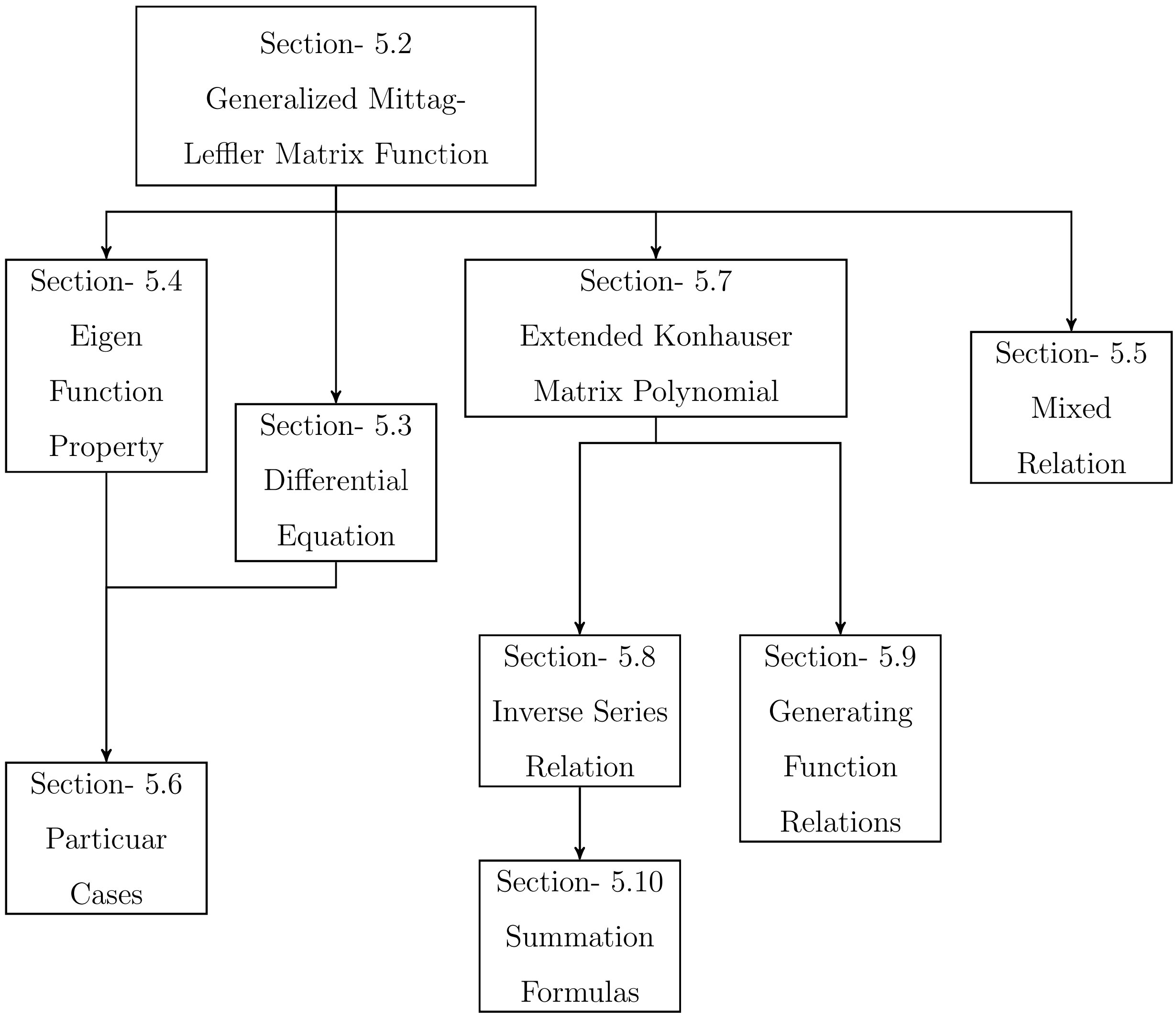
Chapter 5

Generalized Mittag-Leffler Matrix Function



5.1 Introduction

In this chapter, the Mittag-Leffler function is considered with the point of view of providing the matrix form and studying certain properties. Recenlty, a generalized Mittag-Leffler function is introduced in the form [73]:

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r} \frac{z^n}{n!}, \tag{5.1.1}$$

where $\alpha,\ \beta,\ \gamma,\ \lambda\ \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \ \delta,\ \mu > 0,\ \ r \in \mathbb{N} \cup \{-1, 0\}$ and $s \in \mathbb{N} \cup \{0\}$.

We define its matrix version and derive the eigen function property and the differential equation. Followed by this, we deduce a general form of the Konhause matrix polynomial and derive the inverse matrix series, generating function relations and summation fomulas.

5.2 Generalized Mittag-Leffler Matrix Function

We propose a matrix analogue of the generalized function (5.1.1) as follows.

Definition 5.2.1. *For* A, B, C *to be positive stable matrices in* $\mathbb{C}^{p\times p}$, $\alpha, \lambda, z \in \mathbb{C}$ *with* $\Re(\alpha) > 0$, $\delta, \mu > 0, r \in \{-1, 0\} \cup \mathbb{N}$ *and* $s \in \{0\} \cup \mathbb{N}$,

$$E^{B,C,A}_{\alpha I,\mu I,\delta I}(\lambda z;s,r) = \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha nI + B)\ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^s\ \frac{(\lambda z)^n}{n!}. \tag{5.2.1}$$

The parameters r, s, δ, μ are such that the series converges for either $|z| < \infty$ or $|z| < 1$ (see [6, 53]).

We refer to this function as *generalized Mittag-Leffler matrix function.* As a particular case, the above matrix function yields the matrix analogues of

- Mittag-Leffer matrix function (cf. (1.2.19)):

  In (5.2.1), if we put $s = 1, \delta = s, B = A + I, r = 0, \mu = 0, \lambda = 1$ then we get $E^{B,A+I,C}_{rI,sI,O}(z; 1, 0) = E^{B,sI}_{rI,A+I}(z)$ that is,

$$E^{B,sI}_{rI,A+I}(z) = \sum_{n=0}^{\infty}(B)_{sn}\Gamma^{-1}(A + rnI + I)\frac{z^n}{n!}. \tag{5.2.2}$$

- Bessel-Maitland matrix function (cf. (1.2.21)):

  With B replaced by $B + I$, and $\lambda = -1, \delta = 1 = \mu, s = r = 1$, the function

$$E^{I,B+I,I}_{\alpha,1,1}(z; 1, 1) = J^B_\alpha(z),$$

  thus we get

$$J^B_\alpha(z) = \sum_{n=0}^{\infty}(-1)^n\ \Gamma^{-1}(\alpha nI + B + I)\ \frac{z^n}{n!}.$$

- Dotsenko matrix function (cf. (1.2.22)):

  The substitutions $\lambda = 1, \alpha = \mu = \omega/\nu,\ \delta = 1 = s, r = -1$ in (5.2.1) will give $E^{B,C,A}_{\omega/\nu,1,\omega/\nu}(z; 1, -1) = {}_2R_1(C, A; B; \omega, \nu; z)$. Thus,

$$\begin{aligned} {}_2R_1(C, A; B, \omega, \nu; z) &= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\Gamma^{-1}(B + n\frac{\omega}{\nu}I)\ \Gamma(B)\ \Gamma^{-1}(C)\Gamma(C + n\frac{\omega}{\nu}I) \\ &\quad \times\Gamma(A + nI)\Gamma^{-1}(A)\ \frac{z^n}{n!}. \end{aligned}$$

- Saxena and Nishimoto's matrix function (cf. (1.2.23)):

  The function $E^{B_1,B_2,A}_{\alpha_1,\alpha_2,\kappa}(z; 1, 1) = E_{A,\kappa}[(\alpha_j I, B_j)_{1,2}; z]$. This is given as

$$E_{A,\kappa}[(\alpha_j I, B_j)_{1,2}; z] = \sum_{n=0}^{\infty}\Gamma^{-1}(\alpha_1 nI + B_1)\Gamma^{-1}(\alpha_2 nI + B_2)\ (A)_{\kappa n}\ \frac{z^n}{n!},$$

  where $z, \alpha_1, \alpha_2 \in \mathbb{C}, \Re(\alpha_1 + \alpha_2) > \Re(\kappa) - 1, \Re(\kappa) > 0$.

