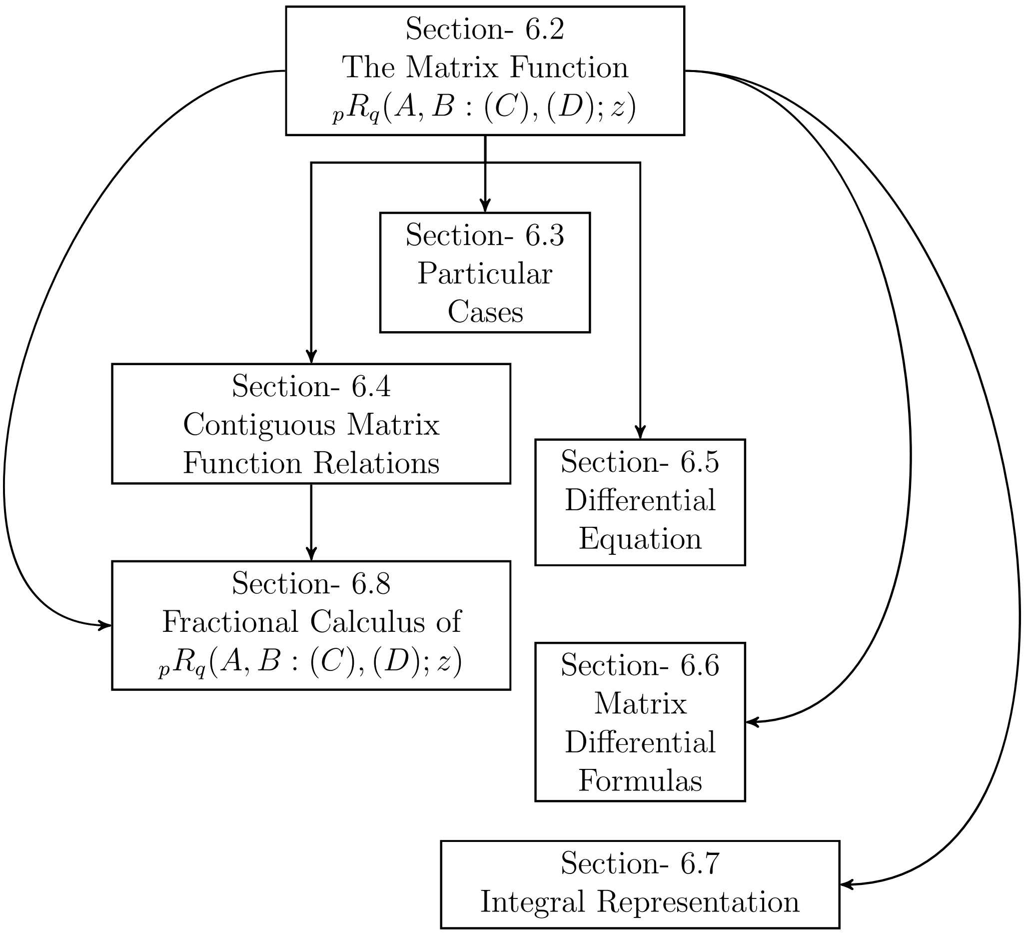
Chapter 6

A Generalized Matrix Function

$_pR_q(A, B : (C), (D); z)$



6.1 Introduction

In this chapter, the generalized function [23]:

$$
\begin{aligned}
{}_pR_q(\alpha, \beta; z) &= {}_pR_q\left(\begin{array}{cc} \gamma_1, \ldots, \gamma_p & \\ & \mid \alpha, \beta; z \\ \delta_1, \ldots, \delta_q & \end{array} \right) \\
&= \sum_{n \geq 0} \frac{(\gamma_1)_n \ldots (\gamma_p)_n}{\Gamma(\alpha n + \beta)(\delta_1)_n \ldots (\delta_q)_n} \frac{z^n}{n!}, \qquad (6.1.1)
\end{aligned}
$$

is provided the matrix extension and some of its properties will be derived.
Here $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha), \Re(\beta), \Re(\gamma_i), \Re(\delta_j) > 0$ $\forall i$ and $\forall j$. The infinite series on the right hand side converges absolutely if
(i) $p \leq q + 1$ $\forall z \in \mathbb{C}$,
(ii) $p = q + 2$ for those $z \in \mathbb{C}$ such that $|z| < 1$ and
(iii) $p = q + 2$ and $|z| = 1$ for $\Re(\sum \delta_j - \sum \gamma_i) > 0$.
As the special cases, this function yields the Gauss function ${}_2F_1(a, b; c; z)$, the Bessel function $J_n(z)$, Elliptic modular functions, Incomplete Beta functions $B(z; a, b)$, the complete elliptic integral of first kind given by

$$K(k) = (\pi/2)F(1/2, 1/2; 1; k^2)$$

and second kind given by

$$E(k) = (\pi/2)F(1/2, -1/2; 1; k^2),$$

the generalized Hypergemetric function: ${}_pF_q[z]$, M-series: ${}_pM_q^{\alpha,\beta}$, Mittag-Leffeler function: $E_\alpha(z)$ etc. Also it contains the Jacobi polynomial: $P_n^{(\alpha,\beta)}(x)$, Legendre polynomial: $P_n(x)$, Chebyshev polynomial: $U_n(x)$, Gegenbauer polynomial: $C_n^\alpha(x)$, Meixner–Pollaczek polynomial: $P_n^\lambda(x, \phi)$, Meixner polynomial: $M_n(x, \beta, \gamma)$, etc.
For this function, the following properties are obtained [23].

6.2 The Matrix Function ${}_pR_q(A,B:(C),(D);z)$

We define the matrix analogue of the function (6.1.1) as follows. We make a slight change in the notation of the matrix function: ${}_pR_q(*)$ with regard to the matrices occuring in its series representation. We use the notation (P) to denote the array of $p\times p$ matrices $P_1,P_2,\ldots,P_k$ for some $k\in\mathbb{N}$.

Definition 6.2.1. *For* $1\leq i\leq p$, $1\leq j\leq q$, *let* A, B, C_i *and* D_j, *be the positive stable matrices in* $\mathbb{C}^{p\times p}$ *such that* D_j+kI *are invertible for all integers* $k\geq 0$, *then the matrix function denoted by* ${}_pR_q(A,B:(C),(D);z)$ *is defined as*

$$\begin{aligned}{}_pR_q(A,B:(C),(D);z) &= {}_pR_q\left(\begin{array}{ll} C_1,\ldots,C_p & \\ & \mid A,B;z \\ D_1,\ldots,D_q & \end{array}\right)\\ &= \sum_{n\geq 0}\Gamma^{-1}(nA+B)(C_1)_n\ldots(C_p)_n\\ &\quad \times(D_1)_n^{-1}\ldots(D_q)_n^{-1}\frac{z^n}{n!}, \end{aligned} \tag{6.2.1}$$

whenever the series converges absolutely.

