

Chapter 5

Core-Envelope Models of Superdense Stars on Paraboloidal Spacetime

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In this chapter, we study two core-envelope models of superdense stars on based on paraboloidal spacetime metric. Both the models satisfy all the physically plausible conditions. Core-envelope models with thin envelope are useful in the study of glitches and star quakes. The comparative study of both the models is done. A noteworthy feature of these models is that they admit thin envelope.

5.1 Introduction

The non-linear nature of Einstein's field equations is a consequence of the self interaction of the gravitational field. This makes it difficult to obtain relativistic models of spherical stars based on exact solutions of Einstein's field equations. The standard method for studying cold compact stars consists of integrating the Tolman, Oppenheimer and Volkoff (TOV) equation assuming an equation of state $p = p(\rho)$, where p is the proper pressure and ρ is the proper density, for the matter distribution. The integration continues till pressure drops down to zero for some value $r = a$ which is taken as the radius of the spherical distribution.

When the density exceeds twice nuclear density, the equation of state becomes uncertain. A widely accepted alternative approach to deal with such situations is the one suggested by Vaidya and Tikekar [99]. In this approach one assigns a geometry to the physical three space in place of the equation of state.

Tikekar and Jotania [89] have shown that the paraboloidal spacetime metric is suitable for describing relativistic models of strange stars and hybrid neutron stars. In this chapter we present two core-envelope models on paraboloidal spacetime.

The assumption of taking isotropic pressure distribution in the core and anisotropic pressure distribution in the envelope may not be unphysical in the case of core consisting of degenerate fermi fluid while its outer envelope may consist of fluid having anisotropic pressure. Further the study of glitches and quakes is important in stars having thin envelope. We investigate whether paraboloidal spacetime is useful in describing spherical distribution of matter with isotropic pressure in the core and anisotropic pressure in the thin envelope.

The field equations for anisotropic models are described in section 5.2. Two different core-envelope models are discussed in sections 5.3 and 5.4. The physical plausibility conditions are checked in section 5.5 which also includes comparative study of the thickness of core and envelope of both the models.

5.2 The Core and Envelope of the Star

We consider the static spherically symmetric paraboloidal spacetime metric

$$ds^2 = e^{\nu(r)} dt^2 - \left(1 + \frac{r^2}{R^2}\right) dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (5.2.1)$$

Following Maharaj and Martens [59], we write the energy momentum tensor for anisotropic fluid distribution in the form:

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij} + \pi_{ij}, \quad u_i u^i = 1, \quad (5.2.2)$$

$$\pi_{ij} = \sqrt{3} S \left[C_i C_j - \frac{1}{3} (u_i u_j - g_{ij}) \right]. \quad (5.2.3)$$

The magnitude of anisotropic stress tensor is $S = S(r)$ and $C^i = \left(0, -\frac{1}{\sqrt{1+\frac{r^2}{R^2}}}, 0, 0\right)$ is a radial vector. For equilibrium models $u_i = (e^{\nu/2}, 0, 0, 0)$ and the non-vanishing components of energy momentum tensor are

$$T_0^0 = \rho, \quad T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right), \quad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right). \quad (5.2.4)$$

The pressure along the radial and tangential direction respectively are given by

$$p_r = p + \frac{2S}{\sqrt{3}}, \quad (5.2.5)$$

and

$$p_\perp = p - \frac{S}{\sqrt{3}}. \quad (5.2.6)$$

Hence the magnitude of anisotropy is given by

$$8\pi\sqrt{3}S = p_r - p_\perp. \quad (5.2.7)$$

The field equations for the spacetime metric (5.2.1) and energy-momentum tensor (5.2.2) are equivalent to the following three equations

$$8\pi\rho = \frac{3 + \frac{r^2}{R^2}}{R^2 \left[1 + \frac{r^2}{R^2}\right]^2}, \quad (5.2.8)$$

$$8\pi p_r = \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \quad (5.2.9)$$

and

$$8\pi\sqrt{3}S = \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} + \frac{1}{R^2} \left(1 + \frac{r^2}{R^2}\right)^{-2} \left[1 + \frac{\nu'r}{2}\right] - \left(1 + \frac{r^2}{R^2}\right)^{-1} \left[\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r}\right]. \quad (5.2.10)$$

