligned} Q_{ms} &= \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,\lambda)}(x^k; r) \\ &= \frac{(ms)!\ \Gamma^{-1}(A + rmI + I)}{m!} (\lambda x^k)^m. \end{aligned}$$

Consequently, the inverse pair (4.3.4) and (4.3.5) assume the form:

$$\begin{aligned} \frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A + rmI + I)}{(ms)!} \sum_{j=0}^{ms} (-msI)_j\ \Gamma^{-1}(A + rjI + I) \\ &\quad \times Z_{j^*}^{(A,\lambda)}(x^k; r) \\ \Rightarrow \\ Z_{m^*}^{(A,\lambda)}(x^k; r) &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}}{(sj)!} Q_{sj} \\ &= \frac{\Gamma(A + rmI + I)}{m!} \sum_{j=0}^{\lfloor m/s \rfloor} \frac{(-mI)_{sj}\ \Gamma^{-1}(A + rjI + I)}{j!} (\lambda x^k)^j, \end{aligned}$$

subject to the condition (4.3.3). □

4.4 Mittag-Leffler Matrix Function

In 2007, a generalization of the Mittag-Leffler function was introduced in the form [98]:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{4.4.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C},\ \Re(\alpha, \beta, \gamma) > 0,\ q \in (0, 1) \cup \mathbb{N}$. Here we allow q to take value 0 in which case the series retains convergence behavior. Also, if α is allowed to assume value 0 then with $q = 0$ and $\beta = 1$, the reducibility of (4.4.1) to the exponential function e^z occurs. Thus, for $q \geq 0,\ \Re(\alpha) \geq 0, \Re(\beta, \gamma) > 0$ and

$z \in \mathbb{C}$, the function (4.4.1) yields an instance

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)\ n!}. \tag{4.4.2}$$

We define here the matrix analogues of (4.4.1) and (4.4.2) as follows.

Definition 4.4.1. *For $A, B \in C^{p\times p}$, $\Re(\mu) > -1$ for all eigen values $\mu \in \sigma(A), r \in \mathbb{C}$ and $s \in \mathbb{N}$,*

$$E_{rI,A+I}^{B,sI}(z) = \sum_{n=0}^{\infty} (B)_{sn} \Gamma^{-1}(A + rnI + I) \frac{z^n}{n!}. \tag{4.4.3}$$

Definition 4.4.2. *For $A \in C^{p\times p}$, $r \in \mathbb{C}$, $\Re(\mu) > -1$ for all eigen values $\mu \in \sigma(A)$,*

$$E_{rI,A+I}(z) = \sum_{n=0}^{\infty} \Gamma^{-1}(A + rnI + I) \frac{z^n}{n!}. \tag{4.4.4}$$

It is interesting to note that putting $B = -mI$, where $m \in \mathbb{N}$ and $z = \lambda x^k$ in (4.4.3), and comparing it with the function in (4.2.1), we obtain the relation:

$$E_{rI,A+I}^{-mI,sI}(\lambda x^k) = m!\ \Gamma^{-1}(A + rmI + I) Z_{m^*}^{(A,\lambda)}(x^k; r).$$

The functions (4.4.3) and (4.4.4) will be used in the generating function relations derived in the following section.

4.5 Generating Function Relations

We derive the generating function relations for the matrix polynomial $Z_{m^*}^{(A,\lambda)}(x^k; r)$ in the form of Theorems 4.5.1, 4.5.2 and 4.5.3.