In view of theory [6] and [53], it can be shown that if $p\leq q+1$, the series converges absolutely for all finite $|z|$ and if $p=q+2$,the series converges for $|z|<1$ and diverges for $|z|>1$. For $|z|=1$, the series converges absolutely when $\beta(D_1)+\cdots+\beta(D_q)>\alpha(C_1)+\cdots+\alpha(C_p)$.

6.3 Particular Cases

The matrix function ${}_pR_q(A,B:(C),(D);z)$ contains several special matrix functions such as the generalized hypergeometric matrix function, the matrix M-series, the Wright matrix function and the Mittag-Leffler matrix function and its various generalizations. We also take into account some matrix polynomials as the particular cases.

We Start with the special case, $A = B = I$ and $C_p = I$. Then the matrix function ${}_pR_q(A, B : (C), (D); z)$ reduces to

$$
\begin{aligned}
{}_pR_q\left(\begin{matrix} C_1, \ldots, C_{p-1}, I \\ \\ D_1, \ldots, D_q \end{matrix} \mid I, I; z\right) &= \sum_{n\geq 0} (C_1)_n \ldots (C_{p-1})_n \, (D_1)_n^{-1} \ldots (D_q)_n^{-1} \, \frac{z^n}{n!} \\
&= {}_{p-1}F_q(C_1, \ldots, C_{p-1}, D_1, \ldots, D_q; z), \quad (6.3.1)
\end{aligned}
$$

which is generalized hypergeometric matrix function with $p-1$ matrix parameters in the numerator and q in the denominator [26].
The Gauss hypergeometric matrix function

$$
{}_2F_1(A_1, B_1; C; z) = \sum_{n\geq 0} (A_1)_n (B_1)_n (c)_n^{-1} \, \frac{z^n}{n!}
$$

and the confluent hypergeometric matrix function

$$
{}_1F_1(A_1; C; z) = \sum_{n\geq 0} (A_1)_n (C)_n^{-1} \, \frac{z^n}{n!}
$$

are the immediate consequences of the function (6.3.1).

- The generalized matrix M-series:
  For $C_p = I$, the function ${}_pR_q(A, B : (C), (D); z)$ yields the matrix analogue of the generalized M-series (cf. [91]) as stated below.

$$
\begin{aligned}
&{}_pR_q\left(\begin{matrix} C_1, \ldots, C_{p-1}, I \\ \\ D_1, \ldots, D_q \end{matrix} \mid A, B; z\right) \\
&= \sum_{n\geq 0} \Gamma^{-1}(nA + B)(C_1)_n \ldots (C_{p-1})_n (D_1)_n^{-1} \ldots (D_q)_n^{-1} \, z^n \\
&= {}_{p-1}M_q^{(A,B)}(C_1, \ldots, C_{p-1}, D_1, \ldots, D_q; z). \quad (6.3.2)
\end{aligned}
$$

- The classical Mittag-Leffler matrix function:

  If we put $p = 1, q = 0$, $C_1 = I$ and $B = I$, the function ${}_pR_q(A, B : (C), (D); z)$ reduces to the matrix version of the classical Mittag-Leffler function (cf. [71]):

$$ {}_1R_0\left(\begin{matrix} I \\ \\ - \end{matrix} \quad \middle| \; A, I; z\right) = \sum_{n\geq 0} \Gamma^{-1}(nA + I) z^n = E_A(z), \tag{6.3.3}$$

- Wiman's *matrix* function:

  For $p = 1, q = 0$ and $C_1 = I$, it gives Wiman's *matrix* function (cf. [110]):

$$ {}_1R_0\left(\begin{matrix} I \\ \\ - \end{matrix} \quad \middle| \; A, B; z\right) = \sum_{n\geq 0} \Gamma^{-1}(nA + B)\, z^n = E_{A,B}(z), \tag{6.3.4}$$

- The generalized Mittag-Leffler matrix function in three parameters:

  The matrix analogue of Prabhaker's generalized Mittag-Leffler function in three parameters (cf. [77]) occurs when $C_1 = C$ which is given by

$$ {}_1R_0\left(\begin{matrix} C \\ \\ - \end{matrix} \quad \middle| \; A, B; z\right) = \sum_{n\geq 0} \Gamma^{-1}(nA + B)\, (C)_n \frac{z^n}{n!} = E^C_{A,B}(z). \tag{6.3.5}$$

- The generalized Mittag-Leffler matrix function in four parameters:

  Two matrices $C_1 = C$, $C_2 = I$ and with a matrix $D_1 = D$, it yields the generalized Mittag-Leffler *matrix* function in four parameters (cf. [84]):

$$ \begin{aligned} {}_2R_1\left(\begin{matrix} C, I \\ \\ D \end{matrix} \quad \middle| \; A, B; z\right) &= \sum_{n\geq 0} \Gamma^{-1}(nA + B)\, (C)_n\, (D)_n^{-1} z^n \\ &= E^{C,D}_{A,B}(z). \end{aligned} \tag{6.3.6}$$

- The generalized Bessel-Maitland matrix function :

  For $p=q=0$ and replacing B by $B+I$ and z by $-z$, we obtain (cf. [51])

$$
\begin{aligned}
{}_0R_0\left(\begin{array}{c} - \\ \\ - \end{array} \mid A, B+I; -z\right) &= \sum_{n\geq 0} \frac{\Gamma^{-1}(nA+B+I)\,(-z)^n}{n!} \\
&= J_A^B(z). \qquad (6.3.7)
\end{aligned}
$$

We further proceed to enlist the polynomial particular cases.

