Chapter 3

Non-Adiabatic Gravitational Collapse with Anisotropic Core

Contents

In this chapter, we study the non-adiabatic gravitational collapse of a spherical distribution of matter accompanied by radial heat flux on the background of a pseudo spheroidal spacetime. The spherical distribution is divided into two regions: a core consisting of anisotropic pressure distribution and envelope consisting of isotropic pressure distribution. Various aspects of the collapse have been studied using both analytic and programming approach.

3.1 Introduction

When a thermonuclear sources of energy in a star are exhausted it begins to collapse due to absence of outward force to balance the inward gravitational force. The final stage of such massive stars is either a white dwarf, a neutron star or a black hole, depending upon the mass of the configuration.

The gravitational collapse problems of spherical distribution of matter, like stars, are important problems in relativistic astrophysics. If the collapse is accompanied by heat flux which radiates out through the surface of the star, then the problem is realistic in nature. The first attempt in this direction was made by Oppenheimer and Snyder [67], when they studied an idealized problem of the gravitational collapse of a spherical dust distribution for adiabatic flow. Thereafter several authors have worked on problems of gravitational collapse considering different idealized situations.

The junction conditions for a more relativistic gravitational collapse with non - adiabatic heat flow has been first studied by Santos [77]. An important consequence of this study was that, at the boundary, the pressure is proportional to the magnitude of the heat flow vector. Based on this approach Oliveira, Santos and Kolassis [66] proposed mathematical model for a collapsing radiating star with unpolarized outgoing radiation and studied the physical conditions and thermodynamic relations. Gravitational collapse solutions with shear and radial heat flow were first obtained by Glass [27]. Dynamical equations governing the gravitational non-adiabatic collapse of a shear-free spherical distribution of anisotropic matter in the presence of charge has been studied by Tikekar and Patel [92].

When we consider the gravitational collapse of spherical distributions consisting of superdense matter distribution, the pressure may not be isotropic throughout the distribution. For such stars the core region may be anisotropic in pressure (Ruderman [76] and Canuto [8]). Therefore study of models by with isotropic pressure throughout the distribution may not be physically realistic. Keeping this in view we have studied the gravitational collapse of spherical distribution of perfect fluid accompanied by heat flux in the radial direction. The whole region is divided into two parts - a core surrounded by an envelope. The core consisting of matter with anisotropic pressure distribution and the envelope with isotropic pressure distribution.

3.2 The Interior Spacetime

The spacetime in the interior of non-adiabatically collapsing fluid sphere with outgoing radial heat-flow is denoted by $\mathbb{M}_{(i)}$. The spacetime metric of $\mathbb{M}_{(i)}$ is taken as spherically symmetric metric in the form:

$$ds^2_{(i)} = e^{\nu(r,t)}dt^2 - e^{\mu(t)}\left(e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{3.2.1}$$

with an ansatz

$$e^{\lambda(r)} = \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} \tag{3.2.2}$$

where K and R are geometric parameters.

When $\mu = 0$, the spacetime reduces to the static pseudo spheroidal spacetime used for describing static core-envelope models of superdense fluid spheres described by Thomas, Ratanpal and Vinodkumar [87].

The energy-momentum tensor for the interior spacetime $\mathbb{M}_{(i)}$ is taken as

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij} + \pi_{ij} + q_i u_j + q_j u_i \tag{3.2.3}$$

where ρ, p, u^i and q^i denote the matter density, isotropic fluid pressure, components of unit time-like flow vector field of matter and components of space-like radial heat flux vector orthogonal to u^i, respectively. The anisotropic stress tensor π_{ij} is given by:

$$\pi_{ij} = \sqrt{3}S\left[C_i C_j - \frac{1}{3}\left(u_i u_j - g_{ij}\right)\right], \tag{3.2.4}$$

where $S = S(r)$ denotes the magnitude of anisotropy and

$$C^i = \left(0, -e^{-\lambda/2}, 0, 0\right), \tag{3.2.5}$$

which is a radial vector. For a comoving observer

$$u^i = \left(e^{-\nu/2}, 0, 0, 0\right) \tag{3.2.6}$$

and

$$q^i = (0, q, 0, 0). \tag{3.2.7}$$

The heat flux vector q^i is orthogonal to u^i with magnitude $q = q(r,t)$.

The energy-momentum tensor (3.2.3) has the following non-vanishing components

$$T_0^0 = \rho \quad T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right) \quad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right) \quad T_0^1 = qe^{\nu/2}. \tag{3.2.8}$$

