Chapter 4

Stability of Superdense Star on Paraboloidal Spacetime

Contents

In this chapter, we have discussed dynamical stability of superdense stars on paraboloidal spacetime under radial modes of pulsation. The paraboloidal spacetime metric is a particular case of Duorah and Ray [20] spacetime metric. Duorah and Ray spacetime metric was discussed in detail by Finch and Skea [26]. Our analysis indicates models with $0.26 \leq \frac{m}{a} \leq 0.36$ are stable for radial modes of pulsation. Here mass m and radius a are in kilometers as per geometrization convention. The paraboloidal geometry for its spatial sections $t = constant$ thus admits the possibilities of describing spacetime of superdense stars in equilibrium. The field equation and its solution is discussed in section 4.2. The dynamical stability of superdense star is discussed in section 4.3.

4.1 Introduction

The solution of Einstein's field equations for a perfect fluid sphere in thermodynamic equilibrium is not sufficient as the equilibrium may be stable equilibrium or unstable equilibrium. Buchdahl's theorem for stable star says that

$$a \geq \frac{8}{9} R_s,$$

where a is radius of star and R_s is the Schwarzschild radius. Hence the radius of a stable star exceeds the Schwarzschild radius. Buchdahl's theorem is independent of any equation of state $p = p(\rho)$.

Chandrasekhar considered the perturbation the solution of the star in stellar equilibrium resulting in non-zero off-diagonal elements in energy-momentum tensor by considering non-zero radial velocity for the fluid. Chandrasekhar assumed amplitude of oscillations ξ in the time dependent form $e^{i\sigma t}$ and applying Rayleigh-Ritz method of variational approach, Chandrasekhar then obtained pulsation equation. In that Pulsation equation if the frequency σ^2 is negative then the amplitude ξ does not have an upper bound. Hence for stable stars frequency must be positive.

We investigate the stability of models of superdense stars on paraboloidal spacetime under radial modes of pulsation. Tikekar and Jotania [89] have shown that the paraboloidal spacetime metric

$$ds^2 = e^{\nu(r)} dt^2 - \left(1 + \frac{r^2}{R^2}\right) dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.1.1}$$

is suitable for describing the interior of strange stars (star consisting of strange matter). The spacetime metric (4.1.1) is a particular case of spacetime metric used by Duorah and Ray [20] which has the form

$$ds^2 = A^2 y^2 (x)\, dt^2 - Z^{-1}(x)\, dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.1.2}$$

with

$$x \equiv Cr^2.$$

In spacetime metric (4.1.2) if we set $A^2 y^2(x) = e^{\nu(r)}$, $C = \frac{1}{R^2}$ and $Z^{-1}(x) = 1 + x$, we get spacetime metric (4.1.1).

Finch and Skea [26] showed that stellar models of Duorah and Ray [20] do not satify Einstein's field equations and they obtained solution satisfying Einstein's field equation.

4.2 Solution of Field Equations

We take the energy-momentum tensor of the form

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij}, \qquad u^i = \left(e^{-\nu/2}, 0, 0, 0\right), \tag{4.2.1}$$

where ρ and p respectively denote the matter density and fluid pressure. The Einstein's field equations for spacetime metric (4.1.1) in view of (4.2.1) takes the following form:

$$8\pi\rho = \frac{1}{R^2}\left(1+\frac{r^2}{R^2}\right)^{-2}\left(3+\frac{r^2}{R^2}\right), \tag{4.2.2}$$

$$8\pi p = \left(1+\frac{r^2}{R^2}\right)^{-1}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}, \tag{4.2.3}$$

$$\left(1+\frac{r^2}{R^2}\right)\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right]-\frac{r\nu'}{2R^2}+\frac{r^2}{R^4}=0. \tag{4.2.4}$$

Following Finch and Skea [26] and Tikekar and Jotania [89], the solution of field equations (4.2.2) - (4.2.4) is given by,

$$e^{\nu/2} = (B - Az)\cos z + (A + Bz)\sin z, \tag{4.2.5}$$

where $z = \sqrt{1+\frac{r^2}{R^2}}$ and A and B are constants of integration. Hence the matter density and fluid pressure take the following explicit forms,

$$8\pi\rho = \left(\frac{1}{R^2}\right)\left(\frac{z^2+2}{z^4}\right), \tag{4.2.6}$$

$$8\pi p = \frac{1}{z^2R^2}\left[\frac{(A-Bz)\, sinz + (Az+B)\, cosz}{(A+Bz)\, sinz - (Az-B)\, cosz}\right], \tag{4.2.7}$$