- Jacobi matrix polynomial :

  The Jacobi matrix polynomial (1.4.4) is the special case of ${}_pR_q(A,B:(C),(D);z)$ for $p=2$, $q=1$, $C_1=A+C+(k+1)I$, $C_2=-kI$, $D_1=C+I$, $A=0$, $B=C+I$ and $z=\dfrac{1+x}{2}$ which is given as

$$
\begin{aligned}
P_k^{(A,C)}(x) &= \frac{(-1)^k}{k!}\,{}_2R_1\left(\begin{array}{c} A+C+(k+1)I, -kI \\ \\ C+I \end{array} \mid 0, C+I; \frac{1+x}{2}\right) \\
&\quad \times\Gamma(C+(k+1)I). \qquad (6.3.8)
\end{aligned}
$$

- Legendre matrix polynomial :

  For $p=2$, $q=1$, $C_1=(k+1)I$, $C_2=-kI$, $D_1=D$, $A=0$ and $z=\dfrac{1-x}{2}$, we get the Legendre matrix polynomial (1.4.6):

$$
P_k(x,D) = {}_2R_1\left(\begin{array}{c} (k+1)I, -kI \\ \\ D \end{array} \mid 0, B; \frac{1+x}{2}\right). \qquad (6.3.9)
$$

- Gegenbauer matrix polynomial (1.4.5):

$$C_k^D(x) = \frac{(2D)_k}{k!} {}_2R_1 \left(\begin{array}{cc} 2D + kI, -kI & \\ & \mid 0, B; \frac{1+x}{2} \\ D + \frac{1}{2}I & \end{array} \right). \tag{6.3.10}$$

- Konhauser matrix polynomial (1.4.2):

$$Z_m^C(x,k) = \frac{\Gamma(C + (km+1)I)}{\Gamma(m+1)} {}_1R_0 \left(\begin{array}{cc} -mI & \\ & \mid kI, C+I; x^k \\ - & \end{array} \right). \tag{6.3.11}$$

- Laguerre matrix polynomial (1.4.1):

  It is the instance $k = 1$ of (6.3.11).

$$L_m^C(x) = \frac{\Gamma(C + (m+1)I)}{\Gamma(m+1)} {}_1R_0 \left(\begin{array}{cc} -mI & \\ & \mid I, C+I; x \\ - & \end{array} \right) \tag{6.3.12}$$

are the particular cases of ${}_pR_q(A, B : (C), (D); z)$.

6.4 Contiguous Matrix Function Relations

In this section, we obtain contiguous matrix function relations and general order derivative formulas of the function ${}_pR_q(A, B : (C), (D); z)$, where $A, B, C_1, \ldots,$ C_p, $D_1, \ldots, D_q$ are positive stable matrices in $\mathbb{C}^{p \times p}$. The following abbreviated notations will be used here. We take

$$R =_p R_q(A, B : (C), (D); z) = {}_pR_q \left(\begin{array}{cc} C_1, \ldots, C_p & \\ & \mid A, B; z \\ D_1, \ldots, D_q & \end{array} \right),$$

$$R(C_i+) = {}_pR_q\left(\begin{matrix} C_1,\ldots,C_{i-1},C_i+I,C_{i+1},\ldots,C_p \\ \\ D_1,\ldots,D_q \end{matrix} \mid A,B;z\right),$$

$$R(C_i-) = {}_pR_q\left(\begin{matrix} C_1,\ldots,C_{i-1},C_i-I,C_{i+1},\ldots,C_p \\ \\ D_1,\ldots,D_q \end{matrix} \mid A,B;z\right),$$

$$R(D_j-) = {}_pR_q\left(\begin{matrix} C_1,\ldots,C_p \\ \\ D_1,\ldots,D_{j-1},D_j-I,D_{j+1},\ldots,D_q \end{matrix} \mid A,B;z\right),$$

$${}_pR_q(A,B+I;z) = {}_pR_q\left(\begin{matrix} C_1,\ldots,C_p \\ \\ D_1,\ldots,D_q \end{matrix} \mid A,B+I;z\right),$$

$${}_pR_q(A,B-I;z) = {}_pR_q\left(\begin{matrix} C_1,\ldots,C_p \\ \\ D_1,\ldots,D_q \end{matrix} \mid A,B-I;z\right). \tag{6.4.1}$$

There are $(p+q-1)$ number of contiguous matrix function relations occurring which connect either R, $R(C_1+)$ and $R(C_i+)$ or R, $R(C_1+)$ and $R(D_j-)$,$1 \leq i \leq p$ and $1 \leq j \leq q$.

A contiguous function relation involving the matrix C_1 is obtained below.

$$\begin{aligned} R(C_1+) &= \sum_{n\geq 0}\Gamma^{-1}(nA+B)(C_1+I)_n(C_2)_n\cdots(C_p)_n \\ &\quad \times(D_1)_n^{-1}\cdots(D_q)_n^{-1}\frac{z^n}{n!} \end{aligned}$$

$$= \sum_{n\geq 0}\Gamma^{-1}(nA+B)C_1^{-1}(C_1+nI)(C_1)_n\cdots(C_p)_n$$
$$\times(D_1)_n^{-1}\cdots(D_q)_n^{-1}\frac{z^n}{n!}. \tag{6.4.2}$$

In general for a matrix C_i with $C_iC_k = C_kC_i,\ i\neq k,\ 1\leq k\leq p$, we have the general form:

$$R(C_i+) = \sum_{n\geq 0}\Gamma^{-1}(nA+B)C_i^{-1}(C_i+nI)(C_1)_n\cdots(C_i)_n\cdots(C_p)_n$$
$$\times(D_1)_n^{-1}\cdots(D_q)_n^{-1}\ \frac{z^n}{n!}. \tag{6.4.3}$$

If $\theta = z\frac{d}{dz}$, then we get

$$(\theta+C_1)R = \sum_{n\geq 1}\Gamma^{-1}(nA+B)(C_1+nI)\,(C_1)_n\cdots(C_p)_n$$
$$\times(D_1)_n^{-1}\cdots(D_q)_n^{-1}\ \frac{z^n}{n!}. \tag{6.4.4}$$

In general for a matrix C_i , we get the general form:

$$(\theta+C_i)R = \sum_{n\geq 1}\Gamma^{-1}(nA+B)(C_i+nI)\,(C_1)_n\cdots(C_p)_n$$
$$\times(D_1)_n^{-1}\cdots(D_q)_n^{-1}\ \frac{z^n}{n!}. \tag{6.4.5}$$

The relations (6.4.2) and (6.4.4) together yield

$$(\theta+C_1)R = C_1\,R(C_1+). \tag{6.4.6}$$

and in general from equations (6.4.3) and (6.4.5), we have

$$(\theta+C_i)R = C_i\,R(C_i+),\quad i=1,\ldots,p. \tag{6.4.7}$$

Similarly for matrices $D_j \in \mathbb{C}^{p\times p}, 1 \leq j \leq q$ with
$D_j D_k = D_k D_j, 1 \leq k \leq q, k > j$, we obtain a set of q equations, given by

$$\theta\, R + R\,(D_j - I) = R(D_j-)(D_j - I). \tag{6.4.8}$$

Now, eliminating θ from (6.4.7) and (6.4.8), we find $(p + q - 1)$ contiguous matrix function relations:

$$C_i\, R - R\,(D_j - I) = C_i\, R(C_i+) - R(D_j-)(D_j - I),\ 1 \leq i \leq p, 1 \leq j \leq q. \tag{6.4.9}$$