Theorem 4.5.1. *Let $r \in \mathbb{C}$, $s \in \mathbb{N}$ and A, B be the matrices in $C^{p\times p}$, $\Re(\mu) > -1$ for all eigenvalues $\mu \in \sigma(A)$, then for $|t| < 1$,*

$$\sum_{m=0}^{\infty} (B)_m\ \Gamma^{-1}(A + rmI + I)\ Z_{m^*}^{(A,\lambda)}(x^k; r)\ t^m$$

$$= \quad (1-t)^{-B} \; E^{B,sI}_{rI,A+I}\left(\lambda x^k(-t)^s(1-t)^{-sI}\right).$$

Proof. Here, substituting the series for $Z^{(A,\lambda)}_{m^*}(x^k;r)$ from (4.2.1) on the left hand side, we get

$$\begin{aligned}
&\sum_{m=0}^{\infty}(B)_m \; \Gamma^{-1}(A+rmI+I) \; Z^{(A,\lambda)}_{m^*}(x^k;r) \; t^m \\
= \; & \sum_{m=0}^{\infty}(B)_m \; \Gamma^{-1}(A+rmI+I)\frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{m!(-1)^{sn}I\Gamma^{-1}(A+rnI+I)}{n!(m-sn)!} \\
& \times(\lambda x^k)^n t^m \\
= \; & \sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-1)^{sn}(B)_m\Gamma^{-1}(A+rnI+I)}{n! \; (m-sn)!} \; (\lambda x^k)^n t^m.
\end{aligned}$$

In view of the double series relation (1.3.26), we further get

$$\begin{aligned}
&\sum_{m=0}^{\infty}(B)_m \; \Gamma^{-1}(A+rmI+I) \; Z^{(A,\lambda)}_{m^*}(x^k;r) \; t^m \\
= \; & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{sn}(B)_{m+sn}\Gamma^{-1}(A+rnI+I)}{n! \; m!} \; (\lambda x^k)^n t^{m+sn} \\
= \; & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(B+snI)_m t^m}{m!}\frac{(-1)^{sn}(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!} \; (\lambda x^k)^n t^{sn} \\
= \; & \sum_{n=0}^{\infty}(1-t)^{-B-snI}\frac{(-1)^{sn}(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!} \; (\lambda x^k)^n t^{sn} \\
= \; & (1-t)^{-B}\sum_{n=0}^{\infty}\frac{(B)_{sn}\Gamma^{-1}(A+rnI+I)}{n!} \; (\lambda x^k(-t)^s(1-t)^{-sI})^n \qquad (4.5.1) \\
= \; & (1-t)^{-B} \; E^{B,sI}_{rI,A+I}\left(\lambda x^k(-t)^s(1-t)^{-sI}\right).
\end{aligned}$$

□

Corollary 4.5.1. *If $r\in\mathbb{N}$, then for $s\le r$ or $s=r+1$,*

$$\begin{aligned}
&\sum_{m=0}^{\infty}(B)_m \; (A+I)^{-1}_{rm} \; Z^{(A,\lambda)}_{m^*}(x^k;r) \; t^m \\
= \; & (1-t)^{-B} \\
& \times {}_sF_r\left(\frac{B}{s},\frac{B+I}{s},\ldots,\frac{B+(s-1)I}{s};\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{s^s}{r^r}\lambda x^k R^s\right),
\end{aligned}$$

where $R=(-t)(1-t)^{-I}$.

Proof. For $r \in \mathbb{N}$, the infinite series on the right hand side in (4.5.1), takes the form

$$(1-t)^{-B}\Gamma^{-1}(A+I)\sum_{n=0}^{\infty}(B)_{sn}(A+I)_{rn}^{-1}\frac{(\lambda x^k R^s)^n}{n!}.$$

This in view of the formula (1.3.23) and the matrix function (4.2.2) leads us to the corollary. □

If $(B)_m$ is dropped from the left hand side of this theorem, then we obtain the following form.