The pressure along the radial direction

$$p_r = p + \frac{2S}{\sqrt{3}}, \tag{3.2.9}$$

is different from the pressure along the tangential direction

$$p_\perp = p - \frac{S}{\sqrt{3}}. \tag{3.2.10}$$

The magnitude of anisotropy is given by:

$$S = \frac{p_r - p_\perp}{\sqrt{3}}. \tag{3.2.11}$$

Equations (1.1.2) corresponding to metric (3.2.1) with the energy-momentum tensor (3.2.3) is equivalent to the following set of four equations:

$$8\pi\rho = \frac{K-1}{R^2}\left(3 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2} e^{-\mu} + \frac{3}{4}e^{-\nu}\dot{\mu}^2, \tag{3.2.12}$$

$$8\pi p_r = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1} e^{-\mu} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right), \tag{3.2.13}$$

$$\begin{aligned} 8\pi\sqrt{3}S &= -\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right)\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1}+ \\ &\quad \frac{(K-1)}{R^2}r\left(\frac{\nu'}{2}+\frac{1}{r}\right)\left(1+K\frac{r^2}{R^2}\right)^{-2}- \\ &\quad \frac{K-1}{R^2}\left(1+K\frac{r^2}{R^2}\right)^{-1}e^{-\mu}, \end{aligned} \tag{3.2.14}$$

$$8\pi q = -\frac{1}{2}\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1}e^{-\nu/2}e^{-\mu}\nu'\dot{\mu}. \tag{3.2.15}$$

Here a dot and a prime denote differentiation with respect to t and r, respectively.

We shall divide the interior $\mathbb{M}_{(i)}$ of the star into two parts core and envelope of the interior.

1. The region $0 \leq r \leq b$ as the core of the star characterized by anisotropic fluid distribution.
2. The region $b \leq r \leq a$ as the envelope of the interior of the star characterized by isotropic fluid distribution.

Further discussion of the core and envelope of the interior spacetime is done in sections 3.6 and 3.7 respectively.

3.3 The Exterior Spacetime

The spacetime in the exterior of non-adiabatically collapsing fluid sphere with outgoing radial heat-flow is denoted by $\mathbb{M}_{(e)}$. We consider Vaidya [98] metric in the exterior $\mathbb{M}_{(e)}$ of the star

$$ds^2_{(e)} = \left(1-\frac{2m(v)}{y}\right)dv^2 - 2dvdy - y^2d\theta^2 - y^2\sin^2\theta d\phi^2, \tag{3.3.1}$$

where $m = m(v)$ denotes the total mass enclosed within the spherical region.

The energy-momentum tensor in $\mathbb{M}_{(e)}$ is given by

$$T_i^j = \varepsilon\zeta^j\zeta_i, \tag{3.3.2}$$

where $\zeta_i = (1, 0, 0, 0)$. The non-vanishing components of T_i^j is

$$T_0^1 = \varepsilon, \tag{3.3.3}$$

where ε is the radiation density.

A time-like 3-hypersurface $\Sigma_{(b)}$ separates the interior from the exterior and the dividing hypersurface Σ deistinguishes the two spacetime manifolds $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$, both of which contain $\Sigma_{(b)}$ as a part of their boundaries. The intrinsic metric on $\Sigma_{(b)}$ is:

$$ds_\Sigma^2 = d\tau^2 - \mathbb{R}(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.3.4}$$

3.4 Boundary Conditions

We follow the method of Israel [38] in matching the interior with the exterior on the boundary $\Sigma_{(b)}$. In order to have a unique intrinsic geometry at the boundary hypersurface $\Sigma_{(b)}$, we must have

$$ds_{(i)}^2 = ds_{(e)}^2 = ds_{(b)}^2. \tag{3.4.1}$$

This conditions guarantee the continuity of metric coefficients across the boundary surface $\Sigma_{(b)}$.

The second boundary condition imposed on $\Sigma_{(b)}$ is

$$\kappa_{ij(e)} - \kappa_{ij(i)} = 0, \tag{3.4.2}$$

where κ_{ij} are components of the extrinsic curvature. This guarantees the continuity of the first derivatives of the metric coefficients in $\mathbb{M}_{(e)}$ and $\mathbb{M}_{(i)}$ across $\Sigma_{(b)}$. The components of the extrinsic curvature κ_{ij} are given by (Eisenhart [22]):

$$\kappa_{ij} = -\eta_\alpha \frac{\partial^2 \chi^\alpha}{\partial \xi^i \partial \xi^j} - \eta_\alpha \Gamma^\alpha_{ab} \frac{\partial \chi^a}{\partial \xi^i} \frac{\partial \chi^b}{\partial \xi^j}, \tag{3.4.3}$$

where ξ^i are coordinates θ, ϕ, τ on $\Sigma_{(b)}$, χ^α are coordinates appropriate to $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$ and η_α are unit normals to $\Sigma_{(b)}$ in $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$.

The equation of the boundary surface $\Sigma_{(b)}$ in terms of the interior coordinates is given by

$$f(r,t) = r - r_{(b)}, \tag{3.4.4}$$

where r_b is a constant. The equation of the boundary surface in terms of the exterior coordinates is given by

$$f(y,v) = y - r_{(b)}(v). \tag{3.4.5}$$

The unit space-like normals $\eta_{\alpha(i)}$ and $\eta_{\alpha(e)}$ to $\Sigma_{(b)}$ in $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$, respectively, are given by

$$\eta_{\alpha(i)} = \left(0, e^{\frac{\mu+\lambda}{2}}, 0, 0\right), \tag{3.4.6}$$

$$\eta_{\alpha(e)} = \left(-\frac{\partial y}{\partial \tau}, \frac{\partial v}{\partial \tau}, 0, 0\right). \tag{3.4.7}$$