- The Elliptic matrix function(cf. (1.2.24)):

  The special case $E^{I/2,I/2,I}_{1,1,1}(k^2; 1, -1) = K(I; k)$, is

$$K(I;k) = \frac{\pi}{2}\ {}_2F_1\left(\begin{array}{ccc} \frac{1}{2}I, & \frac{1}{2}I; & k^2 \\ & & \\ I; & & \end{array}\right).$$

5.3 Differential Equation

We derive the differential equation of the matrix function (5.2.1) in this section. For that we take

$$\frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}} = u,\ \frac{d}{dz} = D,\ zD = \theta,\ \prod_{m=0}^{\delta-1}\left[\left(\theta I + \frac{A+mI}{\delta}\right)\right]^{s} = \Delta_m^{(\delta,A;s)}, \tag{5.3.1}$$

and

$$\prod_{j=0}^{\alpha-1}\left(\theta I + \frac{T+jI}{\alpha} - I\right)^{m} = \Upsilon_j^{(\alpha,T;m)}. \tag{5.3.2}$$

In these notations, the differential equation satisfied by (5.2.1) is derived as

Theorem 5.3.1. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $\boldsymbol{Y} = E^{B,C,A}_{\alpha I,\mu I,\delta I}(\lambda z; s, r)$ satisfies the equation*

$$\Upsilon_j^{(\alpha,B;1)}\ \theta\ \boldsymbol{Y}\ \Upsilon_k^{(\mu,C;r)} - (u\lambda z)\,\boldsymbol{Y}\ \Delta_m^{(\delta,A;s)} = O, \tag{5.3.3}$$

wherein the matrices $AC = CA$.

Proof. We first assume that the matrices occurring here are commutative with one another, then we have

$$\begin{aligned}
\mathbf{Y} &= \sum_{n=0}^{\infty}\Gamma^{-1}(\alpha n I + B)\ \ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^{s}\ \frac{(\lambda z)^n}{n!} \\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}[(B)_{\alpha n}]^{-1}\ \ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^{s}\ \frac{(\lambda z)^n}{n!} \\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\alpha^{-\alpha n}\left[\left(\frac{B}{\alpha}\right)_n\right]^{-1}\left[\left(\frac{B+I}{\alpha}\right)_n\right]^{-1}\cdots\left[\left(\frac{B+(\alpha-1)I}{\alpha}\right)_n\right]^{-1} \\
&\quad\times\mu^{-r\mu n}\left[\left(\frac{C}{\mu}\right)_n\right]^{-r}\left[\left(\frac{C+I}{\mu}\right)_n\right]^{-r}\cdots\left[\left(\frac{C+(\mu-1)I}{\mu}\right)_n\right]^{-r}\ \frac{(\lambda z)^n}{n!}
\end{aligned}$$

$$\times\delta^{s\delta n}\left[\left(\frac{A}{\delta}\right)_n\right]^s\left[\left(\frac{A+I}{\delta}\right)_n\right]^s\cdots\left[\left(\frac{A+(\delta-1)I}{\delta}\right)_n\right]^s.$$

Thus,

$$\begin{aligned}\mathbf{Y} &= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\frac{\delta^{s\delta n}}{\alpha^{\alpha n}\ \mu^{r\mu n}}\left\{\prod_{j=0}^{\alpha-1}\left(\frac{B+jI}{\alpha}\right)_n^{-1}\right\} \\ &\times\left\{\prod_{m=0}^{\delta-1}\left[\left(\frac{A+mI}{\delta}\right)_n\right]^s\right\}\left\{\prod_{k=0}^{\mu-1}\left[\left(\frac{C+kI}{\mu}\right)_n^{-1}\right]^r\right\}\frac{(\lambda z)^n}{n!}.\end{aligned} \tag{5.3.4}$$

Now let us take

$\prod_{m=0}^{\delta-1}\left[\left(\frac{A+mI}{\delta}\right)_n\right]^s = P_n$, $\prod_{j=0}^{\alpha-1}\left(\frac{B+jI}{\alpha}\right)_n = Q_n$ and $\prod_{k=0}^{\mu-1}\left[\left(\frac{C+kI}{\mu}\right)_n^{-1}\right]^r = R_n^{-1}$,

then the function (5.2.1) takes the form

$$\mathbf{Y} = \Gamma^{-1}(B)\sum_{n=0}^{\infty}Q_n^{-1}\ R_n^{-1}\ P_n\ \frac{(u\lambda z)^n}{n!}.$$

Now,

$$\begin{aligned}\theta\ \mathbf{Y} &= \Gamma^{-1}(B)\sum_{n=0}^{\infty}Q_n^{-1}\ R_n^{-1}\ P_n\ \frac{1}{n!}\ \theta\ (u\lambda z)^n \\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}Q_n^{-1}\ R_n^{-1}\ P_n\ \frac{(u\lambda z)^n}{(n-1)!}.\end{aligned}$$

Further, since the matrices commute with one another, we have

$$\begin{aligned}\Upsilon_j^{(\alpha,B;1)}\ \theta I\ \mathbf{Y} &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\prod_{j=0}^{\alpha-1}\left(\theta I+\frac{B+jI}{\alpha}-I\right)\frac{Q_n^{-1}\ R_n^{-1}\ P_n}{(n-1)!} \\ &\quad\times(u\lambda z)^n \\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\prod_{j=0}^{\alpha-1}\left(nI+\frac{B+jI}{\alpha}-I\right)\frac{Q_n^{-1}\ R_n^{-1}\ P_n}{(n-1)!} \\ &\quad\times(u\lambda z)^n \\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_n^{-1}\ P_n}{(n-1)!}\ (u\lambda z)^n.\end{aligned}$$

Finally, if $AC = CA$ then we have

$$\begin{aligned}
&\Upsilon_j^{(\alpha,B;1)}\ \theta I\ \mathbf{Y}\ \Upsilon_k^{(\mu,C;r)} \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_n^{-1}\ P_n}{(n-1)!}\prod_{k=0}^{\mu-1}\left[\left(\theta I+\frac{C+kI}{\mu}-I\right)\right]^r (u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_n^{-1}\ P_n}{(n-1)!}\prod_{k=0}^{\mu-1}\left[\left(nI+\frac{C+kI}{\mu}-I\right)\right]^r (u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_n^{-1}\ P_n}{(n-1)!}\ (u\lambda z)^n.
\end{aligned}$$

Thus,

$$\Upsilon_j^{(\alpha,B;1)}\ \theta\ \mathbf{Y}\ \Upsilon_k^{(\mu,C;r)} = \Gamma^{-1}(B)\sum_{n=0}^{\infty} Q_n^{-1}\ R_n^{-1}\ P_{n+1}\frac{(u\lambda z)^{n+1}}{n!}. \tag{5.3.5}$$

On the other hand,

$$\begin{aligned}
\mathbf{y}\ \Delta_m^{(\delta,A;s)} &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_n^{-1}\ R_n^{-1}\ P_n}{n!}\prod_{m=0}^{\delta-1}\left[\left(\theta I+\frac{A+mI}{\delta}\right)\right]^s (u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\frac{Q_n^{-1}\ R_n^{-1}\ P_n}{n!}\prod_{m=0}^{\delta-1}\left[\left(nI+\frac{A+mI}{\delta}\right)\right]^s (u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty} Q_n^{-1}\ R_n^{-1}\ P_n\ \frac{(u\lambda z)^n}{n!},
\end{aligned}$$

that is,

$$(u\lambda z)\ \mathbf{Y}\ \Delta_m^{(\delta,A;s)} = \Gamma^{-1}(B)\sum_{n=0}^{\infty} Q_n^{-1}\ R_n^{-1}\ P_{n+1}\ \frac{(u\lambda z)^{n+1}}{n!}. \tag{5.3.6}$$

On comparing (5.3.5) and (5.3.6), we get (5.3.3). □

Theorem 5.3.2. *Let $\alpha, \mu, \delta \in \mathbb{N}$ then $\boldsymbol{Y} = E_{\alpha I,\mu I,\delta I}^{B,C,A}(\lambda z; s, r)$ satisfies the equation*

$$\Upsilon_j^{(\alpha,B;1)}\ \theta\ \Upsilon_k^{(\mu,C;r)}\ \boldsymbol{Y} - (u\lambda z)\,\boldsymbol{Y}\ \Delta_m^{(\delta,A;s)} = O, \tag{5.3.7}$$

assuming $BC = CB$.