Next the equations (6.4.6) and (6.4.7) produce $(p - 1)$ contiguous matrix function relations:

$$(C_1 - C_i)R = C_1 R(C_1+) - C_i\, R(C_i+), \quad i = 2, \ldots, p. \tag{6.4.10}$$

Furthermore, the equations (6.4.6) and (6.4.8) give rise to q contiguous matrix function relations:

$$C_1\, R - R\,(D_j - I) = C_1\, R(C_1+) - R(D_j-)(D_j - I),\ 1 \leq j \leq q. \tag{6.4.11}$$

In above contiguous function relations, if the matrices $A, B, C_1, \ldots, C_p, D_1, \ldots,$ D_q are of order 1×1, then we get the corresponding contiguous function relations for the scalar case which are given below in the same order [23].

$$R = {}_pR_q(\alpha, \beta; z) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_q \end{array} \mid \alpha, \beta; z\right).$$

Then the following are the identities.

$$R(\gamma_i+) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_{i-1}, \gamma_i + 1, \gamma_{i+1}, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_q \end{array} \mid \alpha, \beta; z\right),$$

$$R(\gamma_i-) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_{i-1}, \gamma_i - 1, \gamma_{i+1}, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_q \end{array} \mid \alpha, \beta; z\right),$$

$$R(\delta_j-) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_{j-1}, \delta_j - 1, \delta_{j+1}, \ldots, \delta_q \end{array} \mid \alpha, \beta; z\right),$$

$$R(\beta+) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_q \end{array} \mid \alpha, \beta + 1; z\right),$$

$$R(\beta-) = {}_pR_q\left(\begin{array}{c} \gamma_1, \ldots, \gamma_p \\ \\ \delta_1, \ldots, \delta_q \end{array} \mid \alpha, \beta - 1; z\right). \tag{6.4.12}$$

The scalar cases of (6.4.10) and (6.4.11) are ([23])

$$(\gamma_1 - \gamma_i)R = \gamma_1 R(\gamma_1+) - \gamma_i\, R(\gamma_i+), \quad i = 2, \ldots, p, \tag{6.4.13}$$

and

$$(\gamma_1\; -\; (\delta_j - 1))R = \gamma_1\, R(\gamma_1+) - (\delta_j - 1)R(\delta_j-),\; 1 \leq j \leq q, \tag{6.4.14}$$

respectively.

6.5 Differential Equation

We derive the differential equation of the matrix function (6.2.1) in this section. For that we take

$$\frac{d}{dz} = D,\ zD = \theta,\ \prod_{i=1}^{p} (\theta I + C_i) = \Delta_i^C,$$
$$\prod_{k=1}^{q} (\theta I + D_k - I) = \Upsilon_k^D, \prod_{j=0}^{\alpha-1} \left(\theta I + \frac{B + jI}{\alpha} - I\right) = \Upsilon_j^{(\alpha,B)}. \quad (6.5.1)$$

In these notations, the differential equation satisfied by (6.2.1) is derived as

Theorem 6.5.1. *Let* $\alpha \in \mathbb{N}, A = \alpha I,$ *then* $\boldsymbol{Y} =_p R_q(\alpha I, B : (C), (D); z)$ *satisfies the equation:*

$$\Upsilon_j^{(\alpha,B)}\ \theta\ \boldsymbol{Y}\, \Upsilon_k^D - \boldsymbol{Y}\, \Delta_i^C = O. \quad (6.5.2)$$

wherein the matrices $C_i D_k = D_k C_i$ *for all* $i = 1, 2, \ldots, p$ *and* $k = 1, 2, \ldots, q$.

Proof. We first assume that the matrices occurring here are commutative with one another, then we have

$$\begin{aligned}
\mathbf{Y} &= \sum_{n=0}^{\infty} \Gamma^{-1}(\alpha n I + B) \left\{\prod_{i=1}^{p} (C_i)_n\right\} \left\{\prod_{j=1}^{q} (D_j)_n^{-1}\right\} \frac{z^n}{n!} \\
&= \Gamma^{-1}(B) \sum_{n=0}^{\infty} [(B)_{\alpha n}]^{-1} \left\{\prod_{i=1}^{p} (C_i)_n\right\} \left\{\prod_{j=1}^{q} (D_j)_n^{-1}\right\} \frac{z^n}{n!} \\
&= \Gamma^{-1}(B) \sum_{n=0}^{\infty} \alpha^{-\alpha n} \left[\left(\frac{B}{\alpha}\right)_n\right]^{-1} \left[\left(\frac{B+I}{\alpha}\right)_n\right]^{-1} \cdots \left[\left(\frac{B+(\alpha-1)I}{\alpha}\right)_n\right]^{-1} \\
&\quad \times \left\{\prod_{i=1}^{p} (C_i)_n\right\} \left\{\prod_{j=1}^{q} (D_j)_n^{-1}\right\} \frac{z^n}{n!}
\end{aligned}$$

Thus,

$$\mathbf{Y} = \Gamma^{-1}(B) \sum_{n=0}^{\infty} \alpha^{-\alpha n} \left\{\prod_{j=0}^{\alpha-1} \left(\frac{B+jI}{\alpha}\right)_n^{-1}\right\} \left\{\prod_{i=1}^{p} (C_i)_n\right\} \left\{\prod_{j=1}^{q} (D_j)_n^{-1}\right\} \frac{z^n}{n!}. \quad (6.5.3)$$

Now let us take

$u=\alpha^{-\alpha}$, $\prod_{i=1}^{p}(C_i)_n=P_n$, $\prod_{j=0}^{\alpha-1}\left(\frac{B+jI}{\alpha}\right)_n=Q_n$, and $\prod_{k=1}^{q}(D_k)_n^{-1}=R_n^{-1}$,

then the function (6.5.3) takes the form:

$$\mathbf{Y}=\Gamma^{-1}(B)\sum_{n=0}^{\infty}Q_n^{-1}\;P_n\;R_n^{-1}\frac{(uz)^n}{n!}.$$

Now,

$$\begin{aligned}\theta\,\mathbf{Y} &= \Gamma^{-1}(B)\sum_{n=0}^{\infty}Q_n^{-1}\;P_n\;R_n^{-1}\frac{1}{n!}\;u^n\;\theta z^n\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}Q_n^{-1}\;P_n\;R_n^{-1}\frac{(uz)^n}{(n-1)!}.\end{aligned}$$

