Chapter 3. A wavelet based MRA technique for approximating a discrete signal

3.1. Introduction

Discrete Wavelet Transform and Multi Resolution Analysis using Wavelet Filters are now very widely used in the areas of feature extraction [3,4]. But very little work is found in the literature [14, 15, 32] about making use of these techniques for signal approximation or interpolation.

The approximation of a signal is relevant in the context of OCR system development also. Digital images are two dimensional digital signals. In cases where there is insufficient number of images of symbols, these methods of interpolation can be used to generate additional images for the test set or the training set.

Approximating a given signal to double its length for better identification of the locations of the peaks in the signal is a common procedure in industrial applications for the local analysis of the signal as explained below:

To take a specific industrial application, we may consider the problem of identifying the existence and quantity of a particular element in a metal block by the use of atomic mass spectroscopy where a CCD camera is used for obtaining the spectrum. The presence of a peak in the spectrum may indicate the presence of a particular substance/element but this peak may fall in between two pixels in the spectral image. The idea is to upsample the spectral data corresponding to subsampling in CCD. This can be done in either of the following two ways:

1. The device with higher resolution of CCD camera may be used
2. Some techniques of digital signal processing can be applied to the current data.

Since the first option is not always feasible, the second option is often preferred. In this paper we discuss a novel approach for interpolating a given digital signal to double its length by using Multi Resolution Analysis of Discrete Wavelet Transform.

This chapter is divided in to six sections. After giving an introduction in the first section, we discuss the analysis of Fourier and Wavelet transform in signal processing terms in the second section. The effect of applying low-pass filters and high-pass filters to the discretized (finite length) signal is described in the third section. In the fourth section, we highlight the characteristics of the approach presented earlier by applying it on various numerical examples. While the properties of this approach are to be discussed in the fifth section. At the end we conclude the results in the sixth section.

3.2 Analysis of Fourier and Wavelet transform in signal processing

It is well known from Fourier theory that a signal can be expressed as the sum of a, possibly infinite, series of sine and cosines. This sum is also referred to as a Fourier expansion. The big disadvantage of a fourier expansion however is that it has only frequency resolution and no time resolution. This means that although we might be able to determine all the frequencies present in a signal, we do not know when they are present. To overcome this problem in the past 4 to 5 decades several solutions (Fast Fourier Transform (FFT), Windowed Fourier Transform (WFT) etc.) have been developed which are more or less able to represent a signal in the time and frequency domain at the same time.

The idea behind these time-frequency joint representations is to cut the signal of interest into several parts and then analyze the parts separately. It is clear that analyzing a signal this way will give more information about the when and where of different frequency components, but it leads to a fundamental problem as well: how to cut the signal? Suppose that we want to know exactly all the frequency components present at a certain moment in time.

The problem here is that cutting the signal corresponds to a convolution between the signal and the cutting window. Since convolution in the time domain is identical to multiplication in the frequency domain and since the Fourier transform of a Dirac pulse contains all possible frequencies the frequency components of the signal will be smeared out all over the frequency axis. In fact this situation is the opposite of the standard Fourier transform since we now have time resolution but no frequency resolution whatsoever.

The underlying principle of the phenomena just described is due to Heisenberg's uncertainty principle, which, in signal processing terms, states that it is impossible to know the exact frequency and the exact time of occurrence of this frequency in a signal. In other words, a signal can simply not be represented as a point in the time-frequency space. The uncertainty principle shows that it is very important how one cuts the signal.

The wavelet transform or wavelet analysis is probably the most recent solution to overcome the shortcomings of the Fourier transform. In wavelet analysis the use of a fully scalable modulated window solves the signal-cutting problem. The window is shifted along the signal and for every position the spectrum is calculated. Then this process is repeated many times with a slightly shorter (or longer) window for every new cycle. In the end the result will be a collection of time-frequency representations of the signal, all with different resolutions. Because of this collection of representations we can speak of a multiresolution analysis.

3.3. Discretization and Filter Process

In this section, we present the basic result of signal transformation and signal reconstruction using MRA. Consider standard lemma [1.3] regarding the reconstruction of a signal using MRA (figure-1.5) discussed in the chapter 1. In the following section, we introduce the use of wavelets for the Approximation of a signal to double its length.