Theorem 4.5.2. *In the usual notations and meaning, there holds the generating function relation:*

$$\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\ Z_{m^*}^{(A,\lambda)}(x^k;r)\ t^m = e^t\ E_{rI,A+I}\left(\lambda x^k(-t)^s\right).$$

Proof. The proof follows in a straight forward manner. In fact, by using the double series relation (1.3.26), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\Gamma^{-1}(A+rmI+I)\ Z_{m^*}^{(A,\lambda)}(x^k;r)\ t^m\\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\ (m-sn)!}(\lambda x^k)^n t^m\\
=&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!\ m!}(\lambda x^k)^n t^{m+sn}\\
=&\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{(-1)^{sn}\Gamma^{-1}(A+rnI+I)}{n!}(\lambda x^k)^n t^{sn}\\
=&\ e^t\ E_{rI,A+I}\left(\lambda x^k(-t)^s\right).
\end{aligned}$$

□

Again for $r \in \mathbb{N}$, we have (cf. [94, Eq. (3.5), p. 619])

Corollary 4.5.2.

$$\begin{aligned}&\sum_{m=0}^{\infty}(A+I)_{rm}^{-1}\ Z_{m^*}^{(A,\lambda)}(x^k;r)\ t^m\\ =\ & e^t\ {}_0F_r\left(--;\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{\lambda x^k(-t)^s}{r^r}\right).\end{aligned}$$

Here the proof follows by proceeding as in corollary 6.2.

Next, in the notations and meaning of Theorem 4.5.1, we have

Theorem 4.5.3. *Let a and b be complex constants which are not zero simultaneously, then the generating function relation holds.*

$$\begin{aligned}&\sum_{n=0}^{\infty}Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\ \Gamma^{-1}(A+rnI+I)\ t^n\\ &=\ e^{ax}\ (1-bte^{bx})^{-1}\ E_{rI,A+I}(\lambda x^k(-t)^s e^{bsx}).\end{aligned}$$

Proof. Beginning with the left hand side, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I)\ t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/s\rfloor}\frac{(-1)^{sj}\ \Gamma^{-1}(A+rjI+I)(\lambda x^k)^j}{(n-sj)!\ j!}(a+bn)^{n-sj}t^n\\ =\ &\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{((-t)^s\lambda x^k)^j\Gamma^{-1}(A+rjI+I)}{j!}\frac{(a+bn+bsj)^n}{n!}t^n. \qquad (4.5.2)\end{aligned}$$

We use here the Lagrange expansion formula [82, Eq. (18), p. 146]:

$$\frac{f(x)}{1-tg'(x)}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[D^n f(x)(g(x))^n\right]_{x=0},\quad (t=x/g(x))$$

by taking $f(x)=e^{(a+bsj)x}$ and $g(x)=e^{bx}$. Then we find

$$\frac{e^{(a+bsj)x}}{1-bte^{bx}}=\sum_{n=0}^{\infty}(a+bsj+bn)^n\ \frac{t^n}{n!}.$$

Thus (4.5.2) simplifies to

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\Gamma^{-1}(A+rnI+I)\,t^n$$
$$=\sum_{j=0}^{\infty}\frac{\Gamma^{-1}(A+rjI+I)}{j!}\,((-t)^s\lambda x^k)^j\,\frac{e^{(a+bsj)x}}{1-bte^{bx}}.$$

This in view of (4.4.4), yields the desired generating function relation. □

Here also for $r\in\mathbb{N}$, we have (cf. [94, Eq. (3.14), p. 621])

Corollary 4.5.3. *There holds the matrix generating function relation:*

$$\sum_{n=0}^{\infty} Z_{n^*}^{(A,\lambda)}\left(\frac{x^k}{(a+bn)^s};r\right)(a+bn)^n\;(A+I)_{rn}^{-1}\;t^n$$
$$=e^{ax}\,(1-bte^{bx})^{-1}$$
$$\times{}_0F_r\left(--;\frac{A+I}{r},\frac{A+2I}{r},\ldots,\frac{A+rI}{r};\frac{\lambda x^k(-t)^s e^{bsx}}{r^r}\right).$$

4.6 Summation Formulas

We illustrate in this section, the use of the inverse series in deriving some summation formulas. The inverse series of $Z_{m^*}^{A,\lambda}(x^k;r)$ is

$$\frac{(\lambda x^k)^m}{m!}I=\frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r). \quad (4.6.1)$$