By applying transformation

$$z = \sqrt{1 + \frac{r^2}{R^2}}, \quad (5.2.11)$$

and

$$F_1 = e^{\nu/2}, \quad (5.2.12)$$

The nonlinear equation (5.2.10) takes the form

$$\frac{d^2 F_1}{dz^2} - \frac{2}{z} \frac{dF_1}{dz} + \left(\frac{8\pi\sqrt{3}SR^2 z^4 + z^2 - 1}{z^2 - 1}\right) F_1 = 0. \quad (5.2.13)$$

The core of the star extends up to the radius $r = b (< a)$, where $S(r) = 0$ and the radius of the star is taken as $r = a$.

The isotropic pressure distribution is considered in the core region $0 \leq r \leq b$, hence magnitude of anisotropy parameter $S = 0$ in the core and the solution of field equations lead to the spacetime metric (4.2.8) and the expressions of density and pressure respectively are given by (4.2.2), (4.2.3).

We choose anisotropic pressure distribution in the envelope, hence the anisotropic parameter $S(r) \neq 0$ for $b \leq r \leq a$, where a is the boundary of the star.

5.3 Core-Envelope Model - 1

On prescribing

$$8\pi\sqrt{3}S = \frac{(z^2 - 1)(9 - 4z^2)}{4z^6 R^2}, \quad (5.3.1)$$

the equation (5.2.13) takes the form

$$4z^2 \frac{d^2 F_1}{dz^2} - 8z \frac{dF_1}{dz} + 9F_1 = 0, \quad (5.3.2)$$

which admits the closed form solution as

$$e^{\nu/2} = Cz^{3/2} \log z + Dz^{3/2}, \quad (5.3.3)$$

where C and D are constants of integration. Therefore the spacetime metric in the envelope region $b \leq r \leq a$ is described by:

$$ds^2 = (Cz^{3/2} \log z + Dz^{3/2})^2 dt^2 - z^2 dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5.3.4)$$

The matter density, radial pressure and tangential pressure take the following forms:

$$8\pi\rho = \frac{2 + z^2}{R^2 z^4}, \quad (5.3.5)$$

$$8\pi p_r = \frac{(3 \log z + 2 - z^2 \log z) C + (3 - z^2) D}{R^2 z^4 (C \log z + D)}, \quad (5.3.6)$$

$$8\pi p_{\perp} = \frac{(3 \log z + 2 - z^2 \log z) C + (3 - z^2) D}{R^2 z^4 (C \log z + D)} - \frac{9 - 4z^2}{4R^2 z^6}. \quad (5.3.7)$$

At the boundary of the star $r = a$, the spacetime metric in the envelope (5.3.4) should continuously match with Schwarzschild exterior spacetime metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5.3.8)$$

Also radial pressure (5.3.6) must vanish at the boundary of the star $r = a$. The conditions yields the following relationships:

$$\frac{m}{a} = \frac{z_a^2 - 1}{2z_a^2}, \quad (5.3.9)$$

$$C = \frac{z_a^2 - 3}{2z_a^{5/2}}, \quad (5.3.10)$$

and

$$D = \frac{3 \log z_a + 2 - z_a^2 \log z_a}{2z_a^{5/2}}. \quad (5.3.11)$$

where $z_a = \sqrt{1 + \frac{a^2}{R^2}}$. Substituting the value of C and D in (5.3.6) and (5.3.7) we get radial and tangential pressure in the envelope of the star.

At the core-envelope boundary $r = b$, due to assumption (5.3.1) gives core radius as $b = \frac{\sqrt{5}}{2}R$. Also at the core-envelope boundary, coefficients of the spacetime metric (4.2.8) must continuously match with spacetime metric (5.3.4) and $p(b) = p_r(b) = p_\perp(b)$. These conditions lead to the following values for A and B in terms of C and D as

$$A = 1.3367C + 1.1928D, \quad (5.3.12)$$

$$B = -0.2850C + 0.4938D. \quad (5.3.13)$$

Substituting the values of A and B in equation (4.2.3), we get pressure in the core of the star.