3.3.1. Signal Approximation Using Wavelets

In this section we recall the lemma-1.3 of chapter 1 which demonstrates the analysis phase (discrete wavelet transform) and synthesis phase (inverse discrete wavelet transform) for a given function $f \in L^2(R)$. Consider the resolution level of the function is *(j+1)*, so we denote this function by y_{j+1}. For each $j \in z$, define sequences $x_j = (x_j(k))_{k\in z}$ and $y_j = (y_j(k))_{k\in z}$ by $x_j = D(y_{j+1} * \tilde{v})$ and $y_j = D(y_{j+1} * \tilde{u})$. Where D is the downsampling operator on $l^2(z)$. $u = (u_j(k))_{k\in z}$ and $v = (v_j(k))_{k\in z}$ are scaling and wavelet sequence respectively. And $\tilde{u}$ and $\tilde{v}$ are the dual sequences of u (approximation coefficients) and v (detailed coefficients) defined as $\tilde{u}(n) = u(N-n)$ and $\tilde{v}(n) = v(N-n)$ where N is the length of the signal.

The reconstruction of y_{j+1} using one analysis phase and one synthesis phase can be given by

$$y_{j+1} = U(y_j) * u + U(x_j) * v$$

Where U is the upsampling operator on $l^2(z)$.

We extend this result by adding one more level of synthesis phase for the purpose of getting approximated double length signal as follows:

The use of wavelets to interpolate a given digital signal is a less studied area. The pictorial representation of the above equation is shown in figure-1.5 of chapter-1 which is made up of one analysis and one synthesis phase and yields reconstruction of a signal. In order to approximate the signal to its double length, one more level of synthesis phase is applied to figure-1.5.

Consider a signal $f \in L^2(R)$ of length m (number of sample points). Upon applying one analysis phase and two synthesis phases on *f*, we get one level higher resolution of f (say $w_{j+2}(2m)$) which will be of doubled length compared to the original signal values of length m.

Mathematical form of w_{j+2} in terms of detail and approximation components, x_j *and* y_j respectively, can be obtained from figure- 3.1 as follows :

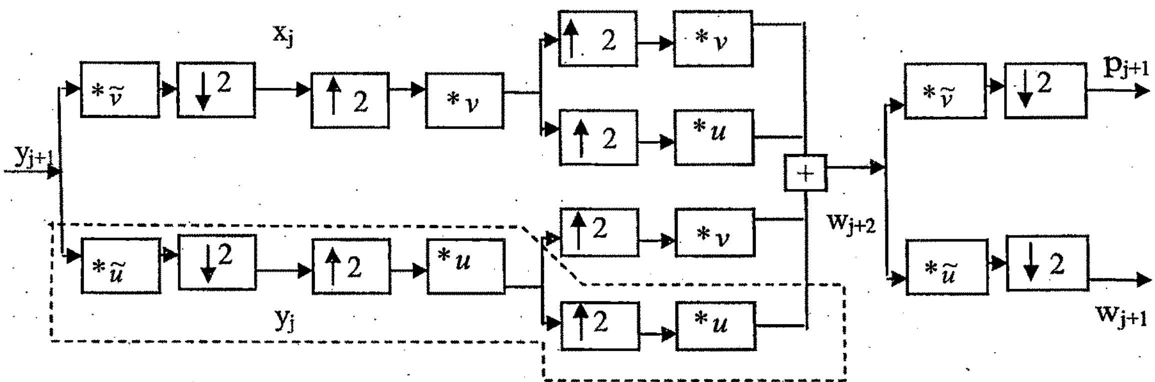


Fig. 3.1. Approximation using filter bank of DWT

$$w_{j+2} = U(U(y_j)*u)*u + U(U(x_j)*v)*u] + U(U(y_j)*u)*v + U(U(x_j)*v)*v \quad (3.1)$$

In section 3.4, we will prove that w_{j+2} is equal to y_{j+2} (the actual higher resolution of y_{j+1}). These computations were implemented in Java.

Following section demonstrates the applicability of a new approach using the equation (3.1) of Approximation using numerical examples of some finite length signals.

3.4. Numerical Examples

We have carried out some experiments with a few discretized signals and applied Haar, Daubechies D4, D6 discrete wavelet transforms on them with the objective of Approximation. The following sections present the results of the numerical experiments.

3.4.1 Approximation of a sinusoidal function

Consider the function sin(x) where $x \in [0,2\pi]$. The interval was divided in to 127 equal sub-intervals and the function was sampled at the 128 points (say m = 128) of discretization.

$$f(x) \begin{cases} = \sin(x) \ , & x\in[0,2\pi] \\ = 0 \ , & otherwise \end{cases}$$

A graph of the function with the sampled values is as shown in figure-3.2. The graph of the w_j values obtained from equation 3.1 is shown in figure-3.3(a).