5.4 Eigen Function Property

In this section, the operators: $D = d/dz,\ \theta = z\ D,\ \frac{\delta^{s\delta}}{\alpha^{\alpha}\ \mu^{r\mu}} = u$ and

$$\Theta_m^{(\delta,A;-s)} = \prod_{m=0}^{\delta-1}\left[\left(-\theta I + \frac{A+mI}{\delta} - I\right)\right]^{-s} \tag{5.4.1}$$

will be used along with the operator (5.3.2) for deriving the Eigen function property. In these notations, we have

Theorem 5.4.1. *Let* $\alpha, \mu, \delta \in \mathbb{N}$ *then* $\boldsymbol{W} = E^{B,C,A}_{\alpha I,\mu I,\delta I}(\zeta\lambda z; s, r)$ *possesses the eigen function property given by*

$$(\lambda u)^{-1}\ D\ \Upsilon_j^{(\alpha,B;1)}\ \boldsymbol{W}\ \Upsilon_k^{(\mu,C;r)}\ \Theta_m^{(\delta,A;-s)} = \zeta\ E^{B,C,A}_{\alpha I,\mu I,\delta I}(\zeta\lambda z; s, r), \tag{5.4.2}$$

where $AC = CA$.

Proof. We begin with

$$\mathbf{W} = E^{B,C,A}_{\alpha I,\mu I,\delta I}(\zeta\lambda z; s, r) = \Gamma^{-1}(B)\sum_{n=0}^{\infty} Q_n^{-1}\ R_n^{-1}\ P_n\ \frac{(\zeta u\lambda z)^n}{n!}.$$

Then using the notations from (5.3.2), we have

$$\begin{aligned}
\Upsilon_j^{(\alpha,B;1)}\ \mathbf{W} &= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\prod_{j=0}^{\alpha-1}\left(\theta I + \frac{B+jI}{\alpha} - I\right)\frac{Q_n^{-1}\ R_n^{-1}}{n!}\ P_n\ (\zeta u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\prod_{j=0}^{\alpha-1}\left(nI + \frac{B+jI}{\alpha} - I\right)\frac{Q_n^{-1}\ R_n^{-1}\ P_n}{n!}\ (\zeta u\lambda z)^n \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty} Q_{n-1}^{-1}\ R_n^{-1}\ P_n\ \frac{(\zeta u\lambda z)^n}{n!}.
\end{aligned}$$

Next if $AC = CA$, then

$$\begin{aligned}
&\Upsilon_j^{(\alpha,B;1)}\mathbf{W}\ \Upsilon_k^{(\mu,C;r)} \\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_n^{-1}\ P_n}{n!}\prod_{k=0}^{\mu-1}\left[\left(\theta I + \frac{C+kI}{\mu} - I\right)\right]^{r}(\zeta u\lambda z)^n
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_{n}^{-1}\ P_{n}}{n!}\ \prod_{k=0}^{\mu-1}\left[\left(nI+\frac{C+kI}{\mu}-I\right)\right]^{r}(\zeta u\lambda z)^{n}\\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n}\ \frac{(\zeta u\lambda z)^{n}}{n!}.
\end{aligned}
$$

Further, from (5.4.1), we have

$$
\begin{aligned}
&\Upsilon_{j}^{(\alpha,B;1)}\ \mathbf{W}\ \Upsilon_{k}^{(\mu,C;r)}\ \Theta_{m}^{(\delta,A;-s)}\\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n}}{n!}\ \Theta_{m}^{(\delta,A;-s)}(\zeta u\lambda z)^{n}\\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n}}{n!}\prod_{m=0}^{\delta-1}\left[\left(-\theta I+\frac{A+mI}{\delta}-I\right)\right]^{-s}(\zeta u\lambda z)^{n}\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\frac{Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n}}{n!}\prod_{m=0}^{\delta-1}\left[\left(nI+\frac{A+mI}{\delta}-I\right)\right]^{-s}(\zeta u\lambda z)^{n}\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n-1}\ \frac{(\zeta u\lambda z)^{n}}{n!}.
\end{aligned}
$$

Finally, applying $(\lambda u)^{-1}\ D$, we get

$$
\begin{aligned}
&(\lambda u)^{-1}\ D\ \Upsilon_{j}^{(\alpha,B;1)}\ \mathbf{W}\ \Upsilon_{k}^{(\mu,C;r)}\ \Theta_{m}^{(\delta,A;-s)}\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\zeta^{n}(\lambda u)^{n-1}\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n-1}\ \frac{1}{n!}\ Dz^{n}\\
&= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\zeta^{n}(\lambda u)^{n-1}\ Q_{n-1}^{-1}\ R_{n-1}^{-1}\ P_{n-1}\ \frac{z^{n-1}}{(n-1)!}\\
&= \Gamma^{-1}(B)\sum_{n=0}^{\infty}\zeta^{n+1}u^{n}\ \ Q_{n}^{-1}\ R_{n}^{-1}\ P_{n}\ \frac{(\lambda z)^{n}}{n!}\\
&= \Gamma^{-1}(B)\zeta\ \sum_{n=0}^{\infty}Q_{n}^{-1}\ R_{n}^{-1}\ P_{n}\ \frac{(\zeta u\lambda z)^{n}}{n!}\\
&= \zeta\ E_{\alpha I,\delta I,\mu I}^{A,B,C}(\zeta\lambda z;s,r).
\end{aligned}
$$

□

Theorem 5.4.2. *Let $\alpha, \mu, \delta \in \mathbb{N}$ and $BC = CB$, then $\boldsymbol{W} = E^{B,C,A}_{\alpha I,\mu I,\delta I}(\lambda z; s, r)$ possesses the eigen function property:*

$$(\lambda u)^{-1}\ D\ \Upsilon_j^{(\alpha,B;1)}\ \Upsilon_k^{(\mu,C;r)}\ \boldsymbol{W}\,\Theta_m^{(\delta,A;-s)} = \zeta\ E^{B,C,A}_{\alpha I,\mu I,\delta I}(\zeta\lambda z; s, r). \quad (5.4.3)$$

5.5 Mixed Relation

We rearrange the matrix functions at the coefficient place as follows.

$$E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z; s, r) = \sum_{n=0}^{\infty} z^n\ \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s}\ \frac{\lambda^n}{n!}.$$

To this function, we we apply $D = d/dz$ and assume that $BC = CB$, then we get

$$\begin{aligned}
&\frac{d}{dz} E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\\
&= \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s}\ (\alpha n)\ z^{(\alpha n-1)}\ \frac{\lambda^n}{n!}\\
&= \alpha\ \sum_{n=1}^{\infty} \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s}\ z^{(\alpha n-1)}\ \frac{\lambda^n}{(n-1)!}\\
&= \alpha\ \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + \alpha I + B)\ [(C)_{\mu n+\mu}]^{-r}\ [(A)_{\delta n+\delta}]^{s}\ z^{(\alpha n+\alpha-1)}\ \frac{\lambda^n}{n!}\\
&= \alpha\ \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + \alpha I + B)\ [(C)_{\mu}]^{-r}\ [(C+\mu I)_{\mu n}]^{-r}\ [(A)_{\delta}]^{s}\\
&\times [(A+\delta I)_{\delta n}]^{s} z^{(\alpha n+\alpha-1)}\ \frac{\lambda^n}{n!}\\
&= \alpha\ [(C)_{\mu}]^{-r} \left\{ \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + \alpha I + B)\ [(C+\mu I)_{\mu n}]^{-r} \right.\\
&\left. \times [(A+\delta I)_{\delta n}]^{s} z^{(\alpha n+\alpha-1)}\ \frac{\lambda^n}{n!} \right\}\ [(A)_{\delta}]^{s}\\
&= \alpha\ z^{\alpha-1}\ [(C)_{\mu}]^{-r}\ E^{B+\alpha I, C+\mu I, A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}. \qquad (5.5.1)
\end{aligned}$$