The boundary conditions (3.4.1) give the following relations:

$$\left(e^{\nu/2}\right)_{(b)} \frac{dt}{d\tau} = 1, \tag{3.4.8}$$

$$\left(re^{\mu/2}\right)_{(b)} = y_{(b)}(v) = \mathbb{R}(\tau), \tag{3.4.9}$$

$$\left[\frac{dv}{d\tau}\right]^{-2}_{(b)} = \left[2\frac{dy}{dv} + 1 - \frac{2m}{y}\right]_{(b)}. \tag{3.4.10}$$

Equations (3.4.2) and (3.4.3) yield the following equations

$$\left[-e^{-\left(\frac{\mu+\lambda}{2}\right)}\frac{\nu'}{2}\right]_{(b)} = \left[\frac{\frac{d^2v}{d\tau^2}}{\frac{dv}{d\tau}} - \frac{m}{y^2}\frac{dv}{d\tau}\right]_{(b)}, \tag{3.4.11}$$

$$\left[re^{\frac{\mu-\lambda}{2}}\right]_{(b)} = \left[y\frac{dy}{d\tau} + y\left(1 - \frac{2m}{y}\right)\frac{dv}{d\tau}\right]_{(b)}. \tag{3.4.12}$$

Using (3.4.10) and (3.4.12), we get the total mass $m(v)$ of the collapsing fluid sphere as

$$m(v) = \left[\frac{re^{\mu/2}}{2}\left(1 - e^{-\lambda} + \frac{r^2e^{\mu-\nu}}{4}\dot{\mu}\right)\right]_{(b)}. \tag{3.4.13}$$

Substituting equations (3.4.10) and (3.4.11) in equations (3.2.13) and (3.2.15), we can obtain

$$p_{(b)} = \left[qe^{\frac{\mu+\lambda}{2}}\right]_{(b)}. \tag{3.4.14}$$

This relation shows that the radial pressure at the boundary is directly related to

the heat flux q at the boundary. The pressure at the boundary becomes zero only when there is no heat flux along the radial direction across the boundary.
The energy density of radiation measured by an observer on $\Sigma_{(b)}$ with four-velocity u^α is given by (Lindquist, Schwartz and Misner [54])

$$\varepsilon = u^\alpha u^\beta T_{\alpha\beta}, \tag{3.4.15}$$

where

$$u^\alpha = \left(\frac{dv}{d\tau}, \frac{dy}{d\tau}, 0, 0\right). \tag{3.4.16}$$

The Einstein's field equations (1.1.2) for the spacetime metric (3.3.1) and energy-momentum tensor (3.3.2) yield

$$8\pi T_{00} = -\frac{2}{y^2}\frac{dm}{dv} \tag{3.4.17}$$

as the only surviving component of Einstein tensor. Now equation (3.4.15) becomes

$$8\pi\varepsilon = \left[-\frac{2}{y^2}\frac{dm}{dv}\left(\frac{dv}{d\tau}\right)^2\right]_{(b)}. \tag{3.4.18}$$

The total luminosity for an observer at rest at infinity (Lindquist, Schwartz and Misner [54]) is:

$$L_\infty = \lim_{y\to\infty, \frac{dy}{d\tau}\to 0} 4\pi y^2 \varepsilon = -\frac{dm}{dv}. \tag{3.4.19}$$

The luminosity observed on $\Sigma_{(b)}$ is:

$$L = 4\pi y^2 \varepsilon. \tag{3.4.20}$$

The boundary red shift $Z_{(b)}$ is given by

$$\frac{dv}{d\tau} = 1 + Z_{(b)}. \tag{3.4.21}$$

From (3.4.18) and (3.4.19), we can write

$$L_\infty = 4\pi y^2 \varepsilon = \frac{1}{\dot{v}^2}. \tag{3.4.22}$$

From (3.4.12) and (3.4.9), we get

$$\dot{v} = \left[\frac{re^{\frac{\mu-\lambda}{2}} - r^2 e^{\mu}\frac{\dot{\mu}}{2}}{re^{\mu/2} - 2m}\right]_{(b)}. \tag{3.4.23}$$

It is observed from equation (3.4.22) that $L_\infty \to 0$ as $\dot{v} \to \infty$. That is when $re^{\mu/2} \to 2m$. Thus when the collapsing star becomes a black hole i.e. when $re^{\mu/2} = 2m$, the boundary redshift becomes infinity.

3.5 Solution of Field Equations

If $\mu(t) = 0$ and $\nu(r,t) = \nu(r)$ in the spacetime metric (3.2.1), we get the usual spherically symmetric static metric in Schwarzschild coordinates. If the matter content of the spacetime is in the form of perfect fluid, then the field equations are:

$$8\pi\rho_0 = \frac{K-1}{R^2}\left(3 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2}, \tag{3.5.1}$$

$$8\pi p_0 = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1}, \tag{3.5.2}$$

where ρ_0 and p_0 are the proper density and radial pressure of the fluid.