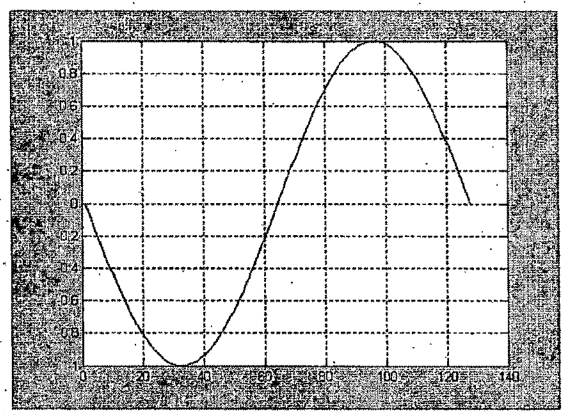


Fig.3.2. Original sin(x) at 128 sample points

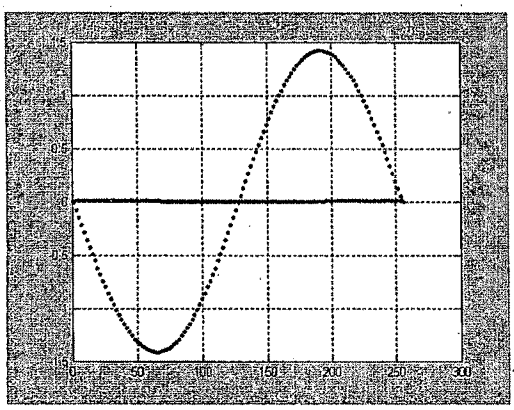


Fig. 3.3(a) Graph of sin(x) with destorption

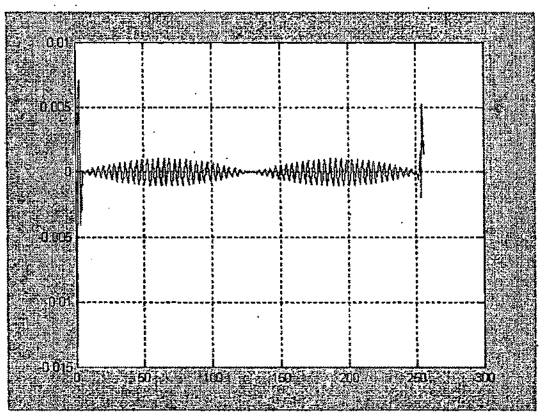


Fig. 3.3(b). Graph of high pass components

It is observed from these graphs that the process specified in equation (3.1) leads to two types of distortions to the original signal, namely (i) there is a distortion in the graph due to the presence of values near horizontal axis and (ii) the amplitude of the signal increases by a factor of about $\sqrt{2}$. It is further observed that these distortions occur due to the presence of high frequency components (figure-3.3(b)). Due to these observations we propose to have only the low frequency components (highlighted region in figure-3.1) for the purpose of approximating the signal to double its length (*2m*).

Therefore we consider the following expression r_{j+2} extracted from equation (3.1) for being the form for the interpolated signal :

$$r_{j+2} = \sqrt{2}[U(U(y_j)*u)*u] \qquad (3.2)$$

where $\sqrt{2}$ is a scaling parameter (estimated from the above mentioned observations) .

Equation (3.2) yields, desired Approximation of *sin(x)* to its double length ($2m = 256$ sample points), where $x \in [0,2\pi]$. Figure-3.4 , 3.5(a) and 3.5(b) shown below demonstrate the graphs of 256 sample points vs. *sin(x)* with the help of equation (3.2) by using Daubechies D4, Haar and D6 wavelets coefficients respectively.