If $BC \neq CB$, but $AC = CA$, then we get

$$\begin{aligned}
&\frac{d}{dz} E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \\
&\quad = \alpha \left\{ \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + \alpha I + B)\ [(C + \mu I)_{\mu n}]^{-r} \right. \\
&\quad \left. \times [(A + \delta I)_{\delta n}]^{s} z^{(\alpha n + \alpha - 1)}\ \frac{\lambda^n}{n!} \right\}\ [(A)_{\delta}]^{s}\ [(C)_{\mu}]^{-r} \\
&\quad = \alpha\ z^{\alpha - 1}\ E^{B+\alpha I, C+\mu I, A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}\ [(C)_{\mu}]^{-r}.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{d}{dz}\left[z^{B-I}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \right] \\
&\quad = \sum_{n=0}^{\infty} \left\{ \frac{d}{dz} z^{\alpha n I + B - I} \right\}\ \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^{s}\ \frac{\lambda^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \left\{ (\alpha n I + B - I) z^{\alpha n I + B - 2I} \right\}\ (\alpha n I + B - I)^{-1} \\
&\quad \times \Gamma^{-1}(\alpha n I + B - I)\ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^{s}\ \frac{\lambda^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} z^{\alpha n I + B - 2I}\ \Gamma^{-1}(\alpha n I + B - I)\ [(C)_{\mu n}]^{-r}\ \ [(A)_{\delta n}]^{s}\ \frac{\lambda^n}{n!} \\
&\quad = z^{B-2I}\ E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r).
\end{aligned}$$

Now, the left hand side being derivative of product of two matrix function, we have

$$\begin{aligned}
&\frac{d}{dz}\left[z^{B-I}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \right] \\
&\quad = (B - I) z^{B-2I}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + z^{B-I}\ \frac{d}{dz} E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r).
\end{aligned}$$

Hence if $BC = CB$, then the mixed relation occurs in the form:

$$\begin{aligned}
&z^{B-2I}\ E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \\
&\quad = (B - I) z^{B-2I}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + \alpha\ z^{\alpha I + B - 2I}\ [(C)_{\mu}]^{-r} \\
&\quad \times E^{B+\alpha I, C+\mu I, A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}. \qquad (5.5.2)
\end{aligned}$$

If $AC = CA$, then we get

$$\begin{aligned}
& z^{B-2I}\ E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \\
& = (B-I) z^{B-2I}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + \alpha\ z^{\alpha I+B-2I} \\
& \times E^{B+\alpha I,C+\mu I,A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}\ [(C)_{\mu}]^{-r}.
\end{aligned} \tag{5.5.3}$$

On the other hand,

$$\begin{aligned}
& (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + z \frac{d}{dz} E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \\
& = (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s} \\
& \times (\alpha n) z^{\alpha n}\ \frac{\lambda^{n}}{n!} \\
& = \sum_{n=0}^{\infty} (\alpha n I + B - I)\ \Gamma^{-1}(\alpha n I + B)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s}\ z^{\alpha n}\ \frac{\lambda^{n}}{n!} \\
& = \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + B - I)\ [(C)_{\mu n}]^{-r}\ [(A)_{\delta n}]^{s}\ z^{\alpha n}\ \frac{\lambda^{n}}{n!} \\
& = E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r).
\end{aligned} \tag{5.5.4}$$

From (5.5.1), with $BC = CB$ we get

$$\begin{aligned}
& (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + z \frac{d}{dz}\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) \\
& = (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + z\alpha\ z^{\alpha-1}\ [(C)_{\mu}]^{-r} \\
& \times E^{B+\alpha I,C+\mu I,A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}.
\end{aligned}$$

Finally, using (5.5.4) we find the mixed relation:

$$\begin{aligned}
& (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + \alpha z^{\alpha}\ [(C)_{\mu}]^{-r} E^{B+\alpha I,C+\mu I,A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s} \\
& = E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r).
\end{aligned}$$

Instead of $BC = CB$, if $AC = CA$, then we get the mixed relation:

$$\begin{aligned}
& (B-I)\ E^{B,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r) + \alpha z^{\alpha}\ E^{B+\alpha I,C+\mu I,A+\delta I}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r)\ [(A)_{\delta}]^{s}\ [(C)_{\mu}]^{-r} \\
& = E^{B-I,C,A}_{\delta I,\alpha I,\mu I}(\lambda z^{\alpha}; s, r).
\end{aligned}$$

5.6 Particular Cases

The properties corresponding to the special cases listed in section 5.2 may be deduced by suitably specializing the parameters involved in the above derived properties.

5.6.1 Differential Equation

(i) Bessel-Maitland matrix function: Taking

$\lambda = -1, \alpha = \mu \in \mathbb{N}, \delta = 1, \mu = 1, r = 1, s = 1, A = I = C$, and replacing B by $B + I$ in (5.3.1) and (5.3.2), we get

$$u = \frac{\delta^{s\delta}}{\alpha^{\alpha}\mu^{r\mu}} = \mu^{-\mu},\ \Delta_m^{(1,I;1)} = \theta I + I,\ \Upsilon_j^{(\mu,B+I,1)} = \prod_{j=0}^{\mu-1}\left(\theta I + \frac{B+I+jI}{\mu} - I\right),$$

$$\Upsilon_k^{(1,I;1)} = \left(\theta I + \frac{1}{\mu}I - I\right).$$

Thus the differential equation occurs in the form:

$$\Upsilon_j^{(\mu,B+I,1)}\theta\ J_{\mu}(z)\ \Upsilon_k^{(1,I;1)} + \mu^{-\mu}z\ J_{\mu}^{B}(z)\ \Delta_m^{(1,I;1)} = O.$$

Or

$$\Upsilon_j^{(\mu,B+I,1)}\theta\ \Upsilon_k^{(1,I;1)}\ J_{\mu}(z) + \mu^{-\mu}z\ J_{\mu}^{B}(z)\ \Delta_m^{(1,I;1)} = O.$$

(ii) Dotsenko matrix function: The substitutions

$\lambda = 1, \alpha = \mu = \frac{\omega}{\mu} = \ell \in \mathbb{N}, \delta = 1, r = -1, s = 1, B = C, C = B$ in (5.3.1) and (5.3.2), gives

$$u = \frac{1}{\ell^{\ell}\ \ell^{-\ell}} = 1,\ \Delta_m^{(1,A;1)} = \theta I + A,\ \Upsilon_j^{(\ell,C;1)} = \prod_{j=0}^{\ell-1}\left(\theta I + \frac{C+jI}{\ell} - I\right),$$

$$\Upsilon_k^{(\ell,B;-1)} = \prod_{k=0}^{\ell-1}\left(\theta I + \frac{B+kI}{\ell}I - I\right)^{-1}.$$

Thus the differential equation with ${}_2R_1[A, C; B, \ell; z] = \mathbf{W}$, becomes

$$\Upsilon_j^{(\ell,C;1)}\ \theta\ \mathbf{W}\ \Upsilon_k^{(\ell,B;-1)} - z\mathbf{W}\Delta_m^{(1,A;1)} = O.$$

Or

$$\Upsilon_j^{(\ell,C;1)}\ \theta\ \Upsilon_k^{(\ell,B;-1)}\ \mathbf{W} - z\mathbf{W}\Delta_m^{(1,A;1)} = O.$$

(iii) <u>Saxena and Nishimoto function</u>: We put

$\lambda = 1, \alpha = \alpha_1 \in \mathbb{N}, \delta = \kappa \in \mathbb{N}, \mu = \alpha_2 \in \mathbb{N}, r = s = 1, B = B_1, C = B_2$ in (5.3.1) and (5.3.2) to get

$$u = \frac{\kappa^\kappa}{(\alpha_1)^{\alpha_1}\ (\alpha_2)^{\alpha_2}},\quad \Upsilon_j^{(\alpha_1,B_1;1)} = \prod_{j=0}^{\alpha_1-1}\left(\theta I + \frac{B_1 + jI}{\alpha_1} - I\right),$$

$$\Delta_m^{(\kappa,A;1)} = \prod_{m=0}^{\kappa-1}\left(\theta I + \frac{A + mI}{\kappa}\right),\ \Upsilon_k^{(\alpha_2,B_2;1)} = \prod_{k=0}^{\alpha_2-1}\left(\theta I + \frac{B_2 + kI}{\alpha_2} I - I\right).$$