5.4 Core-Envelope Model - 2

By choosing

$$8\pi\sqrt{3}S = \frac{(z^2 - 1)(2 - z^2)}{z^6 R^2}, \quad (5.4.1)$$

where z is given by (5.2.11), Tikekar and Jotania [90] have obtained solution of (5.2.13) in the form

$$e^{\nu/2} = Ez^2 - 2Fz, \quad (5.4.2)$$

where E and F are constants of integration. The spacetime metric in the envelope region is described by the metric

$$ds^2 = (Ez^2 - 2Fz)^2 dt^2 - z^2 dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (5.4.3)$$

The expressions for density, radial pressure and tangential pressure in the envelope are respectively given by

$$8\pi\rho = \frac{2 + z^2}{R^2 z^4}, \quad (5.4.4)$$

$$8\pi p_r = \frac{Ez(4 - z^2) - 2F(2 - z^2)}{R^2 z^2 [Ez^3 - 2Fz^2]}, \quad (5.4.5)$$

$$8\pi p_\perp = \frac{Ez(4 - z^2) - 2F(2 - z^2)}{R^2 z^2 [Ez^3 - 2Fz^2]} - \frac{2 - z^2}{R^2 z^6}. \quad (5.4.6)$$

Equation (5.4.1) determines the core boundary as $b = R$. The constants E and F are to be determined by matching the spacetime metric (5.4.3) with Schwarzschild exterior spacetime metric (5.3.8), across the boundary $r = a$, where $p_r(a) = 0$, which gives

$$\frac{m}{a} = \frac{z_a^2 - 1}{2z_a^2}, \quad (5.4.7)$$

$$E = \frac{z_a^2 - 2}{2z_a^3}, \quad (5.4.8)$$

and

$$F = \frac{z_a^2 - 4}{4z_a^2}, \quad (5.4.9)$$

where $z_a = \sqrt{1 + \frac{a^2}{R^2}}$.

Substituting the values of E and F in (5.4.5) and (5.4.6), we get expressions of radial and tangential pressure in the envelope. At the core-envelope boundary $r = b$, the spacetime metric (4.2.8) must continuously match with spacetime metric (5.4.3) and $p(b) = p_r(b) = p_{\perp}(b)$. This gives

$$\left(\sin\sqrt{2} - \sqrt{2}\cos\sqrt{2}\right) A + \left(\cos\sqrt{2} + \sqrt{2}\sin\sqrt{2}\right) B = 2E - 2\sqrt{2}F, \quad (5.4.10)$$

and

$$\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right) A + \left(\cos\sqrt{2} - \sqrt{2}\sin\sqrt{2}\right) B = 2E. \quad (5.4.11)$$

Solving (5.4.10) and (5.4.11) for A and B we get,

$$A = 1.9755E - 1.2410F, \quad (5.4.12)$$

$$B = 0.31185E - 1.2083F. \quad (5.4.13)$$

Substituting the values of A and B in equation (4.2.3), we can obtain the expression of pressure in the core of the star.

5.5 Discussion

Since we have not assumed any equation of state, the matter distribution in the core and envelope should satisfy the following conditions:

- (i) $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$,
- (ii) $p > 0$, $\frac{dp}{dr} < 0$, $\frac{dp}{d\rho} < 1$, $\rho - p > 0$ for $0 \leq r \leq b$,
- (iii) $p_r \geq 0$, $p_\perp > 0$, $\frac{dp_r}{dr} < 0$ for $b \leq r \leq a$,
- (iv) $\frac{dp_r}{d\rho} < 1$, $\frac{dp_\perp}{d\rho} < 1$, $\rho - p_r \geq 0$, $\rho - p_\perp \geq 0$ for $b \leq r \leq a$.