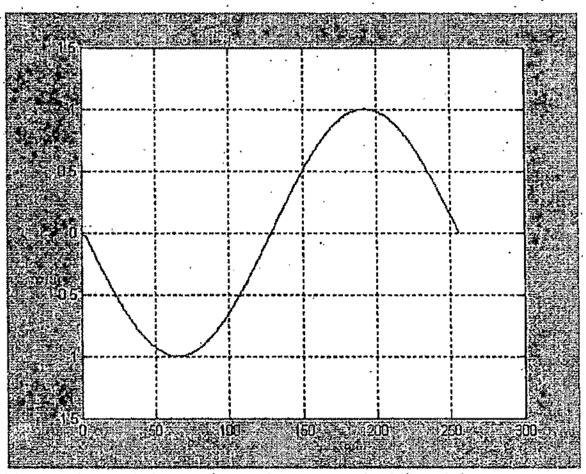


Fig. 3.4. 256 Sample points vs. sin(x) with D4

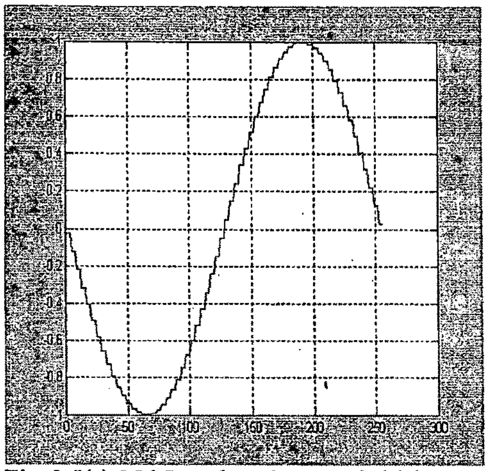


Fig. 3.5(a) 256 Sample points vs. sin(x) by Haar

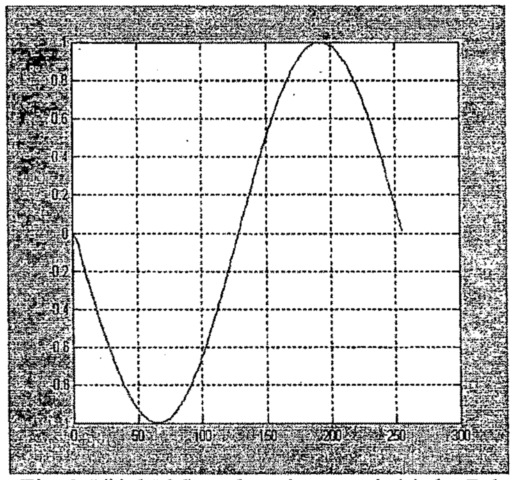


Fig. 3.5(b) 256 Sample points vs. sin(x) by D6

3.4.2 Approximation of a linear signal

Now consider a linear signal $f(x) = x$ in $L^2(R)$ such that,

$$f(x) \begin{cases} = x, & x \in [0,\ 31] \\ = 0, & otherwise \end{cases}$$

Dividing the signal in to 31 equal parts, the 32 discrete values are as shown in figure-3.6(a). Hence the length of the signal is $m=32$. Figure-3.6(b) exhibits the interpolated graph of this signal to double its length ($2m = 64$ sample points) by applying equation 3.2 by using Daubechies D4 wavelets. Similarly figure-3.6(c) and 3.6(d) shown below gives interpolated graph of the signal using Haar and Daubechies D6 respectively.

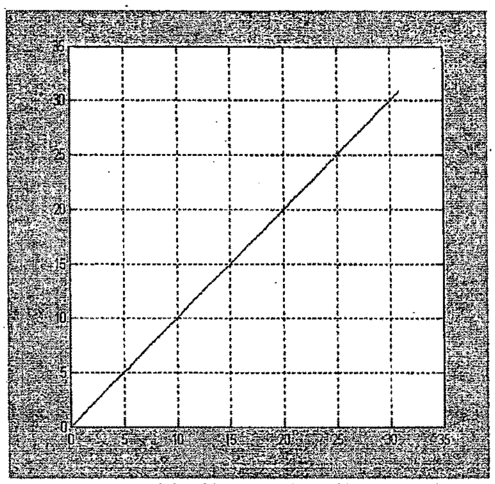


Fig. 3.6(a). f(x) = x is a line graph.