If the summation m from 0 to ∞ is taken on both sides, then we get the expression:

$$e^{\lambda x^k}I=\sum_{m=0}^{\infty}\frac{\Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r).$$

Next, re-writing the inverse series (4.6.1) as

$$(\lambda x^k)^m I = \frac{m!\ \Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r), \tag{4.6.2}$$

and then apllying summation m from 0 to ∞ and assuming $|\lambda x^k| < 1$, we get

$$\frac{1}{1-\lambda x^k}I = \sum_{m=0}^{\infty}\frac{m!\ \Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r),$$

Further, multiplying by $(B)_m$ and then taking infinite series both sides, yields

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(B)_m}{m!}(\lambda x^k)^m &= (1-\lambda x^k)^{-B}\\ &= \sum_{m=0}^{\infty}\frac{(B)_m\ \Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r),\end{aligned}$$

provided that $|\lambda x^k| < 1$. An interesting summation formula occurs if $\lambda = 1$ and the inverse series is multiplied by $(C+I)_m^{-1}$ and then the infinite series is considered. With this, the Bessel matrix function [92, Eq.(1.8), p.267] appears in the summation formula on the left hand side which is stated below (cf. [80, Eq., p.] with $\lambda = k = r = 1$).

$$\begin{aligned}x^{-kC/2}\ J_C(2x^{k/2})\Gamma(C+I) &= \sum_{m=0}^{\infty}\frac{(C+I)_m^{-1}\ \Gamma(A+rmI+I)}{(ms)!}\\ &\quad\times\sum_{j=0}^{ms}(-msI)_j\ \Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r).\end{aligned}$$

Also from (4.6.2), we have the finite product formula:

$$\prod_{m=1}^{M}(\lambda x^k)^m I = \prod_{m=1}^{M}\left\{\frac{m!\ \Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)Z_{j^*}^{A,\lambda}(x^k;r)\right\}.$$

The product on the left hand side when simplified, yields the formula:

$$\begin{aligned}\lambda^M x^{kM(M+1)/2}I &= \prod_{m=1}^{M}\frac{m!\ \Gamma(A+rmI+I)}{(ms)!}\sum_{j=0}^{ms}(-msI)_j\Gamma^{-1}(A+rjI+I)\\ &\quad\times Z_{j^*}^{A,\lambda}(x^k;r).\end{aligned}$$

4.7 Differential Recurrence Relation

We recall the generating function relation of Theorem 4.5.2 and put

$$\Psi \equiv \Psi(x,t) = e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+I)(\lambda x^k t^s)^n.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial x}\Psi &= e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+I)\ (kn)x^{kn-1}(\lambda t^s)^n \\
&= ke^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+I)\ n\ x^{kn-1}(\lambda t^s)^n \\
&= ke^t \sum_{n=1}^{\infty} \frac{(-1)^{sn}}{(n-1)!}\Gamma^{-1}(A+rnI+I)\ x^{kn-1}(\lambda t^s)^n \\
&= (-1)^s k\lambda e^t x^{k-1} t^s \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+(r+1)I) \\
&\quad \times(\lambda x^k t^s)^n, \qquad (4.7.1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi &= \Psi + e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+I)\ (\lambda x^k)^n\ (sn)t^{sn-1} \\
&= \Psi + s\ e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+I)\ (\lambda x^k)^n\ nt^{sn-1} \\
&= \Psi + s\ e^t \sum_{n=1}^{\infty} \frac{(-1)^{sn}}{(n-1)!}\Gamma^{-1}(A+rnI+I)\ (\lambda x^k)^n\ t^{sn-1} \\
&= \Psi + s\ e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn+s}}{n!}\Gamma^{-1}(A+rnI+(r+1)I)\ \lambda^{n+1}x^{kn+k}\ t^{sn+s-1} \\
&= \Psi + (-1)^s s\lambda x^k t^{s-1} e^t \sum_{n=0}^{\infty} \frac{(-1)^{sn}}{n!}\Gamma^{-1}(A+rnI+(r+1)I) \\
&\quad \times \left(\lambda x^k\ t^s\right)^n. \qquad (4.7.2)
\end{aligned}$$