Equations (3.2.12), (3.2.13) will become

$$8\pi\rho = 8\pi\rho_0 e^{-\mu} + \frac{3}{4}e^{-\nu}\dot{\mu}^2, \tag{3.5.3}$$

$$8\pi p = 8\pi p_0 e^{-\mu} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right). \tag{3.5.4}$$

On using the equation (3.4.14) in (3.5.4), we get

$$\left[\frac{1}{2}e^{\frac{\nu-\lambda}{2}}\dot{\mu}\nu'\right]_{(b)} = \left[e^{\mu/2}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{1}{2}\dot{\mu}\dot{\nu}\right)\right]_{(b)}, \tag{3.5.5}$$

if we choose $f(t) = e^{\mu/2}$ equation (3.5.5) reduces to

$$2f\ddot{f} + \dot{f}^2 - 2\alpha\dot{f} = 0, \tag{3.5.6}$$

where

$$\alpha = \left(\frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}}\right)_{(b)}. \quad (3.5.7)$$

Equation (3.5.6) possesses a first integral

$$\dot{f} = -\frac{2\alpha}{b} \frac{1 - b\sqrt{f}}{\sqrt{f}}, \quad (3.5.8)$$

and admits the solution

$$t - t_0 = \frac{f}{2\alpha} + \frac{\sqrt{f}}{\alpha b} + \frac{1}{\alpha b} ln \mid b\sqrt{f} - 1 \mid, \quad (3.5.9)$$

where b and t_0 are arbitrary constants of integration.

We choose $b = 1$ and re-parametrize t, to get

$$\dot{f} = -\frac{2\alpha}{\sqrt{f}} \left(1 - \sqrt{f}\right), \quad (3.5.10)$$

$$t = \frac{f}{2\alpha} + \frac{\sqrt{f}}{\alpha} + \frac{1}{\alpha} ln \left(1 - \sqrt{f}\right). \quad (3.5.11)$$

Here $t = -\infty$ corresponds to $f = 1$. When t gradually increases from $-\infty$ (i.e $f = 1$) the fluid gradually starts collapsing. Further we note that the spacetime metric (3.2.1) corresponds to a static metric when $t = -\infty$.

Equation (3.4.13) gives the total mass of the collapsing fluid sphere as

$$m(v) = \left[2\alpha^2 r^3 e^{-\nu} \left(1 - \sqrt{f}\right)^2 + m_0 f\right]_{(b)}, \quad (3.5.12)$$

where

$$m_0 = \left[\frac{r}{2} \left(1 - e^{-\lambda}\right)\right]_{(b)}, \quad (3.5.13)$$

is the mass inside $\Sigma_{(b)}$ when $t = -\infty$ (i.e. $f = 1$). The expression for luminosity (3.4.22) in this case reads

$$L_\infty = \left\{ \frac{\left[\alpha r^2 e^{-\frac{\lambda+\nu}{2}} \nu' \left(1 - \sqrt{f}\right)\right] \left[r e^{\mu/2} - 2m\right]}{r f \left(e^{-\frac{\lambda}{2}} - r\dot{f}\right)} \right\}_{(b)}. \quad (3.5.14)$$

It follows from equation (3.5.14) that when the collapsing body becomes a black hole, that is, $re^{\mu/2} = 2m$, $L_\infty = 0$.

3.6 The Core of the Star

The core of the spherically symmetric fluid distribution is considered to be anisotropic. Hence in the core region $0 \leq r \leq b$ the radial pressure p_r is different from the tangential pressure $p_\perp$, and hence $S(r) \neq 0$. Defining new variables z and ψ as

$$z = \sqrt{1 + \frac{r^2}{R^2}} \qquad \psi = \frac{e^{\nu/2}}{(1 - K + Kz^2)^{1/4}}, \tag{3.6.1}$$

equation (3.2.14) assumes the closed form

$$\frac{d^2\psi}{dz^2} + \left[\frac{2K\,(2K-1)\,(1-K+Kz^2) - 5K^2z^2}{4\,(1-K+Kz^2)^2} + \frac{8\sqrt{3}\pi R^2 S\,(1-K+Kz^2)}{z^2-1}\right]\psi = 0, \tag{3.6.2}$$

choosing

$$8\pi\sqrt{3}S = -\frac{(z^2-1)\,[2K\,(2K-1)\,(1-K+Kz^2) - 5K^2z^2]}{4R^2\,(1-K+Kz^2)^3}, \tag{3.6.3}$$

the second term of equation (3.6.2) vanishes and solution of the resulting equation takes the simple form

$$\psi = Cz + D, \tag{3.6.4}$$

where C and D are constants of integration. From equation (3.6.1) for a particular choice of curvature parameter $K = 2$ we get

$$e^{\nu/2} = \left(1 + 2\frac{r^2}{R^2}\right)^{\frac{1}{4}} \left(C\sqrt{1 + \frac{r^2}{R^2}} + D\right). \tag{3.6.5}$$

Hence the spacetime metric in the core $(0 \leq r \leq b)$ is described by

$$\begin{aligned} ds^2_{(i)(c)} &= \sqrt{1 + 2\frac{r^2}{R^2}} \left(C\sqrt{1 + \frac{r^2}{R^2}} + D\right)^2 dt^2 - \left(\frac{1 + 2\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}\right) f^2 dr^2 - \\ &\quad r^2 f^2 \left(d\theta^2 + sin^2\theta d\phi^2\right). \end{aligned} \tag{3.6.6}$$