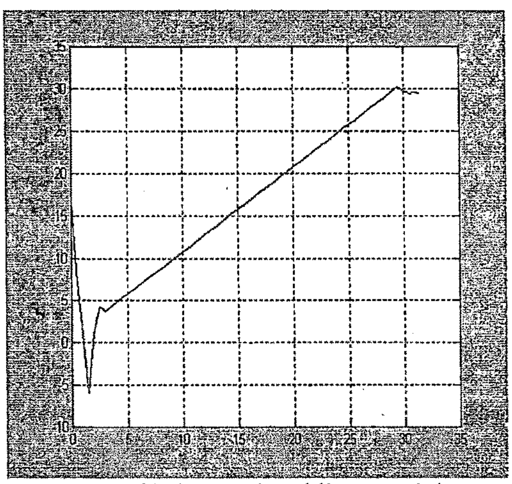


Fig. 3.6(b). interpolated line graph by D4

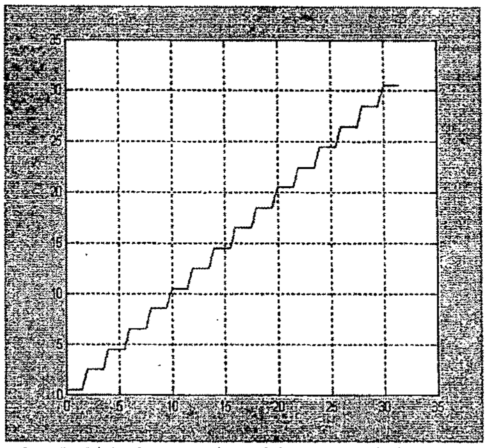


Fig. 3.6(c). interpolated line graph by Haar

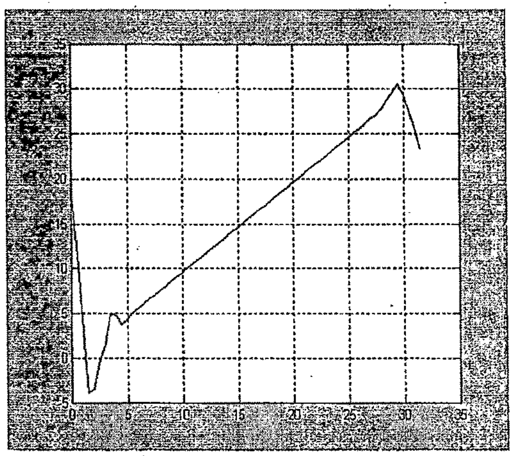


Fig. 3.6(d). interpolated line graph by D6

3.4.3 Approximation of a parabolic signal

Let $f(x) = x^2$ be a parabolic signal in $L^2(R)$ such that,

$$f(x) \begin{cases} = x^2 , & x \in [0,\ 31] \\ = 0 , & otherwise \end{cases}$$

Dividing the signal in to 31 equal parts, the 32 discrete values are as shown in figure-3.7(a). Here also the length of this signal is m=32. Up on applying equation 3.2, the signal is getting interpolated to double its length. Figure-3.7(b) shows the sharp feature of the curvature using Daubechies D4 wavelets.

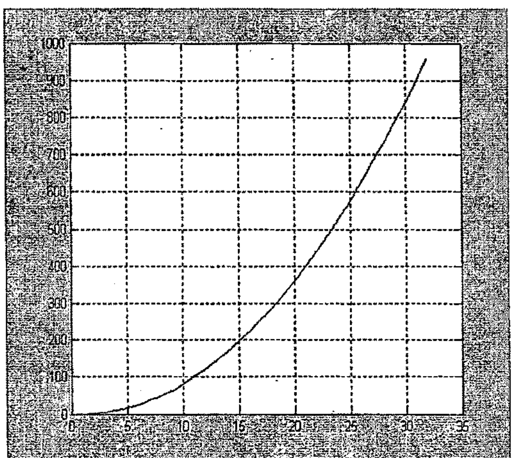


Fig. 3.7(a). Graph of $y = x^2$

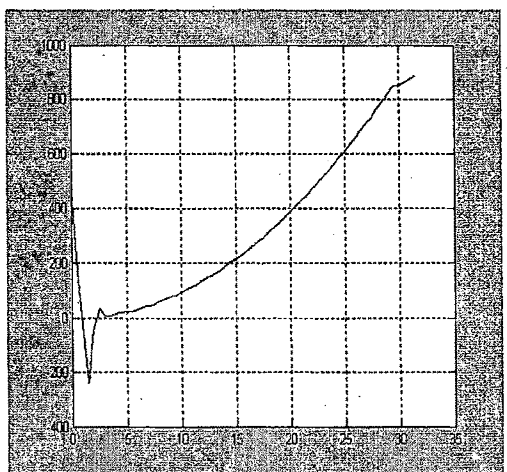


Fig. 3.7(b). interpolated graph of $y = x^2$ by D4

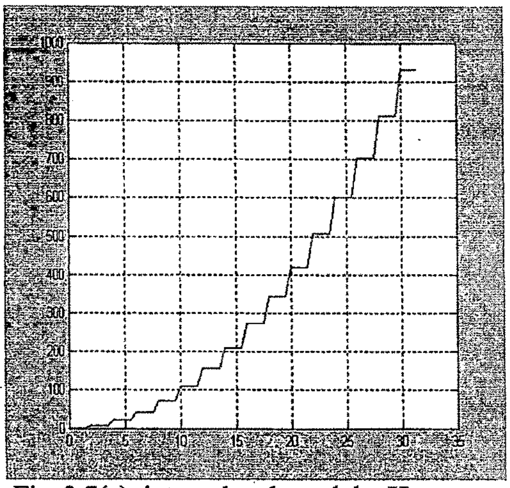


Fig. 3.7(c). interpolated graph by Haar

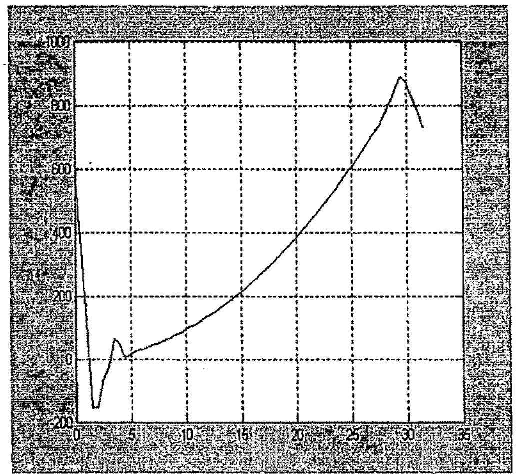


Fig. 3.7(d). interpolated graph of $y=x^2$ by D6

Fig. 3.7(c) and 3.7(d) exhibit the interpolated graphs obtained by equation (3.2) by using Haar and Daubechies D6 wavelets coefficients respectively.