From (4.7.1) and (4.7.2), we find the partial differential matrix equation:

$$\frac{t}{s}\Psi = \frac{t}{s}\left(\frac{\partial}{\partial t}\Psi\right) - \frac{x}{k}\left(\frac{\partial}{\partial x}\Psi\right). \qquad (4.7.3)$$

But since

$$\Psi = \sum_{n=0}^{\infty} \Gamma^{-1}(A+nrI+I) Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n,$$

we have

$$\frac{\partial}{\partial x}\Psi = \sum_{n=0}^{\infty} \Gamma^{-1}(A+nrI+I)\ \frac{\partial}{\partial x} Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n$$

and

$$\frac{\partial}{\partial t}\Psi = \sum_{n=0}^{\infty} n\ \Gamma^{-1}(A+nrI+I) Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^{n-1}.$$

When all the three series are substituted in (4.7.3), we obtain

$$\begin{aligned}
&\frac{t}{s}\ \sum_{n=0}^{\infty} \Gamma^{-1}(A+nrI+I) Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n \\
&= \frac{1}{s}\ \sum_{n=0}^{\infty} n\ \Gamma^{-1}(A+nrI+I) Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n \\
&- \frac{x}{k}\ \sum_{n=0}^{\infty} \Gamma^{-1}(A+nrI+I)\ \frac{\partial}{\partial x} Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{s}\ \sum_{n=1}^{\infty} \Gamma^{-1}(A+nrI-rI+I) Z_{n-1^*}^{(A,\lambda)}(x^k;r)\ t^n \\
&- \frac{1}{s}\ \sum_{n=1}^{\infty} n\ \Gamma^{-1}(A+nrI+I) Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n \\
&+ \frac{x}{k}\ \sum_{n=1}^{\infty} \Gamma^{-1}(A+nrI+I)\ \frac{\partial}{\partial x} Z_{n^*}^{(A,\lambda)}(x^k;r)\ t^n = O.
\end{aligned} \tag{4.7.4}$$

Afetr multiplying both sides by $ks\ \Gamma(A+I)$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big\{ k(A+I)_{r(n-1)}^{-1}\ Z_{n-1^*}^{(A,\lambda)}(x^k;r) - kn\ (A+I)_{rn}^{-1}\ Z_{n^*}^{(A,\lambda)}(x^k;r) \\
&+ sx\ (A+I)_{rn}^{-1}\ \frac{\partial}{\partial x} Z_{n^*}^{(A,\lambda)}(x^k;r) \Big\}\ t^n = O.
\end{aligned}$$

This ultimately leads us to the differential recurrence relation:

$$\begin{aligned}
&sx\ (A+I)_{rn}^{-1}\ \frac{d}{dx} Z_{n^*}^{(A,\lambda)}(x^k;r) + k(A+I)_{r(n-1)}^{-1}\ Z_{n-1^*}^{(A,\lambda)}(x^k;r) \\
&\quad = kn\ (A+I)_{rn}^{-1}\ Z_{n^*}^{(A,\lambda)}(x^k;r),
\end{aligned} \tag{4.7.5}$$

where $n \geq 1, r \in \mathbb{N}$.