Further,

$$\begin{aligned}\Upsilon_j^{(\alpha,B)}\;\theta\,\mathbf{Y} &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\prod_{j=0}^{\alpha-1}\left(\theta I+\frac{B+jI}{\alpha}-I\right)\frac{Q_n^{-1}\;P_n\;R_n^{-1}}{(n-1)!}\;(uz)^n\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\prod_{j=0}^{\alpha-1}\left(nI+\frac{B+jI}{\alpha}-I\right)\frac{Q_n^{-1}\;P_n\;R_n^{-1}}{(n-1)!}\;(uz)^n\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}\frac{Q_{n-1}^{-1}\;P_n\;R_n^{-1}}{(n-1)!}\;(uz)^n.\end{aligned}$$

Finally,

$$\begin{aligned}&\Upsilon_j^{(\alpha,B)}\;\theta\,\mathbf{Y}\;\Upsilon_k^{D}\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}u^n\;\frac{Q_{n-1}^{-1}\;P_n\;R_n^{-1}}{(n-1)!}\;\left\{\prod_{k=1}^{q}\;(\theta I+D_k-I)\right\}\;z^n\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}u^n\;\frac{Q_{n-1}^{-1}\;P_n\;R_n^{-1}}{(n-1)!}\;\left\{\prod_{k=1}^{q}\;(nI+D_k-I)\right\}\;z^n\\ &= \Gamma^{-1}(B)\sum_{n=1}^{\infty}u^n\;\frac{Q_{n-1}^{-1}\;P_n\;R_{n-1}^{-1}}{(n-1)!}\;z^n.\end{aligned}$$

Thus,

$$\Upsilon_j^{(\alpha,B)}\ \theta\ \mathbf{Y}\ \Upsilon_k^D \ = \ \Gamma^{-1}(B)\sum_{n=0}^{\infty} u^{n+1}\ \frac{Q_n^{-1}\ P_{n+1}\ R_n^{-1}}{n!}\ z^{n+1}. \tag{6.5.4}$$

On the other hand, if all $C_iD_k = D_kC_i$, for all $i = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, q$, then

$$\begin{aligned}
&\mathbf{Y}\ \Delta_i^C \\
= \ &\Gamma^{-1}(B)\sum_{n=0}^{\infty} u^n\ \frac{Q_n^{-1}\ P_n\ R_n^{-1}}{n!}\ \left\{\prod_{i=1}^{p}\ (\theta I + C_i)\right\}\ z^n \\
= \ &\Gamma^{-1}(B)\sum_{n=0}^{\infty} u^n\ \frac{Q_n^{-1}\ P_n\ R_n^{-1}}{n!}\ \left\{\prod_{i=1}^{p}\ (nI + C_i)\right\}\ z^n \\
= \ &\Gamma^{-1}(B)\sum_{n=0}^{\infty} u^n\ \frac{Q_n^{-1}\ P_{n+1}\ R_n^{-1}}{n!}\ z^n.
\end{aligned}$$

that is,

$$\Upsilon_j^{(\alpha,B)}\ \theta\ \mathbf{Y}\ \Upsilon_k^D = \ \mathbf{Y}\ \Delta_i^C. \tag{6.5.5}$$

On comparing (6.5.4) and (6.5.5), we get (6.5.6). □

If $D_kB = BD_k$, $k = 1, 2, \ldots, q$, then we have

Theorem 6.5.2. *Let* $\alpha \in \mathbb{N}, A = \alpha I,$ *then* $\boldsymbol{Y} =_p R_q(\alpha I, B : (C), (D); z)$ *satisfies the equation*

$$\Upsilon_j^{(\alpha,B)}\ \theta\ \ \boldsymbol{Y}\,\Upsilon_k^D - \Delta_i^C\ \ \boldsymbol{Y} = O. \tag{6.5.6}$$

We now obtain matrix differential formulas satisfied by the function ${}_pR_q(A, B; z)$ in the next section.

6.6 Matrix Differential Formulas

In this section, we obtain certain matrix differential formulas. We begin with

Theorem 6.6.1. *Let A, B, C_1, $\ldots, C_p$, D_1, $\ldots, D_q \in \mathbb{C}^{p\times p}$ such that each $D_j + kI, 1 \le j \le q$ is invertible for all integers $k \ge 0$ and $C_lC_m = C_mC_l$, $C_lA = AC_l$, $C_lB = BC_l$, $D_iD_j = D_jD_i, 1 \le l, m \le p$, $1 \le i, j \le q$. Then, there hold the matrix differential formula:*

$$\left(\frac{d}{dz}\right)^r {}_pR_q(A,B:(C),(D);z) = (C_1)_r \cdots (C_p)_r \ (D_1)_r^{-1} \cdots (D_q)_r^{-1} \times {}_pR_q\left(\begin{array}{c} C_1+rI,\ldots,C_p+rI \\ \\ D_1+rI,\ldots,D_q+rI \end{array} \mid A, rA+B; z\right). \tag{6.6.1}$$

Proof. We begin with

$$\begin{aligned}
&\frac{d}{dz}\ {}_pR_q(A,B:(C),(D);z)\\
&= \sum_{n\ge1}\Gamma^{-1}(nA+B)\,(C_1)_n\ldots(C_p)_n(D_1)_n^{-1}\ldots(D_q)_n^{-1}\,\frac{z^{n-1}}{(n-1)!}\\
&= \sum_{n\ge0}\Gamma^{-1}(nA+A+B)\,(C_1)_{n+1}\ldots(C_p)_{n+1}(D_1)_{n+1}^{-1}\ldots(D_q)_{n+1}^{-1}\,\frac{z^{n}}{n!}\\
&= \left\{\prod_{1\le k\le p} C_k\right\}\left\{\prod_{1\le j\le q} D_j^{-1}\right\} {}_pR_q\left(\begin{array}{c} C_1+I,\ldots,C_p+I \\ \\ D_1+I,\ldots,D_q+I \end{array} \mid A, A+B; z\right).
\end{aligned} \tag{6.6.2}$$