From this, we obtain the differential equation:

$$\Upsilon_j^{(\alpha_1,B_1;1)}\ \theta\ E_{A,\kappa}[(\alpha_j, B_j)_{1,2}; z]\ \Upsilon_k^{(\alpha_2,B_2;1)} - \frac{\kappa^\kappa\ z}{(\alpha_1)^{\alpha_1}\ (\alpha_2)^{\alpha_2}}\ E_{A,\kappa}[(\alpha_j, B_j)_{1,2}; z]\ \Delta_m^{(\kappa,A;1)} = O.$$

Or

$$\Upsilon_j^{(\alpha_1,B_1;1)}\ \theta\ \Upsilon_k^{(\alpha_2,B_2;1)}\ E_{A,\kappa}[(\alpha_j, B_j)_{1,2}; z] - \frac{\kappa^\kappa\ z}{(\alpha_1)^{\alpha_1}\ (\alpha_2)^{\alpha_2}}\ E_{A,\kappa}[(\alpha_j, B_j)_{1,2}; z]\ \Delta_m^{(\kappa,A;1)} = O.$$

(iv) <u>Elliptic matrix function</u>: Using the substitutions

$\lambda = 1, s = 2, r = 0, \alpha = 1, \delta = 1, \mu = 1, A = I/2, B = I/2, C = I, z = k^2$ in (5.3.1) and (5.3.2) we get

$$u = 1, \Upsilon_k^{(1,I;0)} = (\theta I)^0 = I, \Upsilon_j^{(1,I;1)} = \theta I, \Delta_m^{(1,I/2;2)} = \left(\theta I + \frac{I}{2}\right)^2.$$

Hence, we get the equation:

$$\left[\theta^2 I - k^2\ \left(\theta I + \frac{1}{2} I\right)^2\right]\ K(I; k) = O,$$

that is,
$$\left[(1-k^2)\theta^2 I - k^2\theta I - \frac{1}{4}k^2 I\right] \; K(I;k) = O.$$

5.6.2 Eigen Function Property

(i) Bessel-Maitland matrix function: With the help of the sustitutions $\lambda=-1, \alpha=\mu\in\mathbb{N}, \delta=1, \mu=1, r=1, s=1, A=I=C$, and replacing B by $B+I$ in (5.3.1) and (5.3.2), we get

$$u=\frac{\delta^{s\delta}}{\alpha^{\alpha}\mu^{r\mu}}=\mu^{-\mu},\;\Theta_0^{(1,I;-1)}=-(\theta I)^{-1},\;\Upsilon_0^{(1,I;1)}=\theta I,$$

and
$$\Upsilon_j^{(\mu,B+I;1)}=\prod_{j=0}^{\mu-1}\left(\theta I+\frac{B+I+jI}{\mu}-I\right).$$

Hence,

$$\Omega_{\Theta,\Upsilon}=-\mu^{\mu}\left(\frac{d}{dz}\right)(-1)(\theta I)^{-1}(\theta I)\prod_{j=0}^{\mu-1}\left(\theta I+\frac{B+I+jI}{\mu}-I\right),$$

which finally simplifies to the eigen function property:

$$\mu^{\mu}\;D\;\Upsilon_j^{(\mu,B+I;1)}\;J_{\mu}^{B}(\xi z)=\xi\;J_{\mu}^{B}(z).$$

(ii) Dotsenko matrix function: Using the substitutions $\lambda=1, \alpha=\mu=\frac{\omega}{\mu}=\ell\in\mathbb{N}, \delta=1, r=-1, s=1, B=C, C=B$ in (5.3.1) and (5.3.2), we get

$$u=\frac{1}{\ell^{\ell}\;\ell^{-\ell}}=1,\;\Theta_0^{(1,A;1)}=(A-I-\theta I)^{-1},\;\Upsilon_j^{(\ell,C;1)}=\prod_{j=0}^{\ell-1}\left(\theta I+\frac{C+jI}{\ell}-I\right),$$

$$\Upsilon_k^{(\ell,B;-1)}=\prod_{k=0}^{\ell-1}\left(\theta I+\frac{B+kI}{\ell}-I\right)^{-1}.$$

Hence, with ${}_2R_1[A,C;B,\ell;z] = {}_2R_1[z]$, we get

$$D\ \Upsilon_j^{(\ell,C;1)}\ {}_2R_1[\xi z]\ \Upsilon_k^{(\ell,B;-1)}\ (A-I-\theta I)^{-1} = \xi\ {}_2R_1[A,C;B,\ell;z].$$

(iii) Saxena and Nishimoto matrix function: Taking
$\lambda = 1, \alpha = \alpha_1 \in \mathbb{N}, \delta = \kappa \in \mathbb{N}, \mu = \alpha_2 \in \mathbb{N}, r = s = 1, B = B_1, C = B_2$ in (5.3.1) and (5.3.2), we get

$$u = \frac{\kappa^\kappa}{(\alpha_1)^{\alpha_1}\ (\alpha_2)^{\alpha_2}},\ \Theta_m^{(\kappa,A;1)} = \prod_{m=0}^{\kappa-1}\left(-\theta I + \frac{A+mI}{\kappa} - I\right)^{-1},$$

$$\Upsilon_j^{(\alpha_1,B_1;1)} = \prod_{j=0}^{\alpha_1-1}\left(\theta I + \frac{B_1+jI}{\alpha_1} - I\right),\ \Upsilon_k^{(\alpha_2,B_2;1)} = \prod_{k=0}^{\alpha_2-1}\left(\theta I + \frac{B_2+kI}{\alpha_2} I - I\right).$$

Hence,

$$\left(\frac{(\alpha_1)^{\alpha_1}\ (\alpha_2)^{\alpha_2}}{\kappa^\kappa}\right)\ D\ \Upsilon_j^{(\alpha_1,B_1;1)}\ E_{A,\kappa}[(\alpha_j,B_j)_{1,2};\xi z]\ \Upsilon_k^{(\alpha_2,B_2;1)}\ \Theta_m^{(\kappa,A;1)} = \xi\ E_{A,\kappa}[(\alpha_j,B_j)_{1,2};z].$$

(iv) Elliptic matrix function Putting
$\lambda = 1, s = 2, r = 0, \alpha = 1, \delta = 1, \mu = 1, A = I/2, B = I/2, C = I, z = k^2$, in (5.3.1) and (5.3.2), we get $u = 1$,

$$\Theta_0^{(1,I/2;2)} = \left(\theta' I + \frac{I}{2}\right)^{-2},\ \Upsilon_0^{(1,I/2;1)} = \left(\theta' I + \frac{I}{2}\right),\ \Upsilon_0^{(1,I;0)} = \left(\theta' I + \frac{I}{2}\right)^0 = I,$$

where $\theta' = k^2\ d/dk^2$. From this, we obtain with $D' = d/dk^2$,

$$D'\ \Upsilon_0^{(1,I/2;1)}\ K(I;\xi k)\ \Theta_0^{(1,I/2;2)} = \xi K(I;k).$$

5.7 Extended Konhauser Matrix Polynomial

It is interesting to note that the Konhauser matrix polynomial can be deduced in extended form, from our generalized Mittag-Leffler matrix function defined in (5.2.1). In fact, it is the matrix analogue of a generalized Konhauser polynomial

[73, Eq. (22), p. 71]:

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)=\frac{\Gamma(\alpha n+\beta+1)}{(n!)^s}\sum_{j=0}^{[n/m]}\frac{[(-n)_{mj}]^s\ x^{kj}}{\Gamma(\alpha j+\beta+1)[(\lambda)_{\mu j}]^r\ j!}.$$

An extended Konhauser matrix polynomial which we denote by $Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;s,p)$, is defined as follows.