and the spacetime metric (4.1.1) takes the form

$$ds^2 = \left[(B-Az)\, cosz + (A+Bz)\, sinz\right]^2 dt^2 - z^2 dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.2.8}$$

at the boundary of the star $r = a$, the interior spacetime metric (4.2.8) should continously match with Schwarzschild exterior spacetime metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.2.9}$$

and pressure must vanish at the boundary of the star $r = a$. These two conditions determines the constants of integration A and B as

$$A = \frac{\sqrt{1 - \frac{2m}{a}} \left(z_a sinz_a - cosz_a\right)}{\left(sinz_a + z_a cosz_a\right)\left(cosz_a + z_a sinz_a\right) - \left(sinz_a - z_a cosz_a\right)\left(cosz_a - z_a sinz_a\right)}, \tag{4.2.10}$$

$$B = \left[\frac{sinz_a + z_a cosz_a}{z_a sinz_a - cosz_a}\right] A, \tag{4.2.11}$$

where $z_a = \sqrt{1 + \frac{a^2}{R^2}}$. Substituting the values of A and B in (4.2.3), we get the expression for the pressure profile of the distribution. The expresssion for $\frac{dp}{d\rho}$ is given by

$$\frac{dp}{d\rho} = \frac{z^2 \left(Bcosz + Asinz\right) T\left(z\right)}{\left(z^2 + 4\right)\left(Asinz + Bzsinz + Bcosz - Azcosz\right)^2}, \tag{4.2.12}$$

where $T(z) = \left(z^2 \sin z + \sin z - z \cos z\right) A + \left(z^2 \cos z + \cos z + z \sin z\right) B$ and A, B are given by (4.2.10) and (4.2.11).

The scheme given by Tikekar [88] is used to compute mass and size of the star. The density at the centre of the star is given by

$$8\pi\rho(0) = \frac{3}{R^2}, \tag{4.2.13}$$

and at the surface of the star is

$$8\pi\rho(a) = \frac{1}{R^2}\left(3 + \frac{a^2}{R^2}\right)\left(1 + \frac{a^2}{R^2}\right)^{-2}. \tag{4.2.14}$$

The density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)}$ is then given by

$$\lambda = \frac{\rho(a)}{\rho(0)} = \left(1 + \frac{a^2}{3R^2}\right)\left(1 + \frac{a^2}{R^2}\right)^{-2}. \tag{4.2.15}$$

The continuity of interior spacetime metric (4.2.8) with Schwarzschild exterior spacetime metric (4.2.9) determines the mass radius relation,

$$\frac{m}{a}=\frac{a^2}{2R^2}\left(1+\frac{a^2}{R^2}\right)^{-1}. \tag{4.2.16}$$

Equations (4.2.15) and (4.2.16) give $\frac{a^2}{R^2}$ and $\frac{m}{a}$ in terms of density variation parameter λ as

$$\frac{a^2}{R^2}=\frac{1-6\lambda+\sqrt{1+24\lambda}}{6\lambda}, \tag{4.2.17}$$

and

$$\frac{m}{a}=\frac{1-6\lambda+\sqrt{1+24\lambda}}{2\left(1+\sqrt{1+24\lambda}\right)}. \tag{4.2.18}$$

Further we can express λ in terms of $\frac{m}{a}$ in the form

$$\lambda=\left(1-\frac{2m}{a}\right)\left(1-\frac{4m}{3a}\right). \tag{4.2.19}$$

4.3 Dynamic Stability

A sufficient condition for the dynamic stability of a spherically symmetric distribution of matter under small radial adiabatic perturbations has been developed by Chandrasekhar [9]. A normal mode of radial oscillations for an equilibrium configuration

$$\delta r=\xi\left(r\right)e^{i\sigma t}, \tag{4.3.1}$$

is stable if its frequency σ is real and is unstable if σ is imaginary. Chandrasekhar's pulsation equation for the spacetime metric (4.1.1) is given by

$$\begin{aligned}\sigma^2\int_0^a e^{(3\lambda+\nu)/2}\left(p+\rho\right)\frac{u^2}{r^2}dr^2 = & \int_0^a e^{(\lambda+3\nu)/2}\frac{4}{r^3}\left(\frac{dp}{dr}\right)u^2dr - \\ & \int_0^a e^{(\lambda+3\nu)/2}\frac{1}{r^2}\frac{1}{p+\rho}\left(\frac{dp}{dr}\right)^2u^2dr + \\ & \int_0^a e^{(\lambda+3\nu)/2}\left(\frac{p+\rho}{r^2}\right)8\pi e^{\lambda}pu^2dr + \\ & \int_0^a e^{(\lambda+3\nu)/2}\left(\frac{p+\rho}{r^2}\right)\frac{dp}{d\rho}\left(\frac{du}{dr}\right)^2dr,\end{aligned} \tag{4.3.2}$$