Similarly, second derivative yields

$$\frac{d^2}{dz^2}\ {}_pR_q(A,B:(C),(D);z)$$

$$= \sum_{n\geq 0} \Gamma^{-1}(nA + A + B)\,(C_1)_{n+1} \dots (C_p)_{n+1}\,(D_1)^{-1}_{n+1} \dots (D_q)^{-1}_{n+1} \frac{d}{dz}\left(\frac{z^n}{n!}\right)$$

$$= \sum_{n\geq 1} \Gamma^{-1}(nA + A + B)\,(C_1)_{n+1} \dots (C_p)_{n+1}\,(D_1)^{-1}_{n+1} \dots (D_q)^{-1}_{n+1}\, \frac{z^n}{(n-1)!}$$

$$= \sum_{n\geq 0} \Gamma^{-1}(nA + 2A + B)\,(C_1)_{n+2} \dots (C_p)_{n+2}\,(D_1)^{-1}_{n+2} \dots (D_q)^{-1}_{n+2}\, \frac{z^n}{n!}$$

$$= \left\{\prod_{1\leq k\leq p} (C_k)_2\right\} \left\{\prod_{1\leq j\leq q} D_j^{-1}\right\} {}_pR_q\left(\begin{matrix} C_1 + 2I, \dots, C_p + 2I \\ \\ D_1 + 2I, \dots, D_q + 2I \end{matrix} \;\middle|\; A, 2A + B; z\right). \tag{6.6.3}$$

In general, r^{th} order derivative leads us to (6.6.1). □

Theorem 6.6.2. *Let* $A,\ B,\ C_1,\ \dots, C_p,\ D_1,\ \dots, D_q \in \mathbb{C}^{p\times p}$ *be such that each* $D_j \pm kI, 1 \leq j \leq q$ *is invertible for all integers* $k \geq 0$ *and* $D_iD_j = D_jD_i$ *Then, there hold the matrix differential formula:*

$$\left(\frac{d}{dz}\right)^r ({}_pR_q(A, B; z) z^{D_j - I})$$
$$= {}_pR_q\left(\begin{matrix} C_1, \dots, C_p \\ \\ D_1, \dots, D_{j-1}, D_j - rI, D_{j+1}, \dots, D_q \end{matrix} \;\middle|\; A, B; z\right)$$
$$\times (D_j - I)_r\, \frac{z^{D_j - (r+1)I}}{n!}. \tag{6.6.4}$$

Proof. From (6.2.1) and [6], we have

$$\frac{d}{dz}\,({}_pR_q(A, B : (C), (D); z) z^{D_j - I})$$

$$= \sum_{n\geq 0} \Gamma^{-1}(nA + B)\,(C_1)_n \dots (C_p)_n\,(D_1)^{-1}_n \dots (D_q)^{-1}_n\,(D_j + (n-1)I)\, \frac{z^{D_j + (n-2)I}}{n!}$$

$$= \sum_{n\geq 0}\Gamma^{-1}(nA+B)\,(C_1)_n\ldots(C_p)_n\,(D_1)_n^{-1}\ldots(D_j-I)_n^{-1}\ldots(D_q)_n^{-1}$$

$$\times(D_j-I)\,\frac{z^{D_j+(n-2)I}}{n!}$$

$$= \;{}_pR_q\left(\begin{array}{c} C_1,\ldots,C_p \\ \\ D_1,\ldots,D_{j-1},D_j-I,D_{j+1},\ldots,D_q\end{array}\;\middle|\;A,B;z\right)\,(D_j-I)\,\frac{z^{D_j-2I}}{n!}. \tag{6.6.5}$$

Repeating the application of d/dz, r-times say, we obtain (6.6.4). □

On the other hand, if $C_iA = AC_i$ and $C_iB = BC_i$, faor all $i = 1, 2, \ldots, p$, then we have

Theorem 6.6.3. *Let A, B, C_1, $\ldots$, C_p, D_1, $\ldots$, $D_q \in \mathbb{C}^{p\times p}$ such that each $D_j + kI, 1 \leq j \leq q$ is invertible for all integers $k \geq 0$ and $C_iC_j = C_jC_i$ $C_iD_j = D_jC_i$, and $(\Gamma^{-1}(nA+B))\,C_i = C_i\,(\Gamma^{-1}(nA+B))\,,\; 1\leq i,j\leq p$. Then, there hold the matrix differential formula*

$$\left(z^2\,\frac{d}{dz}\right)^r\left(z^{C_i-(r-1)I}\,{}_pR_q(A,B:(C),(D);z)\right)$$

$$= (C_i)_r\,z^{C_i+rI}\,{}_pR_q\left(\begin{array}{c} C_1,\ldots,C_{i-1},C_i+rI,C_{i+1},\ldots,C_p \\ \\ D_1,\ldots,D_q\end{array}\;\middle|\;A,B;z\right). \tag{6.6.6}$$

Proof. From (6.2.1) and [6], we have

$$\left(z^2\,\frac{d}{dz}\right)\left(z^{C_i-(r-1)I}\,{}_pR_q(A,B:(C),(D);z)\right)$$

$$= \sum_{n\geq 1}\Gamma^{-1}(nA+B)\,(C_1)_n\ldots(C_p)_n\,(D_1)_n^{-1}\ldots(D_q)_n^{-1}\,z^2\,\frac{d}{dz}\,\frac{z^{C_i+nI}}{n!}$$

$$= \sum_{n\geq 1} \Gamma^{-1}(nA+B)\,(C_1)_n \ldots (C_p)_n\,(D_1)_n^{-1} \ldots (D_q)_n^{-1}\,(C_i+nI)\frac{z^{C_i+nI+I}}{n!}$$

$$= C_i\, z^{C_i+I}\,{}_pR_q\left(\begin{array}{c} C_1,\ldots,C_{i-1},C_i+I,C_{i+1},\ldots,C_p \\ \\ D_1,\ldots,D_q \end{array} \mid A,B;z\right). \qquad (6.6.7)$$

Differentiating this r-times, we arrive at (6.6.6). □

Theorem 6.6.4. *Let A, B, C_1, $\ldots, C_p$, D_1, $\ldots, D_q \in \mathbb{C}^{p\times p}$ be such that each $D_j + kI, 1 \leq j \leq q$ is invertible for all integers $k \geq 0$ and A, $B-I$ are positive stable with AB=BA. Then the matrix function ${}_pR_q(A,B;z)$ defined in (6.2.1) satisfies the matrix differential formula:*

$$zA\frac{d}{dz}\,{}_pR_q(A,B:(C),(D);z)$$
$$= {}_pR_q(A,B-I:(C),(D);z) - (B-I)_pR_q(A,B:(C),(D);z). \qquad (6.6.8)$$