Definition 5.7.1. *For the matrices A and B in $C^{p\times p}$,*

$$\begin{aligned} Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;s,p) &= \Gamma(A+rmI+I)\sum_{n=0}^{\lfloor m/\delta\rfloor}(-m)_{\delta n}^s\ [(B)_{\mu n}^p]^{-1} \\ &\quad\times\Gamma^{-1}(A+rnI+I)\ \frac{(\lambda x^k)^n}{n!\ (m!)^s}, \end{aligned} \tag{5.7.1}$$

where $r\in\mathbb{C}$; $\delta,k\in\mathbb{N}$, $s,p\geq 0$, $\Re(\mu)>-1$, for all eigen values $\mu\in\sigma(A)$, λ is a complex number with $\Re(\lambda)>0$ and the floor function $\lfloor u\rfloor = floor\ u$, represents the greatest integer $\leq u$.

It may be seen that this definition provides further extension to our polynomial (4.2.1).

Evidently, this generalized polynomial yields the *extended* Laguerre matrix polynomial:

$$\begin{aligned} L_{m^*}^{(A,B,\mu,r)}(\lambda x;s,p) &= \Gamma(A+rmI+I)\sum_{n=0}^{\lfloor m/\delta\rfloor}(-m)_{\delta n}^s\ [(B)_{\mu n}^p]^{-1} \\ &\quad\times\Gamma^{-1}(A+rnI+I)\ \frac{(\lambda x)^n}{n!\ (m!)^s}. \end{aligned} \tag{5.7.2}$$

Here the substitutions $\mu=0,\delta=r=s=1$ provides the Laguerre matrix polynomial [61, Eq.(10), p. 3]:

$$L_m^{(A,\lambda)}(x) = (A+I)_m\sum_{n=0}^{m}\frac{(-1)^n}{n!(m-n)!}\ [(A+I)_n]^{-1}(\lambda x)^n. \tag{5.7.3}$$

We derive the inverse series and the generating function relations of the polynomial (5.7.1) in the following sections.

5.8 Inverse Series Relations

We now establish the inverse series relation for $s = 1$ in

Theorem 5.8.1. *For $A, B \in C^{p\times p}$ and $\delta = 2, 3, \ldots,$*

$$Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) = \Gamma(A + rmI + I) \sum_{n=0}^{\lfloor m/\delta \rfloor} (-mI)_{\delta n} \left[(B)_{\mu n}^{p}\right]^{-1} \times\Gamma^{-1}(A + rnI + I)\frac{(\lambda x^k)^n}{n!\ m!}, \tag{5.8.1}$$

if and only if

$$\frac{(\lambda x^k)^m}{m!}I = (B)_{\mu m}^{p}\ \Gamma(A + rmI + I) \sum_{j=0}^{m\delta} \frac{(-m\delta I)_j}{(m\delta)!} \times\Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p), \tag{5.8.2}$$

and for $m \neq \delta l,\ l \in \mathbb{N}$,

$$\sum_{j=0}^{m}(-mI)_j\ \Gamma^{-1}(A + rjI + I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) = O. \tag{5.8.3}$$

Proof. We show that the series (5.8.1) implies both (5.8.2) and (5.8.3). The proof of (5.8.1) implies (5.8.2) runs as follows.

Denoting the right hand side of (5.8.2) by the matrix Ξ_m, and then substituting for $Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p)$ from (5.8.1), we get

$$\begin{aligned}
\Xi_m &= \frac{\Gamma(A + rmI + I)}{(m\delta)!}(B)_{\mu m}^{p} \sum_{j=0}^{m\delta}(-m\delta I)_j\ \Gamma^{-1}(A + rjI + I) \\
&\quad \times Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p) \\
&= \frac{\Gamma(A + rmI + I)}{(m\delta)!}(B)_{\mu m}^{p} \sum_{j=0}^{m\delta}(-m\delta I)_j \sum_{i=0}^{\lfloor j/\delta \rfloor} (-jI)_{\delta i}\ (B)_{\mu i}^{-p} \\
&\quad \times\Gamma^{-1}(A + riI + I)\frac{(\lambda x^k)^i}{i!} \\
&= \Gamma(A + rmI + I) \sum_{j=0}^{m\delta} \sum_{i=0}^{\lfloor j/\delta \rfloor} \frac{(-1)^{j+\delta i}(B)_{\mu m}^{p}(B)_{\mu i}^{-p}}{(m\delta - j)!\ (j - \delta i)!\ i!}\Gamma^{-1}(A + riI + I)
\end{aligned}$$

$$
\begin{aligned}
&\times(\lambda x^k)^i \\
= \;& \Gamma(A+rmI+I)\sum_{i=0}^{m}\sum_{j=0}^{m\delta-\delta i}\frac{(-1)^j\;(B)^p_{\mu m}(B)^{-p}_{\mu i}}{(m\delta-\delta i-j)!\;j!\;i!}\Gamma^{-1}(A+riI+I) \\
&\times(\lambda x^k)^i \\
= \;& \frac{(\lambda x^k)^m}{m!}I+\Gamma(A+rmI+I)\sum_{i=0}^{m-1}\frac{(B)^p_{\mu m}(B)^{-p}_{\mu i}}{(m\delta-\delta i)!\;i!}\Gamma^{-1}(A+riI+I) \\
&\times(\lambda x^k)^i\sum_{j=0}^{m\delta-\delta i}(-1)^j\binom{m\delta-\delta i}{j}.
\end{aligned}
$$

Here the inner sum in the second term on the right hand side vanishes, consequently, we arrive at $\Xi_m=\dfrac{(\lambda x^k)^m}{m!}I$.

Further, we show that (5.8.1) also implies (5.8.3). For that we substitute for $Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p)$ from (5.8.1) to the left hand side of (5.8.3) to get

$$
\begin{aligned}
&\sum_{j=0}^{m}(-mI)_j\Gamma^{-1}(A+rjI+I)\;Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p) \\
= \;& \sum_{j=0}^{m}\frac{(-1)^j m!}{(m-j)!}I\sum_{i=0}^{\lfloor j/\delta\rfloor}\frac{(-1)^{\delta i}\;(B)^{-p}_{\mu i}}{(j-\delta i)!\;i!}\Gamma^{-1}(A+riI+I)(\lambda x^k)^i \\
= \;& \sum_{i=0}^{\lfloor m/\delta\rfloor}\frac{m!\;(B)^{-p}_{\mu i}\Gamma^{-1}(A+ri+I)}{(m-\delta i)!\;i!}(\lambda x^k)^i\sum_{j=0}^{m-\delta i}(-1)^j\binom{m-\delta i}{j} \\
= \;& O
\end{aligned}
$$

if $m\neq\delta l,\; l\in\mathbb{N}$. Thus completing the first part. The proof of converse part uses the technique illustrated in [16]. We now show that the series (5.8.2) and the condition (5.8.3) together imply the series (5.8.1). The proof uses the Lemma 4.3.1 with

$$P_j=j!\;\Gamma^{-1}(A+rjI+I)\;Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p),$$

and consider one sided relation in the lemma that is, the series on the left hand side implies the series on the right hand side. Then

$$
Q_m=\sum_{j=0}^{m}(-mI)_j\;\Gamma^{-1}(A+rjI+I)\;Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p) \tag{5.8.4}
$$

$$\Rightarrow$$

$$Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;1,p) = \frac{\Gamma(A+rmI+I)}{m!} \sum_{j=0}^{m} \frac{(-mI)_j}{j!} \; Q_j. \qquad (5.8.5)$$

Since the condition (5.8.3) holds, $Q_m = 0$ for $m \neq \delta l,\; l \in \mathbb{N}$, whereas

$$Q_{m\delta} = \sum_{j=0}^{m\delta} (-m\delta I)_j \; \Gamma^{-1}(A+rjI+I) \; Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).$$

But since the series (5.8.2) holds true,

$$Q_{m\delta} = \frac{(m\delta)! \; \Gamma^{-1}(A+rmI+I)}{m!} [(B)_{\mu m}^p]^{-1} (\lambda x^k)^m.$$