Utilizing equation (3.6.5), the matter density, fluid pressure, anisotropy parameter and heat flux takes the form

$$8\pi\rho = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}\frac{1}{f^2} + \frac{12\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)^2}{f^3}, \tag{3.6.7}$$

$$\begin{aligned} 8\pi p_r &= \frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)}\frac{1}{f^2} + \\ & \frac{4\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.6.8}$$

$$\begin{aligned} 8\pi p_\perp &= \left[\frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}\right]\frac{1}{f^2} + \\ & \frac{4\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.6.9}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}\frac{1}{f^2}, \tag{3.6.10}$$

$$8\pi q = 4\alpha\frac{r}{R^2}\frac{\sqrt{1+\frac{r^2}{R^2}}}{\left(1+2\frac{r^2}{R^2}\right)^{9/4}}\frac{\left[C\left(2+3\frac{r^2}{R^2}\right)+D\sqrt{1+\frac{r^2}{R^2}}\right]}{\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{7/2}}. \tag{3.6.11}$$

Continuity of pressure and metric coefficients across the core boundary $r = b$ will give the expression of the constants C and D, and this is done in section 3.8.

The polytropic index $\gamma = \frac{d(ln\, p)}{d(ln\, \rho)}$ at the centre is given by

$$\gamma_0 = \frac{\left[k_1 k_2 f^{3/2} + \alpha^2 R^2 \left(5 - 4\sqrt{f}\right)\right]\left[3k_2^2 f + 12\alpha^2 R^2 \left(1 - \sqrt{f}\right)^2\right]}{\sqrt{f}\left[k_1 k_2 \sqrt{f} + 4\alpha^2 R^2 \left(1 - \sqrt{f}\right)\right]\left[6k_2^2 f + 12\alpha^2 R^2 \left(1 - \sqrt{f}\right)\left(3 - 2\sqrt{f}\right)\right]}, \tag{3.6.12}$$

where,

$$k_1 = A + B\left(ln[\sqrt{2} + 1] + \frac{1}{\sqrt{2}}\right),$$

$$k_2 = A + B\left(ln[\sqrt{2} + 1] - \frac{1}{\sqrt{2}}\right).$$

3.7 The Envelope of the Star

The envelope of the star is characterized by the isotropic distribution of matter. So throughout the enveloping region $b \leq r \leq a$ the radial pressure p_r is equal to the tangential pressure $p_\perp$, so that $S(r) = 0$. Then equation (3.2.14) reduces to

$$\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right)\left(1 + \frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right) - \frac{(K-1)}{R^2} r\left(\frac{\nu'}{2} + \frac{1}{r}\right) + \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right) = 0. \tag{3.7.1}$$

Choosing new independent variable z and dependent variable F defined by

$$z = \sqrt{1 + \frac{r^2}{R^2}}, \tag{3.7.2}$$

$$F = e^{\nu/2}, \tag{3.7.3}$$

equation (3.7.1) takes the form

$$\left(1 - K + Kz^2\right)\frac{d^2F}{dz^2} - Kz\frac{dF}{dz} + K\left(K-1\right)F = 0. \tag{3.7.4}$$

Equation (3.7.4) admits a closed form solution

$$F = e^{\nu/2} = A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\mathbb{L}\left(r\right) - \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right], \tag{3.7.5}$$

for $K = 2$, where A and B are constants of integration and

$$\mathbb{L}(r) = \sqrt{1+\frac{r^2}{R^2}} ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right). \tag{3.7.6}$$

Hence the spacetime metric in the envelope region $b \leq r \leq a$ is described by

$$\begin{aligned} ds^2_{(i)(e)} &= \left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2 dt^2 - \\ &\left(\frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\right) f^2 dr^2 - r^2 f^2 \left(d\theta^2 + sin^2\theta d\phi^2\right). \end{aligned} \tag{3.7.7}$$

The density, pressure and heat flux are given by the expressions

$$\begin{aligned} 8\pi\rho &= \frac{1}{R^2}\left[3+2\frac{r^2}{R^2}\right]\left[1+2\frac{r^2}{R^2}\right]^{-2}\frac{1}{f^2} + \\ &\frac{12\alpha^2}{\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)^2}{f^3}, \end{aligned} \tag{3.7.8}$$

$$\begin{aligned} 8\pi p &= \frac{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) + \frac{1}{\sqrt{2}}\sqrt{1+\frac{r^2}{R^2}}\right]}{R^2\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}}\frac{1}{f^2} + \\ &\frac{4\alpha^2}{\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.7.9}$$

$$8\pi q = 4\alpha\frac{r}{R^2}\frac{\sqrt{1+\frac{r^2}{R^2}}}{1+2\frac{r^2}{R^2}}\frac{\left\{A + Bln\left[\sqrt{2}\sqrt{1+2\frac{r^2}{R^2}} + \sqrt{1+\frac{r^2}{R^2}}\right]\right\}}{\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)}{f^{7/2}}. \tag{3.7.10}$$

The constants A and B are to be determined by matching the static solution with Schwarzschild exterior spacetime metric

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{3.7.11}$$

at the boundary surface $r = a$ of the distribution when static term of pressure is zero i.e $(p_0)_\Sigma = 0$. The continuity of the metric coefficients give

$$\sqrt{\frac{1+\frac{a^2}{R^2}}{1+2\frac{a^2}{R^2}}} = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\mathbb{L}(a) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right), \tag{3.7.12}$$

where

$$L(a) = \sqrt{1+\frac{a^2}{R^2}} ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right). \tag{3.7.13}$$