It is observed from these three illustrations that sinusoidal function gives perfect Approximation while non periodic functions (linear and parabolic) shows some kind of distortion at the boundary points which arise due to the lack of information near boundary points.

3.4.4 Approximation of a highly Non-linear signal

We have collected a sequence of observations of the length 32 (see figure-3.8(a)) from the industry. Figure-3.8(b) demonstrates the resultant interpolated double length signal of the length 64 by applying equation (3.2).

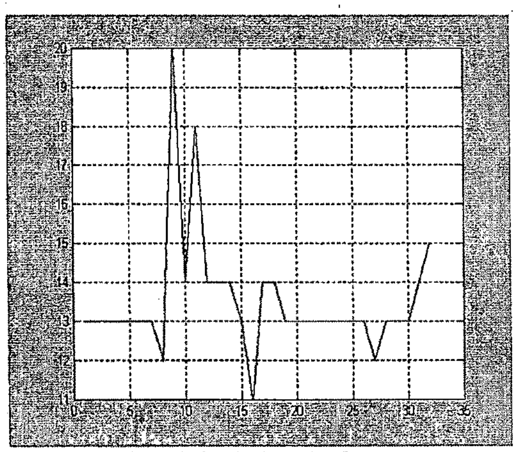


Fig. 3.8(a). original signal of size $n=32$

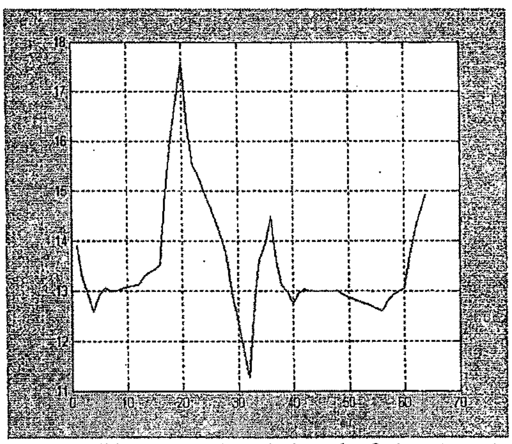


Fig.3.8(b). interpolated signal of size $2n=64$

The graph in figure-3.8(a) is highly nonlinear in nature. Discrete Wavelet Transform captures the features quite efficiently at the time of Approximation.

In the next section, we will discuss an interesting relationship, which is observed during above mentioned process using figure-3.1.

3.5. Properties of the Proposed Approximation Method using Wavelets

If we apply one more level of analysis phase on w_{j+2} ((j+2)th resolution of original signal), we get original signal at (j+1)th resolution (i.e. y_{j+1}) back. Hence we can rewrite equation (3.2) as below:

$$y_{j+2} = U(U(y_j)*u)*u + U(U(x_j)*v)*u + U(U(y_j)*u)*v + U(U(x_j)*v)*v \quad (3.3)$$

Equation (3.2) gives y_{j+2}, which represents the signal at a resolution one level higher than that of y_{j+1} for the original signal *f*. Additionally, we observe that the detail and approximation coefficients obtained when this additional level of analysis phase is applied are equal.

These observations are consolidated in the following lemma.

Lemma 3.1: Let $\{V_j\}_{j\in z}$ be an MRA. Suppose $f \in L^2(R)$ is a signal and for each $j \in z$, sequences $y_{j+1} = (y_{j+1}(k))_{k\in z}$ and $y_{j+2} = (y_{j+2}(k))_{k\in z}$ form (j+1)th and (j+2)th level resolutions of the signal f respectively.

Let $x_j = D(y_{j+1} * \tilde{v})$, $y_j = D(y_{j+1} * \tilde{u})$,

$$w_{j+2} = U(U(y_j)*u)*u + U(U(x_j)*v)*u] + U(U(y_j)*u)*v + U(U(x_j)*v)*v$$

If $p_{j+1} = D(w_{j+2} * \tilde{v})$, $w_{j+1} = D(w_{j+2} * \tilde{u})$,

then, $p_{j+1} = w_{j+1}$ and naturally equal to y_{j+1}.