Consequently, the inverse pair (5.8.4) and (5.8.5) assume the form:

$$\begin{aligned}
\frac{(\lambda x^k)^m}{m!} I &= \frac{\Gamma(A+rmI+I)}{(m\delta)!} \; [(B)_{\mu m}^p]^{-1} \sum_{j=0}^{m\delta} (-m\delta I)_j \; \Gamma^{-1}(A+rjI+I) \\
&\quad \times Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p) &= \frac{\Gamma(A+rmI+I)}{m!} \sum_{j=0}^{\lfloor m/\delta \rfloor} \frac{(-mI)_{\delta j}}{(\delta j)!} \; H_{\delta j} \\
&= \frac{\Gamma(A+rmI+I)}{m!} \sum_{j=0}^{\lfloor m/\delta \rfloor} \frac{(-mI)_{\delta j}}{j!} [(B)_{\mu m}^p]^{-1} \\
&\quad \times \Gamma^{-1}(A+rjI+I) \; (\lambda x^k)^j,
\end{aligned}$$

subject to the condition (5.8.3). □

If $\delta = 1$, then the following inverse series relations hold.

$$\begin{aligned}
Z_m^{(A,B,\mu,r)}(\lambda x^k;1,p) &= \frac{\Gamma(A+rmI+I)}{m!} \sum_{n=0}^{m} (-mI)_n \qquad (5.8.6) \\
&\quad \times [(B)_{\mu n}^p]^{-1} \; \Gamma^{-1}(A+rnI+I) \frac{(\lambda x^k)^n}{n!},
\end{aligned}$$

if and only if

$$\begin{aligned}(\lambda x^k)^m I &= \Gamma(A+rmI+I)(B)^p_{\mu m}\sum_{j=0}^{m}(-mI)_j \\ &\times\Gamma^{-1}(A+rjI+I)\ Z_j^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned} \tag{5.8.7}$$

This is an evident consequence of Lemma 4.3.1 with

$$Q_m = \Gamma^{-1}(A+rmI+I)m!Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)$$

and

$$P_m = \Gamma^{-1}(A+rmI+I)[(B)^p_{\mu m}]^{-1}(\lambda x^k)^m.$$

The Konhauser matrix polynomial and its inverse [94, Eq.(3.2) and (3.29), p.618, 626] follow from (5.8.8) and (5.8.9) when $\mu = 0$ which are stated below.

$$\begin{aligned}Z_m^{(A,r)}(\lambda x^k;1,p) &= \frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{m}(-mI)_n \\ &\times\Gamma^{-1}(A+rnI+I)\frac{(\lambda x^k)^n}{n!},\end{aligned} \tag{5.8.8}$$

if and only if

$$\begin{aligned}(\lambda x^k)^m I &= \Gamma(A+rmI+I)\sum_{j=0}^{m}(-mI)_j \\ &\times\Gamma^{-1}(A+rjI+I)\ Z_j^{(A,r)}(\lambda x^k;1,p).\end{aligned} \tag{5.8.9}$$

Further, if $k = 1$, then we obtain the inverse pair for the Laguerre matrix polynomial (5.7.3) [61, Eq. (26), p. 5]:

$$\begin{aligned}L_m^{(A,r)}(\lambda x;1,p) &= \frac{\Gamma(A+rmI+I)}{m!}\sum_{n=0}^{m}(-mI)_n \\ &\times\Gamma^{-1}(A+rnI+I)\frac{(\lambda x)^n}{n!},\end{aligned} \tag{5.8.10}$$

if and only if

$$(\lambda x)^m I = \Gamma(A + rmI + I) \sum_{j=0}^{m} (-mI)_j \times\Gamma^{-1}(A + rjI + I)\ L_j^{(A,r)}(\lambda x; 1, p). \quad (5.8.11)$$

5.9 Generating Function Relations

We obtain here a matrix generating function relation involving the L-exponential function

$$e_k(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^{k+1}}$$

of order k, due to Ricci and Tavkhelidze [81].

Theorem 5.9.1. *If $A + \ell I, B + \jmath I$ are the matrices in $C^{p\times p}$ which are invertible for all $\ell, \jmath = 0, 1, 2, \ldots$ and if $\mu, r, \delta \in \mathbb{Z}_{>0}$, then there holds the generating function relation:*

$$\sum_{m=0}^{\infty} (A + I)_{rm}^{-1}\ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; s, p)\ t^{ms} = e_{s-1}(t^s)$$

$$\times\ {}_0F_{r+\mu p}\left(-; \left(\frac{B}{\mu}\right)^p, \ldots, \left(\frac{B + (\mu - 1)I}{\mu}\right)^p, \frac{A + I}{r}, \ldots, \frac{A + rI}{r}; \frac{(-t)^{\delta s}\ \lambda x^k}{r^r\ \mu^{\mu p}}\right),$$

Proof. Beginning with the left hand side, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}(A + I)_{rm}^{-1}\ \ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; s, r)t^{ms} \\
= &\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor m/\delta\rfloor} \frac{(-1)^{s\delta n}(A + I)_{rn}^{-1}}{n!\ [(m - \delta n)!]^s}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{ms} \\
= &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}(A + I)_{rn}^{-1}}{n!\ (m!)^s}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{ms+\delta sn} \\
= &\sum_{m=0}^{\infty} \frac{t^{ms}}{(m!)^s} \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}(A + I)_{rn}^{-1}}{n!}(B)_{\mu n}^{-p}(\lambda x^k)^n t^{\delta sn},
\end{aligned}$$

which is the right hand side expression. □

The generating function relation of the Konhauser matrix polynomial occurs when $\mu = 0, \lambda = 1, s = 1, \delta = 1$ which is given by (cf. [106, Ex. 65(ii), p. 198])
$\sum_{m=0}^{\infty} (A+I)_{rm}^{-1}\ Z_m^A(x;r)\ t^m = e^t\ {}_0F_r\left(-; \prec r; A+I \succ; -tx^r\ r^{-r}\right),$
where $\prec m; A \succ = (A)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{A+(i-1)I}{m}\right)_k.$
Further specialization $k = 1$ yields the generating function for $L_n^{(A,\lambda)}(x)$ given by

$$\begin{aligned}\sum_{m=0}^{\infty}(A+I)_m^{-1}\ L_m^{(A,\lambda)}(x)\ t^m &= e^t\ {}_0F_1\left(-; A+I; -\lambda x t\right)\\ &= e^t\ (\lambda x t)^{-A/2}\ \Gamma(A+I)\ J_A(2\sqrt{\lambda x t}). \quad (5.9.0)\end{aligned}$$

This is a matrix version of the known generating function relation of the Laguerre polynomial $L_n^{(\alpha)}(x)$ [80, Eq. (2), p. 201]. Here the function $J_A(z)$ is the Bessel matrix function of first kind defined by [92, Eq. (1.10), p. 268]

$$J_A(z) = \left(\frac{z}{2}\right)^A\ \Gamma^{-1}(A+I)\ {}_0F_1\left(-; A+I; -\frac{z^2}{4}\right). \tag{5.9.1}$$

Theorem 5.9.2. *Let $r \in \mathbb{C},\ s \in \mathbb{N}$ and A, B be the positive matrices in $C^{p\times p}$, then for $|t| < 1$,*

$$\begin{aligned}&\sum_{m=0}^{\infty}(Q)_m\Gamma^{-1}(A+rmI+I)\ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p)\ t^m\\ &\quad= (1-t)^{-Q}\ E_{rI,\mu I,\delta I}^{Q,A+I,B}\left(-\lambda x^l t^\delta (1-t)^{-\delta I}; 1, r\right).\end{aligned}$$