The scheme given by Tikekar [88], which is described in section 4.2 is used to determine the mass and size of the superdense star. It follows from the expression (5.3.5) that $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$ for both core-envelope models. The expressions of $\frac{a^2}{R^2}$, $\frac{m}{a}$ and density variation parameters λ are described by equations (4.2.17) - (4.2.19) respectively.

In Finch and Skea [26] approach, the ratio $\frac{A}{B}$ is restricted by the limits $0.217958 \leq \frac{A}{B} \leq 6.406980$. This restriction, in view of the arguments described in section 4.2 leads to the constraint $\frac{m}{a} < 0.3614955$. If $\frac{m}{a} > 0.3614955$, then $\frac{dp}{d\rho} > 1$ in the core, and therefore the models with $\frac{m}{a} > 0.3614955$, physical plausibility condition (ii) is not satisfied in the core. Further for core-envelope model - 1 having $\frac{m}{a} \geq 0.36$ it is observed that $\frac{dp_r}{d\rho} > 1$ in the envelope violating the condition (iv). Following Sharma *et al.* [81], we choose central density as $\rho(0) = 4.68 \times 10^{15} \text{ gm cm}^{-3}$. From (5.3.5) it is observed that density is decreasing throughout the distribution. It is observed that conditions (ii) - (iv) are satisfied for the stars for which $0.28 \leq \frac{m}{a} \leq 0.35$ for the first model and $0.26 \leq \frac{m}{a} \leq 0.36$ for the second model, using programming and graphical methods.

Numerical estimates of the radius of the star (in kilometers), the core-radius (in kilometers), the mass of the star (in kilometers) and the thickness of the envelope (in kilometers) for the first model are given in table 5.1 and for the second model are given in table 5.2. The mass of the star in grams is obtained as $M = \frac{mc^2}{G}$. These models admit thin envelopes and the thickness of the envelope increases as $\frac{m}{a}$ increases.

Table 5.1: Masses and equilibrium radii of core-envelope model - 1 of superdense stars corresponding to $\rho(0) = 4.68 \times 10^{15} \text{ gm/cm}^3$.

$\frac{m}{a}$	a	m	$b(= \frac{\sqrt{5}}{2} R)$	Thickness of the envelope
0.28	6.617267	1.852835	6.557918	0.059349
0.29	6.892874	1.998934	6.557918	0.334956
0.30	7.183840	2.155152	6.557918	0.625921
0.31	7.492298	2.322612	6.557918	0.934380
0.32	7.820774	2.502648	6.557918	1.262856
0.33	8.172285	2.696854	6.557918	1.614367
0.34	8.550480	2.907163	6.557918	1.992561
0.35	8.959822	3.135938	6.557918	2.401904

Table 5.2: Masses and equilibrium radii of core-envelope model - 2 of superdense stars corresponding to $\rho(0) = 4.68 \times 10^{15} \text{ gm/cm}^3$.

$\frac{m}{a}$	a	m	$b(= R)$	Thickness of the envelope
0.26	6.105090	1.587323	5.865581	0.239509
0.27	6.355196	1.715903	5.865581	0.489616
0.28	6.617267	1.852835	5.865581	0.751687
0.29	6.892874	1.998934	5.865581	1.027294
0.30	7.183840	2.155152	5.865581	1.318259
0.31	7.492298	2.322612	5.865581	1.626718
0.32	7.820774	2.502648	5.865581	1.955194
0.33	8.172285	2.696854	5.865581	2.306705
0.34	8.550480	2.907163	5.865581	2.684899
0.35	8.959822	3.135938	5.865581	3.094242
0.36	9.405854	3.386107	5.865581	3.540274

The plots showing variation of ρ throughout the distribution, p in the core, p_r and p_\perp in the envelope, $\frac{dp}{d\rho}$ in the core, $\frac{dp_r}{d\rho}$ and $\frac{dp_\perp}{d\rho}$ in the envelope against z for core-envelope model - 1 and model - 2 with $\frac{m}{a} = 0.29$ are depicted in Figures 5.1 - 5.5. Figure 5.2 shows that the pressure in the core for core-envelope model - 1 is always greater than that of model - 2. Figure 5.3 shows that tangential pressure is always greater than radial pressure in the envelope for both the models. It is also observed from figures 5.4 and 5.5 that speed of sound is less than speed of light.