U is an upsampling operator and * denotes convolution

Proof:

We have
$w_{j+2} = U(U(y_j)*u)*u + U(U(x_j)*v)*u + U(U(y_j)*u)*v + U(U(x_j)*v)*v$

Therefore, $w_{j+2} = U[U(y_j)*u + U(x_j)*v]*u + [U(U(y_j)*u)*v + U(U(x_j)*v)*v]$
(Distributive property of convolution)

Substituting the value of y_{j+1}, from equation (1.21) of chapter 1 we get,

$$w_{j+2} = U(y_{j+1})*u + U(U(y_j)*u)*v + U(U(x_j)*v)*v \qquad (3.4)$$

It is clear from lemma (1.3) of chapter 1 that the reconstruction of a signal at (j+2)th level of resolution can be expressed as follows :

$$y_{j+2} = U(y_{j+1})*u + U(x_{j+1})*v \qquad (3.5)$$

Using one-one property of wavelet transform [11], it is obvious that both, y_{j+2} and w_{j+2}, form a signal at one higher level of resolution than y_{j+1}.

$$\text{Hence, } y_{j+2} = w_{j+2} \qquad (3.6)$$

Equating (3.4) and (3.5) by canceling identical terms, we get

$$U(x_{j+1})*v = [U(U(y_j)*u)*v] + [U(U(x_j)*v)*v] \qquad (3.7)$$

Application of downsampling operator D on both the sides of equation (3.7), gives

$$D[U(x_{j+1})*v] = (U(y_j)*u)*v + (U(x_j)*v)*v$$

$$= [U(y_j)*u + U(x_j)*v]*v \quad \text{(Distributive property of convolution)}$$

$$\therefore D[U(x_{j+1})*v] = y_{j+1}*v$$

Applying upsampling operator U on both the sides, we get

$$U(x_{j+1})*v = U(y_{j+1})*v$$

Applying deconvolution followed by downsampling, we get

$$x_{j+1} = y_{j+1}. \qquad (3.8)$$

Now,

$$w_{j+1}(k) = D(w_{j+2} * \tilde{u})(k)$$
$$= y_{j+2} * \tilde{u}(2k) \qquad \text{(from 3.6)}$$
$$= \sum_k \tilde{u}(2k-m) y_{j+2}(m)$$
$$= \sum_k \overline{u(m-2k)} \langle f, \phi_{j+2,m} \rangle$$
$$= < f, \sum_k u(m-2k) \phi_{j+2,m} >$$
$$= < f, \phi_{j+1,k} >$$
$$= y_{j+1}(k) \qquad (3.9)$$

Similarly, $p_{j+1}(k) = x_{j+1}(k)$

But, using (3.8), we get

$$p_{j+1}(k) = y_{j+1}(k) \qquad (3.10)$$

Therefore from (3.9) and (3.10) we have

$$p_{j+1} = w_{j+1} = y_{j+1} \qquad \text{Q.E.D.} \qquad (3.11)$$

It is interesting to note that the high pass and low pass coefficients become identical.

For instance consider the following illustrations in which a finite length signal is given as an input and yields double length approximated signal as an output using equation (3.1). Up on allowing this new signal to one more analysis phase as shown in figure-3.1, we get two signals of the same length as the original one. These two signals are found to be identical to each other as well as the original signal.

Illustration 1:

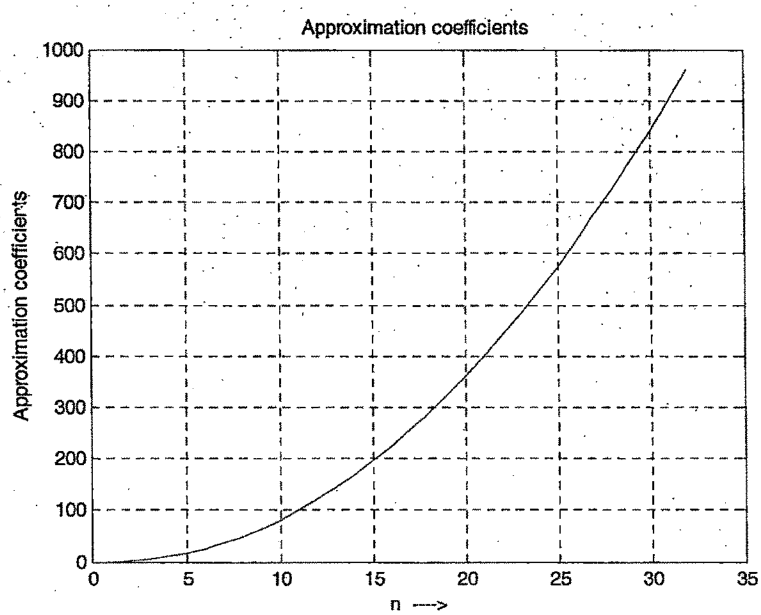
Consider a signal of finite length {1,4,-3,0} to be y_{j+1} [13], the $(j+1)^{th}$ level resolution of size $n = 4$. From equation (3.1), the $(j+2)^{th}$ level resolution of this signal obtained by using Daubechie's D4 wavelets turns out to be y_{j+2} ={0.16485, 2.21605, -1.5783, 4.47608, 1.95723, -4.47607, -0.54301, 0.61237}. Upon passing this new signal through one more analysis phase (see figure-3.1), we get detail coefficients p_{j+1} = {0.99936, 4.0000, -2.99999, 0} and approximation coefficients w_{j+1} = {1.00037, 4.0000, -2.99999, 0} which are identical (ignoring the small variations resulting from round off errors and the method of storing floating point values in computers). These identical signals are also the same as the original signal {1,4,-3,0}.