The continuity of pressure across $r = a$ requires, p_0 to vanish on the boundary implying that

$$A\sqrt{1+\frac{a^2}{R^2}} = -B\left(\mathbb{L}(a) + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right). \tag{3.7.14}$$

The constants A and B are determined from equations (3.7.12) and (3.7.14) as:

$$A = \frac{\mathbb{L}(a) + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}, \tag{3.7.15}$$

$$B = -\frac{\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}. \tag{3.7.16}$$

The polytropic index $\gamma = \frac{d(ln\, p)}{d(ln\, \rho)}$ at the surface is given by

$$\gamma_s = \frac{\left[u_a v_a f^{3/2} + \alpha^2 R^2 w_a^2\left(5 - 4\sqrt{f}\right)\right]\left[\left(2+w_a^2\right)v_a^2 f + 12\alpha^2 R^2 w_a^4\left(1-\sqrt{f}\right)^2\right]}{\sqrt{f}\left[u_a v_a\sqrt{f} + 4\alpha^2 R^2 w_a^2\left(1-\sqrt{f}\right)\right]\left[2\left(2+w_a^2\right)v_a^2 f + 12\alpha^2 R^2 w_a^4\left(1-\sqrt{f}\right)\left(3-2\sqrt{f}\right)\right]}, \tag{3.7.17}$$

where,

$$u_a = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\sqrt{1+\frac{a^2}{R^2}}\ln\left[\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right] + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right),$$

$$v_a = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\sqrt{1+\frac{a^2}{R^2}}\ln\left[\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right] - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right),$$

and

$$w_a = \sqrt{1+2\frac{a^2}{R^2}}.$$

3.8 Physical Plausibility

Since our approach does not assume any equation of state for matter, it is necessary to examine the physical plausibility of the solution in the light of energy conditions. A physically plausible solution for the core-envelope model is expected to fulfil the following requirements:

(i) The spacetime metric (3.6.6) in the core should continuously match with the spacetime metric (3.7.7) in the envelope across the core boundary $r = b$.

(ii) $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$,

(iii) $p_r \geq 0$, $p_\perp > 0$, $\frac{dp_r}{dr} < 0$, $\rho - p_r - 2p_\perp \geq 0$ for $0 \leq r \leq b$,

(iv) $\frac{dp_r}{d\rho} < 1$, $\frac{dp_\perp}{d\rho} < 1$ for $0 \leq r \leq b$,

(v) $p > 0$, $\frac{dp}{dr} < 0$, $\frac{dp}{d\rho} < 1$, $\rho - 3p > 0$ for $b \leq r \leq a$.

At the core boundary $r = b$, the anisotropy parameter vanishes hence from equation (3.6.10) we get $\frac{b^2}{R^2} = 2$. The continuity of matric coefficients and the continuity of static pressure across $r = b$ of the distribution leads to

$$11\sqrt{3}C + D = 5^{\frac{3}{4}}\left[\sqrt{3}A + B\left(\mathbb{L}(b) + \sqrt{2.5}\right)\right], \tag{3.8.1}$$

$$\sqrt{3}C + D = 5^{-\frac{1}{4}}\left[\sqrt{3}A + B\left(\mathbb{L}(b) - \sqrt{2.5}\right)\right], \tag{3.8.2}$$

where

$$\mathbb{L}(b) = \sqrt{3}ln\left(\sqrt{5} + \sqrt{6}\right). \tag{3.8.3}$$

Equations (3.8.1) and (3.8.2) determines C and D in terms of A and B

$$C = \frac{2A + B\left(\frac{2}{\sqrt{3}}\mathbb{L}(b) + \sqrt{7.5}\right)}{5^{\frac{5}{4}}}, \tag{3.8.4}$$

$$D = \frac{3\sqrt{3}A + B\left(3\mathbb{L}(b) - 8\sqrt{2.5}\right)}{5^{\frac{5}{4}}}, \tag{3.8.5}$$

substituting these values of C and D in (3.6.8) and (3.6.9) we get radial and tangential pressure in the core of the star. Owing to the complexity of expressions, programming is used to verify requirements (ii) to (v).

3.9 Discussion

We have discussed certain aspects of a non-adiabatically collapsing spherical distribution of matter associated with radial heat flow. We found that the density $\rho(r,t)$, radial pressure $p_r(r,t)$, tangential pressure $p_\perp(r,t)$ are positive throughout the distribution during its collapse from equilibrium to blackhole. Density and radial pressure and decreases along the radially outward direction.

The plots showing variations of pressure, density and sound speed for the model with $\lambda = 0.05$ and $f = 0.7$ are shown in Figures 3.1 - 3.3 respectively.