Proof. The left hand sides is

$$z\;A\;\frac{d}{dz}\,{}_pR_q(A,B:(C),(D);z)$$
$$= \sum_{n\geq 0} n\;A\;\Gamma^{-1}(nA+B)\,(C_1)_n \ldots (C_p)_n\,(D_1)_n^{-1} \ldots (D_q)_n^{-1}\,\frac{z^n}{n!}$$

$$= \sum_{n\geq 0} [(nA+B-I)-(B-I)]\,\Gamma^{-1}(nA+B)\,(C_1)_n \ldots (C_p)_n$$

$$\times (D_1)_n^{-1} \ldots (D_q)_n^{-1}\,\frac{z^n}{n!}$$

$$= \sum_{n\geq 0} (nA+B-I)\Gamma^{-1}(nA+B)\,(C_1)_n \ldots (C_p)_n\,(D_1)_n^{-1} \ldots (D_q)_n^{-1}\,\frac{z^n}{n!}$$

$$-(B-I)\sum_{n\geq 0} \Gamma^{-1}(nA+B)\,(C_1)_n \ldots (C_p)_n\;(D_1)_n^{-1} \ldots (D_q)_n^{-1}\frac{z^n}{n!},$$

$$= \sum_{n\geq 0} \Gamma^{-1}(nA+B-I)\,(C_1)_n \ldots (C_p)_n\,(D_1)_n^{-1} \ldots (D_q)_n^{-1}\,\frac{z^n}{n!}$$

$$-(B-I)\sum_{n\geq 0}\Gamma^{-1}(nA+B)\,(C_1)_n\dots(C_p)_n\;(D_1)_n^{-1}\dots(D_q)_n^{-1}\frac{z^n}{n!},$$

$$= {}_pR_q(A,B-I:(C),(D);z)-(B-I)_pR_q(A,B:(C),(D);z). \tag{6.6.9}$$

This completes the proof. □

6.7 Integral Representation

In this section, we derive the integral representation of the function ${}_pR_q(A,B:(C),(D);z)$.

Theorem 6.7.1. *Let A, B, C_1, $\dots$, C_p, $D_1,\dots,D_q$ be the matrices in $\mathbb{C}^{p\times p}$ such that $C_pD_j=D_jC_p, 1\leq j\leq q$ and C_p,D_q,D_q-C_p are positive stable. Then, for $|z|<1$, the matrix function ${}_pR_q(A,B:(C),(D);z)$ defined in (6.2.1) possesses the intgral representation:*

$$\begin{aligned}&{}_pR_q(A,B:(C),(D);z)\\ &\quad=\int_0^1 {}_{p-1}R_{q-1}\left(\begin{matrix}C_1,\dots,C_{p-1}\\ \\ D_1,\dots,D_{q-1}\end{matrix}\;\middle|\;A,B;t\,z\right)t^{C_p-I}(1-t)^{D_q-C_p-I}dt\\ &\qquad\times\Gamma(D_q)\Gamma^{-1}(C_p)\Gamma^{-1}(D_q-C_p)\end{aligned} \tag{6.7.1}$$

Proof. Since C_p,D_q,D_q-C_p are positive stable and $C_pD_q=D_qC_p$, we have [53]

$$\begin{aligned}(C_p)_n(D_q)_n^{-1} &= \Gamma(D_q)\Gamma^{-1}(C_p)\Gamma^{-1}(D_q-C_p)\\ &\quad\times\int_0^1 t^{C_p+(n-1)I}(1-t)^{D_q-C_p-I}dt.\end{aligned} \tag{6.7.2}$$

Using (6.7.2) in (6.2.1), we get

${}_pR_q(A,B:(C),(D);z)$

$$= \sum_{n\geq 0}\Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1}\Gamma(D_q)\Gamma^{-1}(C_p)$$

$$\times\Gamma^{-1}(D_q-C_p)\left\{\int_0^1 t^{C_p+(n-1)I}(1-t)^{D_q-C_p-I}\right\}\frac{z^n}{n!}dt. \quad (6.7.3)$$

In order to interchange the integral and summation, we consider the product

$$\begin{aligned} S_n(z,t) &= \Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1} \\ &\quad\times\Gamma(D_q)\Gamma^{-1}(C_p)\Gamma^{-1}(D_q-C_p)\frac{z^n}{n!}\,t^{C_p+(n-1)I} \\ &\quad\times(1-t)^{D_q-C_p-I}. \end{aligned} \quad (6.7.4)$$

For $0<t<1$ and $n\geq 0$, we get

$$\|S_n(z,t)\|$$

$$\leq \left\|\Gamma^{-1}(nA+B)\,(C_1)_n\cdots(C_{p-1})_n\,(D_1)_n^{-1}\cdots(D_{q-1})_n^{-1}\,\frac{z^n}{n!}\right\|\left\|\Gamma\begin{pmatrix} D_q \\ C_p, D_q-C_p\end{pmatrix}\right\|$$

$$\times\,\|t^{C_p-I}\|\|(1-t)^{D_q-C_p-I}\|. \quad (6.7.5)$$

The Schur decomposition (1.3.8) yields

$$\|t^{C_p-I}\|\;\|(1-t)^{D_q-C_p-I}\|$$

$$\begin{aligned} &\leq t^{\alpha(C_p)-1}(1-t)^{\alpha(D_q-C_p)-1}\left(\sum_{k=0}^{r-1}\frac{(\|C_p-I\|\;r^{1/2}\;\ln t)^k}{k!}\right) \\ &\quad\times\left(\sum_{k=0}^{r-1}\frac{(\|D_q-C_p-I\|\;r^{1/2}\;\ln(1-t))^k}{k!}\right). \end{aligned} \quad (6.7.6)$$

Since $0<t<1$, we have

$$\|t^{C_p-I}\|\;\|(1-t)^{D_q-C_p-I}\|\leq\mathcal{A}\;t^{\alpha(C_p)-1}(1-t)^{\alpha(D_q-C_p)-1}, \quad (6.7.7)$$

where

$$\mathcal{A} = \left(\sum_{k=0}^{r-1} \frac{(\|C_p - I\| \; r^{1/2} \;\; \ln t)^k}{k!} \right) \left(\sum_{k=0}^{r-1} \frac{(\|D_q - C_p - I\| \; r^{1/2} \;\; \ln(1-t))^k}{k!} \right) \tag{6.7.8}$$

Thus, the matrix series $\sum_{n\geq 1} \Gamma^{-1}(nA + B)\,(C_1)_n \cdots (C_{p-1})_n\,(D_1)_n^{-1} \cdots (D_{q-1})_n^{-1}\,\frac{z^n}{n!}$ converges absolutely for $p \leq q + 2$ and $|z| < 1$. Suppose it converges to S', then from (6.7.7)and with

$$\left\| \Gamma \begin{pmatrix} D_q \\ \\ C_p, D_q - C_p \end{pmatrix} \right\| = N,$$

we get

$$\sum_{n\geq 0} \|S_n(z,t)\| \leq f(t) = N S' \mathcal{A}\, t^{\alpha(C_p)-1}\,(1-t)^{\alpha(D_q - C_p)-1}. \tag{6.7.9}$$