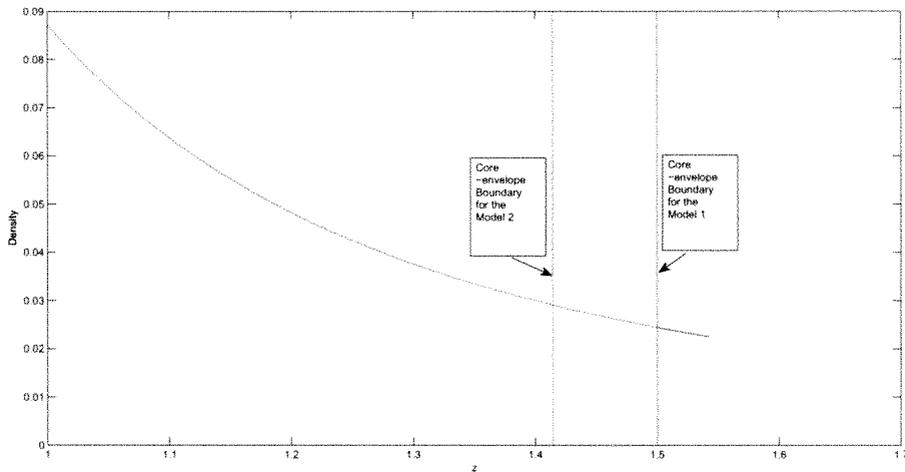


Figure 5.1: Variation of ρ against z throughout the distribution

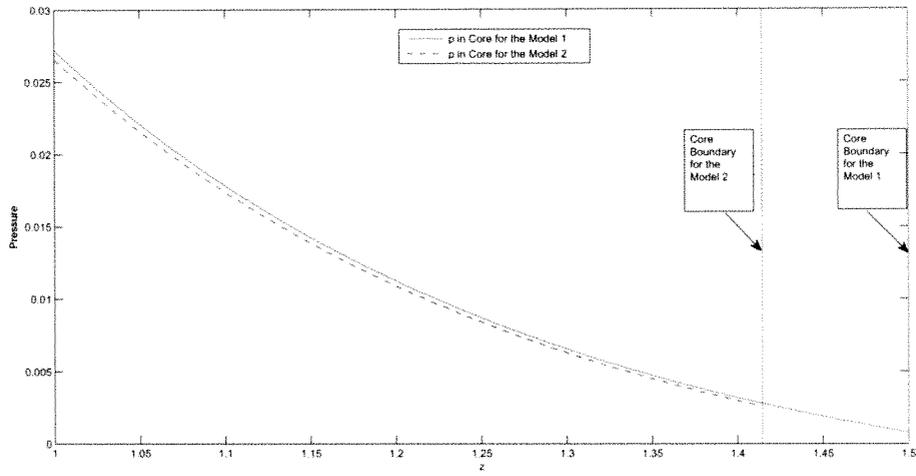


Figure 5.2: Variation of p against z in the core

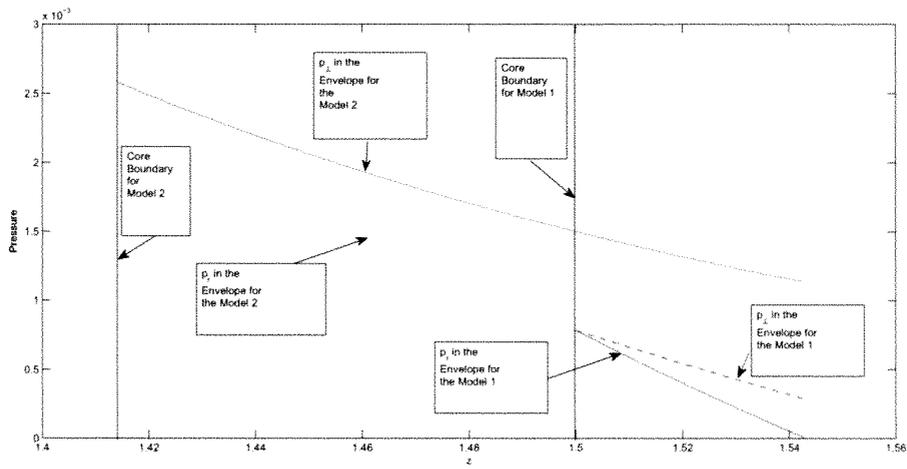


Figure 5.3: Variation of p_r and p_{\perp} in the envelope

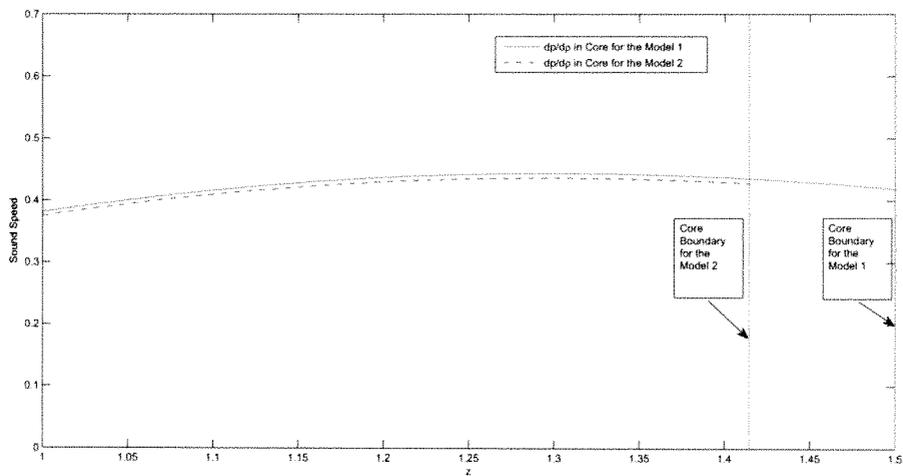


Figure 5.4: Variation of $\frac{dp}{d\rho}$ against z in the core

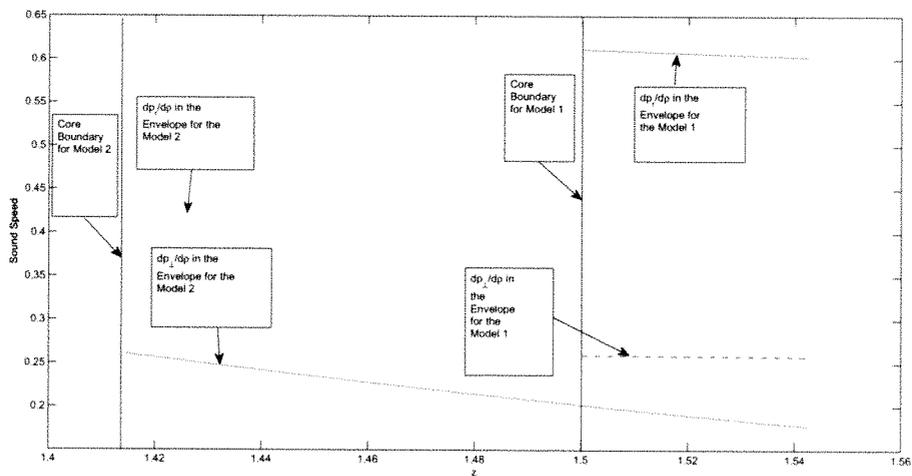


Figure 5.5: Variation of $\frac{dp_r}{d\rho}$ and $\frac{dp_i}{d\rho}$ against z in the envelope

These models are falling under Type I and Type II strange stars (Tikekar and Jotania [89]). From table 5.1 and table 5.2, it is observed that the first core-envelope model has very thin envelope. Thus we have presented core-envelope models with isotropic pressure in the core and anisotropic pressure in the envelope. A noteworthy feature of these models is that they admit thin envelope. Hence these models are significant in the study of glitches and star quakes.