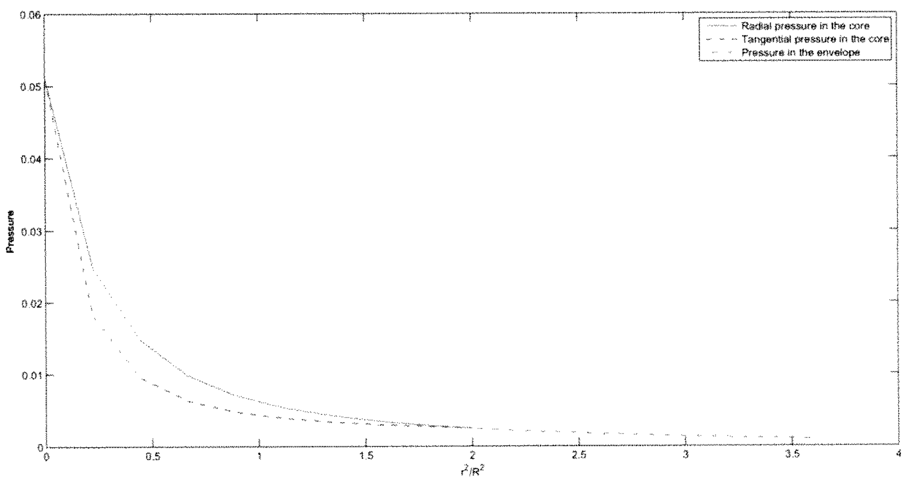

Figure 3.1: Variation of p_r, $p_\perp$ against $\frac{r^2}{R^2}$ in the core and variation of p against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

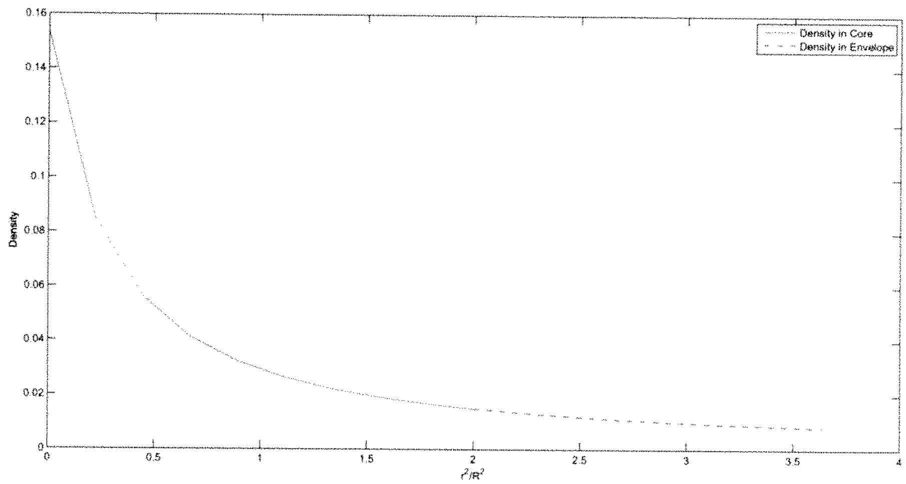

Figure 3.2: Variation of ρ against $\frac{r^2}{R^2}$ throughout the distribution for $f = 0.7$.

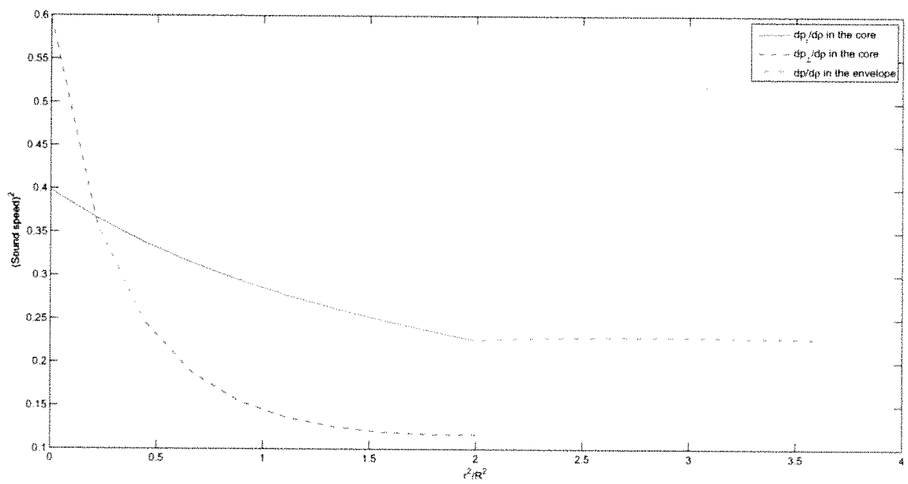

Figure 3.3: Variation of $\frac{dp_r}{d\rho}$, $\frac{dp_\perp}{d\rho}$ against $\frac{r^2}{R^2}$ in the core and variation of $\frac{dp}{d\rho}$ against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

Figure 3.4 indicates that strong energy condition is satisfied throughout the distribution. The variation of the polytropic index γ with respect to time function $f(t)$ is calculated numerically for the model with $\lambda = 0.05$ at centre and on the boundary and these variations are shown in figure 3.5. The polytropic index at the centre is less than $\frac{4}{3}$ and at the boundary is much larger than $\frac{4}{3}$ during the initial stage of collapse. This indicates that the central region is dynamically unstable. The collapsing star becomes a black hole when f takes the value 0.5356.