where $u = \xi r^2 e^{-\nu/2}$ and $e^\lambda = 1 + \frac{r^2}{R^2}$. The boundary condition to be satisfied at $r = a$ is that Lagrangian change in pressure should vanish at $r = a$ that is,

$$\Delta p = -e^{\nu/2}\left(\frac{\gamma p}{r^2}\right)\left(\frac{du}{dr}\right) = 0 \ \ at \ \ r = a,$$

where γ is the adiabatic index. Therefore we must have,

$$\frac{du}{dr} = 0 \ \ at \ \ r = a. \qquad (4.3.3)$$

Following the method of Bardeen *et al.* [4], we choose

$$u = R^3 x^{3/2}\left(1 + a_1 x + b_1 x^2 + ...\right),$$

as a trial function, where $x = \frac{r^2}{R^2}$. The boundary condition $\frac{du}{dr} = 0$ at $r = a$ yields

$$3 + 5a_1 b + 7b_1 b^2 + ... = 0, \qquad (4.3.4)$$

where $b = \frac{a^2}{R^2}$ and a_1, b_1.... are parameters. We consider here a three term approximation of (4.3.4). The pulsation equation (4.3.2) for the metric (4.1.1) now takes the form,

$$\sigma^2 \int_0^a e^{(3\lambda+\nu)/2}(p+\rho)\frac{u^2}{R^2}dr = \int_0^a \{T_1 T_2 (T_3 + T_4)\}\{[(T_5 T_9 T_{10}) - (T_{11} T_{12} T_{13}) + T_{14}] T_{15}\} dr + \int_0^a \{T_1 T_2 (T_3 + T_4)\}\{(T_{16} T_{17})\} dr, \qquad (4.3.5)$$

where,

$T_1 = \frac{1}{2R^2 z^2}$,

$T_2 = z^2 + 3(B - Az)cosz + 3(Bz + A)sinz$,

$T_3 = \left[\frac{(A-Bz)sinz+(Az+B)cosz}{(A+Bz)sinz-(Az-B)cosz}\right]$,

$T_4 = \frac{z^2+2}{z^4}$,

$T_5 = 2T_1$,

$T_6 = Bcosz + Asinz$,

$T_7 = Az^2 sinz + Asinz + Bzsinz - Azcosz + Bz^2 cosz + Bcosz$,

$T_8 = (Asinz + Bzsiz + Bcosz - Azcosz)^2$,

$T_9 = \frac{T_6 T_7}{T_8}$,

$T_{10} = \frac{1}{T_3+T_4}$,

$T_{11} = \frac{4r^2}{R^4 z^4}$,

$T_{12} = T_{10}^2,$
$T_{13} = T_9^2,$
$T_{14} = \frac{8\pi}{R^4} T_3,$
$T_{15} = r^4 \left(1 + a_1 \frac{r^2}{R^2} + b_1 \frac{r^4}{R^4}\right)^2,$
$T_{16} = \left(\frac{z^2}{z^2+4}\right) T_9,$
$T_{17} = r^2 \left(3 + 5a_1 \frac{r^2}{R^2} + 7b_1 \frac{r^4}{R^4}\right)^2.$

4.4 Discussion

We have evaluated the integral on the right side of equation (4.3.5) numerically for different choices small, large, positive and negative values of the constants a_1, b_1. It is found that the integral admits positive value for the strange star models, $0.26 \leq \frac{m}{a} \leq 0.36$. Table 4.1 presents these numerical computations for certain specific choices of a_1 and b_1 for the model with $\frac{m}{a} = 0.27$. This analysis indicates that these models with $0.26 \leq \frac{m}{a} \leq 0.36$ will be stable for radial modes of pulsation. The static paraboloidal spacetime metric (4.2.8) for its spatial sections $t = constant$ admits the possibility of describing spacetime of superdense stars in equilibrium.

Table 4.1: The Values of the integral on the right side of the pulsation equation (4.3.5) for some specific choices of the constants a_1, b_1 with $\frac{m}{a} = 0.27$.

a_1	b_1	*Integral*
0.000	−0.717	10.4332
−0.776	0.000	4.8092
−0.858	1.000	34.8663
1.000	−1.641	22.3521
5.000×10^2	-4.627×10^2	6.6213×10^5
-5.410×10^2	5.000×10^2	7.7009×10^5
1.000×10^5	-9.240×10^4	2.6303×10^{10}
-1.000×10^5	9.240×10^4	2.6303×10^{10}