Illustration 2:

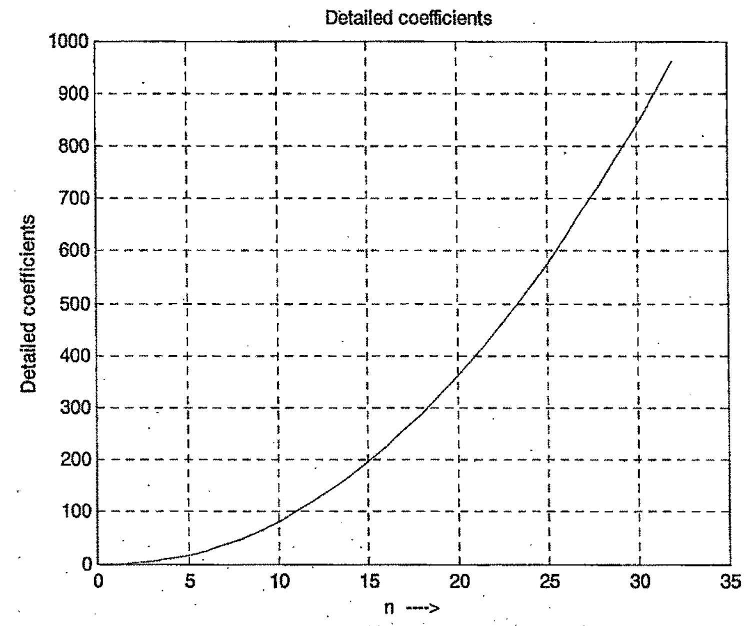
Consider $f(x) = x^2$ be a parabolic signal (as discussed in section 3.4.3) in $L^2(R)$ such that,

$$f(x) \begin{cases} = x^2 , & x \in [0, \ 31] \\ = 0 , & otherwise \end{cases}$$

Dividing the signal in to 31 equal parts, the 32 discrete values are as shown in figure-3.7(a). The double length approximated signal obtained using equation (3.1) is allowed to pass one more analysis phase. The approximation coefficients and detailed coefficients are obtained and portrayed as shown in the figure 3.9(a) and 3.9(b) respectively.



[fig. 3.9(a) The approximation coefficients of $y = x^2$]



[fig. 3.9(b) The detailed coefficients of $y = x^2$]

It is observed from the figures 9(a) and 9(b) that the graphs are identical to each other and same as the figure 3.7(a).

3.6. Conclusions

In this chapter we highlight a novel Approximation technique using MRA with DWT. The method has been verified on finite length signals obtained from standard functions like *sine x*, *cos x*, *exp(x)*, *x*, x^2. Numerical experiments using discrete wavelet transforms like Haar, Daubechies D4, D6 were carried out and it has been observed that this method for interpolating a signal works best with the D4 wavelets.

If the given signal is subjected to analysis phase once and synthesis phases twice, then the signal is decomposed into four components each of size twice that of the original signal. The algebraic sum of all these four components yields a new double length signal. In order to interpret this new signal, it is passed through one more stage of analysis phase. The two branches of this stage produce two signals of length equal to that of the original signal which have been found to have the following properties : (i) the detail and approximation coefficients become identical and same (ii) each of these signals is the same as the original signal. Hence it leads to the conclusion that the intermediate signal w_{j+2} contains the original signal approximated to double its size.

This technique can be applied to industrial problems involving digital signal processing. The method can be extended to two-dimensional signals like digital images. This extended method can be useful in generating more digital images for a glyph from given samples to aid in the development of the OCR systems.