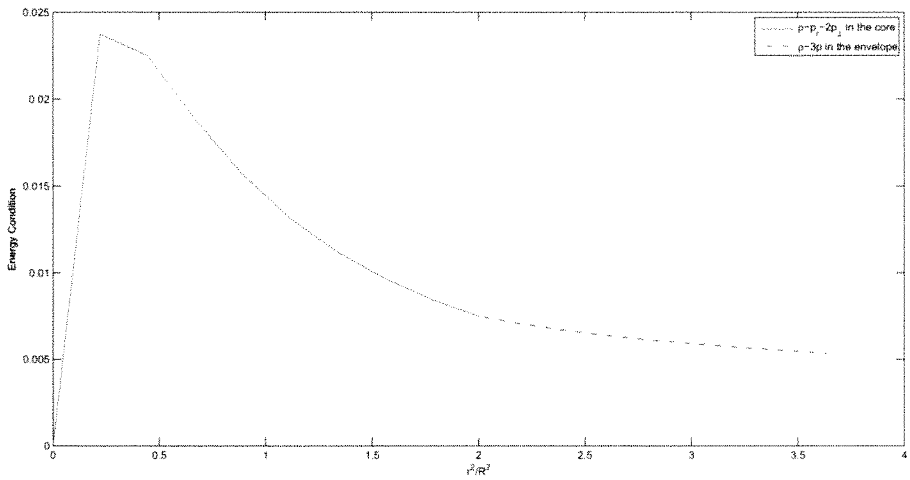

Figure 3.4: Variation of $\rho - p_r - 2p_\perp$ against $\frac{r^2}{R^2}$ in the core and variation of $\rho - 3p$ against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

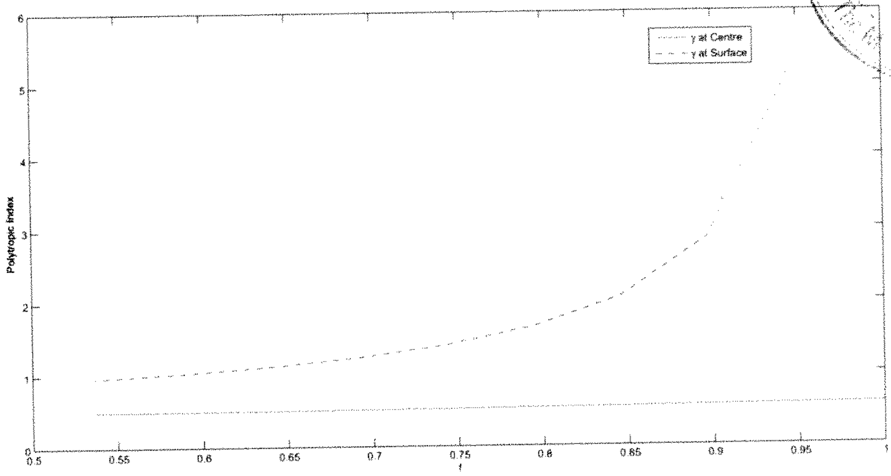

Figure 3.5: Variation of polytropic index γ at the centre and at the surface against f.

For temperature, we assume the evolution of heat flow governed by the Maxwell-Cattaneo transport equation

$$\tau_0 h^{\alpha\beta} u^c q_{\beta;c} + q^\alpha = K h^{\alpha\beta} \left(T_{,\beta} - T\dot{u}_\beta\right), \tag{3.9.1}$$

where $h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$ is the projection tensor, K is the thermal conductivity coefficient, τ_0 is the relaxation time and $\dot{u}_\beta = u_{\beta;c}u^c$. For the spacetime metric (3.2.1), transport equation (3.9.1) takes the form,

$$\tau_0 \dot{q} + e^{\nu/2} q = -K e^{-(\mu+\lambda)/2} \frac{d}{dr}\left(T e^{\nu/2}\right). \tag{3.9.2}$$

If the neutrinos are generated by thermal emission the τ_0 depends on temperature as

$$\tau_0 \propto T^{-3/2}. \tag{3.9.3}$$

Following J. Martínez [62], thermal conductivity $K = \frac{4}{3} b T^3 \tau_0$ and $b_0 = \frac{7a_0}{8}$, where $a_0 = 6.252 \times 10^{-64} cm^{-2} k^{-4}$ is a radiation constant. τ_0 can be expressed in dimensionless form as

$$\tau_0 \approx \mathbb{A} \frac{M_0}{\bar{\rho}\sqrt{Y_e T^3}}, \tag{3.9.4}$$

where $\mathbb{A} = 10^9 K^{3/2} m^{-1}$, M_0 is the initial mass of star in meters, T is the kelvin temperature, $\bar{\rho}$ is dimensionless energy density and $0.2 \leq Y_e \leq 0.3$.
Now equation (3.9.2) takes the form:

$$\frac{\mathbb{A}M_0}{\bar{\rho}\sqrt{Y_e}} T^{-3/2}\dot{q} + e^{\nu/2} q = -\left(\frac{7a}{6}\right)\left(\frac{\mathbb{A}M_0}{\bar{\rho}\sqrt{Y_e}}\right) T^{3/2} e^{-(\mu+\lambda)/2} \frac{d}{dr}\left(T e^{\nu/2}\right). \qquad (3.9.5)$$

The temperature profile can be obtained by solving equation (3.9.5) using appropriate initial conditions.