Since $\alpha(C_p) > 0$ and $\alpha(D_q - C_p) > 0$, the function $f(t)$ is integrable and by the dominated convergence theorem [30], hence the summation and the integral can be interchanged in (6.7.3) to get:

$$\begin{aligned} &{}_pR_q(A, B : (C), (D); z) \\ &\quad = \int_0^1 t^{C_p+(n-1)I}(1-t)^{D_q-C_p-I} \left\{ \sum_{n\geq 0} \Gamma^{-1}(nA + B)\,(C_1)_n \cdots (C_{p-1})_n \right. \\ &\qquad \left. \times (D_1)_n^{-1} \cdots (D_{q-1})_n^{-1} \Gamma(D_q)\Gamma^{-1}(C_p)\Gamma^{-1}(D_q - C_p)\frac{z^n}{n!} \right\} dt. \end{aligned} \tag{6.7.10}$$

This gives (6.7.1). □

The scalar case of (6.7.1) is the following integral representation [23].
If $p \leq q+2$ and if $\Re(\delta_1) > \Re(\gamma_1) > 0$.Also $\Re(\alpha), \Re(\beta) > 0$ and no one of the $\delta_1, \delta_2, ..., \delta_q$ is non negative integer or zero then for $|z| < 1$ [23],

$${}_pR_q(\alpha, \beta; z) \;=\; \frac{\Gamma(\delta_1)}{\Gamma(\gamma_1)\Gamma(\delta_1 - \gamma_1)} \int_0^1 t^{\gamma_1 - 1}(1-t)^{\delta_1 - \gamma_1 - 1}$$

$$\times {}_{p-1}R_{q-1}\left(\begin{array}{c} \gamma_2,\ldots,\gamma_p \\ \\ \delta_2,\ldots,\delta_q \end{array} \mid \alpha,\beta;t\,z\right) dt. \qquad (6.7.11)$$

6.8 Fractional Calculus of ${}_pR_q(A,B:(C),(D);z)$

We establish fractional order integral and derivative of matrix function ${}_pR_q(A,B:(C),(D);z)$ in the following two theorems.

Theorem 6.8.1. *Let A, B, C_1, $\ldots,C_p$, $D_1,\ldots,D_q$ such that $D_iD_j = D_jD_i, 1\le i,j\le q$ be the matrices in $\mathbb{C}^{p\times p}$ with $\mu\in\mathbb{C}$ and $\Re(\mu)>0$. Then the fractional integral of the matrix function ${}_pR_q(A,B;z)$ is given by*

$$\mathbf{I}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]$$

$$= {}_pR_q\left(\begin{array}{c} C_1,\ldots,C_p \\ \\ D_1,\ldots,D_{j-1},D_j+\mu I,D_{j+1},\ldots,D_q \end{array} \mid A,B;z\right) z^{D_j+(\mu-1)I}$$

$$\times\Gamma(D_j)\Gamma^{-1}(D_j+\mu I). \qquad (6.8.1)$$

Proof. From Equation (1.6.1), we have

$$\mathbf{I}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]$$

$$= \frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}_p R_q(A,B:(C),(D);t)t^{D_j-I}dt$$

$$= \frac{1}{\Gamma(\mu)}\sum_{n\ge 0}(C_1)_n\ldots(C_p)_n\left(\int_0^z (z-t)^{\mu-1}t^{D_j+(n-1)I}dt\right)$$

$$\times(D_1)_n^{-1}\ldots(D_q)_n^{-1}\frac{1}{n!}$$

$$= \sum_{n\geq 0} (C_1)_n \dots (C_p)_n \left(\mathbf{I}^{\mu}\, z^{D_j+(n-1)I}\right)$$

$$\times (D_1)_n^{-1} \dots (D_q)_n^{-1} \frac{1}{n!}. \tag{6.8.2}$$

Using the Lemma 1.6.1, we get

$$\mathbf{I}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]$$

$$= \frac{1}{\Gamma(\mu)} \sum_{n\geq 0} (C_1)_n \dots (C_p)_n \Gamma(D_j+nI)\Gamma^{-1}(D_j+nI+\mu I)$$

$$\times z^{D_j+(n+\mu-1)I} (D_1)_n^{-1} \dots (D_q)_n^{-1} \frac{1}{n!}$$

$$= {}_pR_q\left(\begin{array}{c} C_1,\dots,C_p \\ \\ D_1,\dots,D_{j-1},D_j+\mu I, D_{j+1},\dots,D_q \end{array} \mid A,B;z\right)$$

$$\times z^{D_j+(\mu-1)I}\, \Gamma(D_j)\Gamma^{-1}(D_j+\mu I). \tag{6.8.3}$$

This completes the proof. □

Theorem 6.8.2. *Let A, B, C_1, $\dots, C_p$, $D_1,\dots,D_q$ be matrices in $\mathbb{C}^{p\times p}$ and $\mu\in\mathbb{C}$ such that $D_iD_j = D_jD_i, 1\leq i,j\leq q$ and $\Re(\mu)>0$. Then*

$$\mathbf{D}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]$$

$$= {}_pR_q\left(\begin{array}{c} C_1,\dots,C_p \\ \\ D_1,\dots,D_{j-1},D_j-\mu I, D_{j+1},\dots,D_q \end{array} \mid A,B;z\right) z^{D_j-(\mu-1)I}$$

$$\times \Gamma(D_j)\Gamma^{-1}(D_j-\mu I). \tag{6.8.4}$$

Proof. From the fractional derivative operator (1.6.2), we find

$$\mathbf{D}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}] = (\mathbf{I}^{n-\mu}\mathbf{D}^{n}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]).$$

Now using Theorem 6.8.1, we get

$$\mathbf{D}^{\mu}[{}_pR_q(A,B:(C),(D);z)z^{D_j-I}]$$

$$= \left(\frac{d}{dz}\right)^{\mu} {}_pR_q\left(\begin{array}{c} C_1,\dots,C_p \\ \\ D_1,\dots,D_{j-1},D_j+(r-\mu)I,D_{j+1},\dots,D_q \end{array} \mid A,B;z\right)$$

$$\times z^{D_j+(r-\mu-1)I}\Gamma(D_j)\Gamma^{-1}(D_j+(r-\mu)I). \tag{6.8.5}$$

Now, proceeding in the same manner as in Theorem 6.6.2, we get (6.8.4). □