The relation (4.7.4) can also be simplified by multiplying both sides by $\Gamma(A + nrI + I)$. We find the expression:

$$sx \ \frac{d}{dx} Z_{n^*}^{(A,\lambda)}(x^k;r) - kn \ Z_{n^*}^{(A,\lambda)}(x^k;r) + k(A + nrI + I)_{-r}^{-1} \ Z_{n-1^*}^{(A,\lambda)}(x^k;r) = O. \quad (4.7.6)$$

If we insists that $r \in \mathbb{C}$, then the equation (4.7.4) after multiplying by ks, can be re-written as

$$\begin{aligned} &sx \ \Gamma^{-1}(A + nrI + I) \ \frac{d}{dx} Z_{n^*}^{(A,\lambda)}(x^k;r) - kn \ \Gamma^{-1}(A + nrI + I) \ Z_{n^*}^{(A,\lambda)}(x^k;r) \\ &+ k\Gamma^{-1}(A + nrI - rI + I) Z_{n-1^*}^{(A,\lambda)}(x^k;r) = O. \end{aligned}$$

This is a differential recurrence relation.

4.8 General Order Derivative

We obtain a general order derivative of $x^A \ Z_{m^*}^{(A,\lambda)}(x^k, r)$, with the objective of deriving a mixed relation in the subsequent section.

For $l \in \mathbb{N}$, we have

$$\begin{aligned} &\frac{d^l}{dx^l} \left[x^A \ Z_{M^*}^{(A,\lambda)}(x^k, r) \right] \\ &= \Gamma(A + rnI + I) \sum_{j=0}^{[m/s]} \frac{(-1)^{sj}}{(m - sj)!j!} \Gamma^{-1}(A + rjI + I) \lambda^j \frac{d^l}{dx^l} x^{A+kjI}. \end{aligned}$$

Now [6]

$$\frac{d^l}{dx^l} x^{A+kjI} = \Gamma(A + kjI + I)\Gamma^{-1}(A + kjI + I - lI) x^{A+kjI+lI}.$$

Hence,

$$\begin{aligned} &\frac{d^l}{dx^l} \left[x^A \ Z_{m^*}^{(A,\lambda)}(x^k, r) \right] \\ &= \Gamma(A + rmI + I) \sum_{j=0}^{[m/s]} \frac{(-1)^{sj}}{(m - sj)!j!} \Gamma^{-1}(A + rjI + I)\Gamma(A + kjI + I) \\ &\times \Gamma^{-1}(A + kjI + I - lI) \ \lambda^j \ x^{A+kjI+lI}. \end{aligned}$$

In the case when $k = r$, then this further reduces to

$$\begin{aligned}
&\frac{d^l}{dx^l}\left[x^A\ Z_{m^*}^{(A,\lambda)}(x^r,r)\right] \\
&= \Gamma(A+rmI+I)\sum_{j=0}^{[m/s]}\frac{(-1)^{sj}}{(m-sj)!j!}\Gamma^{-1}(A+rjI+I-lI)\ \lambda^j\ x^{A+kjI+lI} \\
&= x^{A-lI}\ \Gamma(A+rmI+I)\ \Gamma(A+rmI+I-lI)\ \Gamma^{-1}(A+rmI+I-lI) \\
&\times\sum_{j=0}^{[m/s]}\frac{(-1)^{sj}}{(m-sj)!j!}\Gamma^{-1}(A+rjI+I-lI)\ \lambda^{rj}\ x^{rj} \\
&= x^{A-lI}\ \Gamma(A+rmI+I)\ \Gamma^{-1}(A+rmI+I-lI)\ Z_{m^*}^{(A-lI,\lambda)}(x^r,r). \qquad (4.8.1)
\end{aligned}$$

4.9 Mixed Relation

Theorem 4.9.1. *The following mixed relation holds.*

$$\begin{aligned}
&sx\ x^{A-I}\ \Gamma(A+rmI+I)\ \Gamma^{-1}(A+rmI)\ Z_{m^*}^{(A-I,\lambda)}(x^r,r) - Ax^{A-I}Z_{m^*}^{(A,\lambda)}(x^r,r) \\
&-rmx^A\ Z_{m^*}^{(A,\lambda)}(x^r;r) + kx^A(A+mrI+I)_{-r}^{-1}\ Z_{m-1^*}^{(A,\lambda)}(x^r;r) = O, \qquad (4.9.1)
\end{aligned}$$

where the notations carry their usual meaning.