Proof. Here, substituting the series for $Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p)$ from (5.7.1) on the left hand side and using (1.3.26), we get

$$\begin{aligned}&\sum_{m=0}^{\infty}(Q)_m\ \Gamma^{-1}(A+rmI+I)\ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k; 1, p)\ t^m\\ &= \sum_{m=0}^{\infty}(Q)_m\ \Gamma^{-1}(A+rmI+I)\frac{\Gamma(A+rmI+I)}{m!}\\ &\quad\times\sum_{k=0}^{\lfloor m/\delta\rfloor}\frac{m!(-1)^{\delta k}\ (B)_{\mu k}^{-p}\ \Gamma^{-1}(A+rkI+I)}{k!(m-\delta k)!}(\lambda x^l)^k t^m\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor m/\delta\rfloor}\frac{(-1)^{\delta k}(Q)_m(B)_{\mu k}^{-p}\ \Gamma^{-1}(A+rkI+I)}{k!\ (m-\delta k)!}\ (\lambda x^l)^k t^m\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{\delta k}(Q)_{m+\delta k}(B)_{\mu k}^{-p}\,\Gamma^{-1}(A+rkI+I)}{k!\ m!}\ (\lambda x^l)^k t^{m+\delta k}\\
&= \sum_{m=0}^{\infty}\left[\sum_{k=0}^{\infty}\frac{(Q+\delta kI)_m t^m}{m!}\right]\frac{(-1)^{\delta k}(Q)_{\delta k}(B)_{\mu k}^{-p}\,\Gamma^{-1}(A+rkI+I)}{k!}\ (\lambda x^l)^k t^{\delta k}\\
&= \sum_{k=0}^{\infty}(1-t)^{-Q-\delta kI}\frac{(-1)^{\delta k}(Q)_{\delta k}(B)_{\mu k}^{-p}\,\Gamma^{-1}(A+rkI+I)}{k!}\ (\lambda x^l)^k t^{\delta k}\\
&= (1-t)^{-Q}\sum_{k=0}^{\infty}\frac{(Q)_{\delta k}(B)_{\mu k}^{-p}\,\Gamma^{-1}(A+rkI+I)}{k!}\ (\lambda x^l(-t)^{\delta}(1-t)^{-\delta I})^k\\
&= (1-t)^{-Q}\ E_{rI,\mu I,\delta I}^{Q,A+I,B}\left(-\lambda x^l t^{\delta}(1-t)^{-\delta I};1,r\right).
\end{aligned}
$$

□

Here, two special cases are worth mentioning; the Konhauser matrix polynomial and Laguerre polynomial. With $r = l$ and $\delta = \lambda = 1$, the Theorem 5.9.2 yields the generating function relation (cf. [106, Ex.66, p.198]):

$$
\begin{aligned}
&\sum_{m=0}^{\infty}(Q)_m\Gamma^{-1}(A+mrI+I)\ Z_{m^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)\ t^m\\
&= (1-t)^{-Q}\ E_{rI,0I,I}^{Q,A+I,-}\left(-x^r t(1-t)^{-I};1,r\right)\\
&= (1-t)^{-Q}\ \sum_{k=0}^{\infty}\frac{(Q)_k\,\Gamma^{-1}(A+krI+I)}{k!}\ (-x^r t(1-t)^{-I})^k\\
&= (1-t)^{-Q}\ {}_1F_r(Q;\prec r;A+I\succ;-x^r\ t\ r^{-r}\ (1-t)^{-I})\ \Gamma^{-1}(A+I).
\end{aligned}
$$

Further, if $r = 1$, then the generating function of Laguerre matrix polynomial occurs in the form (cf. [80, Eq. (3), p. 202]):

$$
\begin{aligned}
&\sum_{m=0}^{\infty}(Q)_m\Gamma^{-1}(A+mI+I)\ L_m^A(x)\ t^m\\
&= (1-t)^{-Q}\ E_{I,0I,I}^{Q,A+I,-}\left(-xt(1-t)^{-I};1,1\right)\\
&= (1-t)^{-Q}\ {}_1F_1(Q;A+I;-x\ t\ (1-t)^{-I})\ \Gamma^{-1}(A+I).
\end{aligned}
$$

5.10 Summation Formulas.

We illustrate the application of inverse series relation to obtain certain summation formulas. The inverse series is re-stated below as

$$\begin{aligned}\frac{(\lambda x^k)^m}{m!}I &= (B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p). \qquad (5.10.1)\end{aligned}$$

Now, considering series from $m=0$ to ∞ both sides, we get

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(\lambda x^k)^m}{m!}I &= \sum_{m=0}^{\infty}(B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned}$$

From this, we obtain the summation formula:

$$e^{\lambda x^k}I=\sum_{m=0}^{\infty}\sum_{j=0}^{m\delta}(B)^p_{\mu m}\ (A+I)_{rm}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)^{-1}_{rj}\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).$$

Next, in (5.10.1), introducing $(P)_m,\ P\in\mathbb{C}^{p\times p}$ and then taking series as above, we find $|\lambda x^k|<1$,

$$\begin{aligned}\sum_{m=0}^{\infty}(P)_m\ \frac{(\lambda x^k)^m}{m!}I &= \sum_{m=0}^{\infty}(P)_m\ (B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned}$$

In view of (1.3.21), this implies the formula:

$$(1-\lambda x^k)^{-P}=\sum_{m=0}^{\infty}\sum_{j=0}^{m\delta}(P)_m\ (B)^p_{\mu m}\ (A+I)_{rm}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)^{-1}_{rj}\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p)$$

for $|\lambda x^k|<1$.

In this, putting $P = I$, we get

$$(1-\lambda x^k)^{-I} = \sum_{m=0}^{\infty}\sum_{j=0}^{m\delta} m!(B)_{\mu m}^{p}\ (A+I)_{rm}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)_{rj}^{-1}\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).$$

Further, multiplying both sides in (5.10.1) by $(P)_m(Q)_m^{-1}, P, Q \in \mathbb{C}^{p\times p}$, and then taking infinite sum, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}(P)_m\ (Q)_m^{-1}\ \frac{(\lambda x^k)^m}{m!}I \quad &= \quad \sum_{m=0}^{\infty}(P)_m(Q)_m^{-1}\ (B)_{\mu m}^{p}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad \times\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned}$$

This may be simplified to

$$\begin{aligned}{}_1F_1\left(P;Q;\lambda x^k\right) \quad &= \quad \sum_{m=0}^{\infty}\sum_{j=0}^{m\delta}(P)_m(Q)_m^{-1}\ (B)_{\mu m}^{p}\ (A+I)_{rm}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)_{rj}\\ &\quad \times\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned}$$

Now from (5.10.1), we have

$$\begin{aligned}\Gamma^{-1}(A+rmI+I)\ \frac{(\lambda x^k)^m}{m!}I \quad &= \quad (B)_{\mu m}^{p}\ \sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad \times\Gamma^{-1}(A+rjI+I)\ Z_{j^*}^{(A,B,\mu,r)}(\lambda x^k;1,p).\end{aligned}$$

To this identity, we multiply both sides by $\Gamma(A+I)$ and put $\lambda = -1$, then applying the infinite series, we obtain

$$\sum_{m=0}^{\infty}(A+I)_{rm}^{-1}\ \frac{(-x^k)^m}{m!} = \sum_{m=0}^{\infty}(B)_{\mu m}^{p}\ \sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)_{rj}^{-1}\ Z_{j^*}^{(A,B,\mu,r)}(-x^k;1,p).$$

Here, the right hand side series represents the ${}_0F_r[*]$ matrix function. Now, if r is taken to be unity, then this function particularized to the Bessel matrix function (5.9.1). The resultant identity assumes the form:

$$x^{-kA/2}\ \Gamma(A+I)\ J_A(2x^{k/2}) = \sum_{m=0}^{\infty}(B)_{\mu m}^{p}\ \sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}(A+I)_{rj}^{-1}\ Z_{j^*}^{(A,B,\mu,r)}(-x^k;1,p).$$

Instead of considering the summation both sides, if we apply the finite product on k, then we have the expression:

$$\begin{aligned}\prod_{k=1}^{N}\frac{(\lambda x^k)^m}{m!}I &= \prod_{k=1}^{N}\Bigg[(B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)\ Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p)\Bigg].\end{aligned}$$

From this, we get

$$\begin{aligned}\left(\frac{\lambda^m}{m!}\right)^N\prod_{k=1}^{N}x^{mk}I &= \Bigg[(B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)\Bigg]^k\ \prod_{k=1}^{N}Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p).\end{aligned}$$

But since

$$\prod_{k=1}^{N}x^{mk}=x^{mN(N+1)/2},$$

we finally find the identity:

$$\begin{aligned}\left(\frac{\lambda^m}{m!}\right)^N\ x^{mN(N+1)/2}I &= [(B)^p_{\mu m}\ \Gamma(A+rmI+I)\sum_{j=0}^{m\delta}\frac{(-m\delta I)_j}{(m\delta)!}\\ &\quad\times\Gamma^{-1}(A+rjI+I)]^k\ \prod_{k=1}^{N}Z^{(A,B,\mu,r)}_{j^*}(\lambda x^k;1,p).\end{aligned}$$

Likewise, from (5.10.1) various summation formulas can be derived by treating the left hand side appropriately.