Proof. We put $l = 1$ in (4.8.1) to get

$$\frac{d}{dx}\left[x^A\ Z_{m^*}^{(A,\lambda)}(x^r,r)\right] = x^{A-I}\ \Gamma(A+rmI+I)\ \Gamma^{-1}(A+rmI)\ Z_{m^*}^{(A-I,\lambda)}(x^r,r).$$

Here the left hand side may be replaced by its actual expression to get

$$\begin{aligned}
&x^A\ \frac{d}{dx}Z_{m^*}^{(A,\lambda)}(x^r,r) + Ax^{A-I}Z_{m^*}^{(A,\lambda)}(x^r,r) \\
&= x^{A-I}\ \Gamma(A+rmI+I)\ \Gamma^{-1}(A+rmI)\ Z_{m^*}^{(A-I,\lambda)}(x^r,r).
\end{aligned}$$

In this, by rearranging the terms we obtain

$$\begin{aligned}
x^A\ \frac{d}{dx}\ &=\ x^{A-I}\ \Gamma(A+rmI+I)\ \Gamma^{-1}(A+rmI)\ Z_{m^*}^{(A-I,\lambda)}(x^r,r) \\
&\quad -Ax^{A-I}Z_{m^*}^{(A,\lambda)}(x^r,r). \qquad (4.9.2)
\end{aligned}$$

Now, re-writing (4.7.6) with $k = r$, we have

$$sx\,\frac{d}{dx}Z_{n^*}^{(A,\lambda)}(x^r;r) - rn\; Z_{n^*}^{(A,\lambda)}(x^r;r) + r(A+nrI+I)_{-r}^{-1}\; Z_{n-1^*}^{(A,\lambda)}(x^r;r) = O. \tag{4.9.3}$$

Using (4.9.2) in (4.9.3) leads us to the desired mixed relation. □

4.10 Contour Integral

Following the work [96, Thm-2.1, p.125], we obtain below the contour integration representation of the polynomial $Z_{m^*}^{(A,\lambda)}(x^k,r)$.

From [96, Eq.(2.3), p.125]), we have

$$\Gamma^{-1}(A+rjI+I) = \frac{1}{2\pi i}\int_{\mathfrak{C}} e^t t^{-A-rjI-I}\,dt,$$

where $\mathfrak{C}$ is the contour coming from $-\infty$, encircling the origin of the complex t-plane in the positive direction and then going back to $-\infty$. This enables us to express the polynomial in integral form as follows.

$$\begin{aligned} Z_{m^*}^{(A,\lambda)}(x^k,r) &= \Gamma(A+rmI+I)\sum_{j=0}^{[m/s]}\frac{(-1)^{sj}}{(m-sj)!j!}\Gamma^{-1}(A+rjI+I)\lambda^j x^{kj}. \\ &= \Gamma(A+rmI+I)\sum_{j=0}^{[m/s]}\frac{(-1)^{sj}}{(m-sj)!j!}\left\{\frac{1}{2\pi i}\int_{\mathfrak{C}} e^t\; t^{-A-rjI-I}\,dt\right\}\lambda^j x^{kj}. \\ &= \frac{\Gamma(A+rmI+I)}{(2\pi i)m!}\int_{\mathfrak{C}} e^t\; t^{-A-I}\left\{\sum_{j=0}^{[m/s]}\frac{(-m)_{sj}}{j!}\;Y^j\right\}dt, \end{aligned} \tag{4.10.1}$$

where $Y = \lambda x^k\; t^{-rI}$. The series occuring in the integrand is nothing but the terminating hypergeometric matrix function ${}_sF_0[*]$. Thus, we have

$$Z_{m^*}^{(A,\lambda)}(x^k,r) = \frac{\Gamma(A+rmI+I)}{(2\pi i)m!}\int_{\mathfrak{C}} t^{-A-I}\;{}_sF_0[--;\prec s;-m\succ;Y]\,dt. \tag{4.10.2}$$

4.11 Matrix Integral Transform

Using the integral formula (4.11.1), we define Euler (Beta) matrix transform as follows.

Definition 4.11.1. *For the matrices $P, Q \in C^{p\times p}$, a Beta matrix transform may be defined as*

$$\mathfrak{B}\{f(x) : P, Q\} = \int_0^1 x^{P-I}(1-x)^{Q-I} f(x)\, dx. \tag{4.11.1}$$

We apply this transform to the polynomial (4.2.1) in the following theorem.

Theorem 4.11.1. *If $A, P, Q \in C^{p\times p}$, P, Q are positive stable matrices, for $q = 0, 1, 2, \ldots,$ the matrices $P + qI,\ Q$ are commutative, $P + qI, Q + qI,\ P + Q + qI$ are invertible and $k, r, s, m \in \mathbb{N}$, then*

$$\begin{aligned}\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k; r) : P, Q\right\} &= \frac{(A+I)_{rm}}{m!}\,\Gamma(Q)\Gamma^{-1}(P)\Gamma^{-1}(P+Q) \\ &\quad\times {}_{s+k}F_{r+k}\left[\begin{matrix}\prec s; -mI \succ, & \prec k; P \succ; & \dfrac{s^s}{r^r}t \\ \prec r; A+I \succ, & \prec k; P+Q \succ; & \end{matrix}\right],\end{aligned}$$

where the notation $\Delta(j; C)$ carries the meaning as in (1.3.23).

Proof. From (4.11.1),

$$\begin{aligned}&\mathfrak{B}\left\{Z_{m^*}^{(A,\lambda)}(tx^k; r) : P, Q\right\} \\ &= \int_0^1 x^{P-I}(1-x)^{Q-I} Z_{m^*}^{(A,\lambda)}(tx^k; r)dx \\ &= \int_0^1 x^{P-I}(1-x)^{Q-I}\frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)(tx^k)^n dx \\ &= \frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)t^n\int_0^1 x^{knI+P-I}(1-x)^{Q-I}dx \\ &= \frac{\Gamma(rmI+A+I)}{m!}\sum_{n=0}^{\lfloor m/s\rfloor}\frac{(-m)_{sn}}{n!}\Gamma^{-1}(rnI+A+I)\ t^n\ \mathfrak{B}(knI+P, Q)\end{aligned}$$

$$= \frac{\Gamma(rmI + A + I)}{m!} \sum_{n=0}^{\lfloor m/s \rfloor} \frac{(-m)_{sn}}{n!} \Gamma^{-1}(rnI + A + I)\ t^n\ \Gamma(knI + P)\ \Gamma(Q)$$
$$\times \Gamma^{-1}(knI + P + Q)$$
$$= \frac{(A + I)_{rm}}{m!} \sum_{n=0}^{\lfloor m/s \rfloor} (-m)_{sn} (A + I)^{-1}_{rn} (P)_{kn} (P + Q)^{-1}_{kn} \Gamma(Q)\Gamma(P)\Gamma^{-1}(P + Q) \frac{t^n}{n!}$$
$$= \frac{(A + I)_{rm}\ \Gamma(P)\Gamma(Q)\Gamma^{-1}(P + Q)}{m!}$$
$$\times\ {}_{s+k}F_{r+k} \left[\begin{array}{ccc} \prec s; -mI \succ, & \prec k; P \succ; & \dfrac{s^s}{r^r} t \\ \prec r; A + I \succ, & \prec k; P + Q \succ; & \end{array} \right]$$

$\square$

This theorem reduces to the Euler (Beta) transform given in [78, Theorem 9.4, p. 649] when the P, Q, A are scalars.