Chapter 1

Wavelets and Multiresolution Analysis

1.1 Historical Perspective

1.1.1 An Overview

The fundamental idea behind studying wavelets is to analyze according to scale. Some researchers in the wavelet field feel that by using wavelets, one is adopting the new perspective in processing the data. Wavelets are functions that satisfy certain mathematical requirements and are used in representing data or other functions. Approximation using superposition of functions as existed since the early eighteen hundreds when Joseph Fourier discovered that he could superpose sines and cosines to represent other functions. However, in wavelet analysis the scale that we use to look at data plays a special role. Wavelet algorithms process data at different scales or resolutions. If we look at signals with large window, we would notice gross features. Similarly, if we look at a signal with a small window, we would notice small features, i.e., the wavelet analysis see both the forest and the trees at one time. This makes wavelets interesting and useful.

For many decades, scientists have approximated functions by sines and cosines terms using the expansion (1.2). By that definition, these functions are non-local and therefore they do a very poor job in approximating sharp spikes. But with wavelet analysis, we can approximate functions that are contained in finite domains. Wavelets are well suited for approximating data with sharp discontinuities.

The wavelet analysis process is to adapt a prototype function, called mother wavelet. Temporal analysis is performed with a contracted, high frequency version of the prototype wavelet while frequency analysis is performed with a dilated, low frequency version of the same wavelet. Since the original signal can be represented as a wavelet expansion, data operations can be performed using the wavelet coefficients. If we further chose the best wavelet according to the data, the data is sparsely represented.

Other applied fields that are making use of wavelets include astronomy, acoustics, nuclear engineering, subband coding, signal and image processing, neurophysiology, magnetic resonance, imaging, speech discrimination, optics, fractals, turbulence, earthquake predictions, radar, human vision, and pure mathematics application such as solving partial differential equations.

1.1.2 History from Eighteenth Century

In the history of Mathematics, wavelet analysis shows many different origins (see [Mey93]). Much of the work performed in the 1930's, and at that time the separate efforts did not appear to be the parts of Coherent theory.

Before 1930

Before 1930, the main branch of mathematics leading wavelets begin with Joseph Fourier with his theories of frequency analysis, now referred to as Fourier synthesis. He asserted that any 2π periodic function $f(x)$ is the sum of its Fourier series given by (1.2). The coefficients are calculated by (1.3). Fourier's assertions played an essential role in the evolution of the ideas. After 1807, by exploring the meaning of functions, Fourier series convergence, and orthogonal systems, mathematicians gradually led from their notion of frequency analysis to the notion of scale analysis. Analyzing $f(x)$ by creating mathematical structures that vary in scales, i.e. construct a function, shift it by some amount, and change its scale. Apply that structure in approximating a signal. Now, repeat the procedures. Take that basic structure, shift it, and scale it again. Apply it to the same signal to get a new approximation. It turns out that this sort of scale analysis is less sensitive to the noise because it measures the average fluctuation of the signal at different scales.

The first wavelet appears in the thesis of A. Harr (1909). One property of the Harr wavelet is that it has compact support. Unfortunately, Harr wavelets are not continuously differentiable which somewhat limits their applications.

Between 1930 - 1980

In the 1930's, several groups working independently found the representation of the function using scale varying basic functions. In 1930, Paul Lavey, investigated Brownian motion used a random signal. He found the Harr basis function superior to the Fourier basis functions for studying small complicated details in the Brownian motion. Another 1930's research effort by Littlewood Paley, and Stein involved computing the energy of a function $f(x)$

$$energy = \frac{1}{2}\int_0^{2\pi} |f(x)|^2 dx \tag{1.1}$$

The computation produce different results if the energy was concentrated around a few points or distributed over a larger interval. This result disturbed the scientists because it indicates that energy might not be conserved. The research discovered a function that can vary in scale and can conserve energy when computing the functional energy. Their work provided David Marr with an effective algorithm for numerical image processing using wavelets in 1980's. Between 1960 and 1980 the mathematician G. Weiss and R. Coifman studied the simplest elements of functions space called atoms, with the goal of finding the atoms for a common function and finding the assembly rules that allow the reconstruction of all the elements of the function space using these atoms. In 1980, Grossmann and Morlet, physicist and engineer, defined wavelets in the context of quantum physics. These two researcher provided a way of thinking the wavelets based on the physical institution.

After 1980

In 1985, Stephan Mallat gave wavelets as pioneer work in the field of digital image processing. He discovered relationships between quadrature mirror filters, pyramid algorithms and orthonormal wavelet basis. Inspired by this, Meyer constructed the first non trivial wavelet. Unlike the Harr wavelets, the Meyer wavelets are continuously differentiable and do not have a compact support. A couple of years later, Ingrid Daubechies used Mallat's work to construct a set of orthonormal basis functions that are perhaps the most elegant and have become the milestone of wavelet application of today.

During 1995 to 1999, G. Kaiser, K. Amartunga, W. Dahman, K. Chen, J. Weiss, R. Coifman, and R. Pownoskwi used the compactly supported wavelets in the field of numerical solution of partial differential equations.

1.2 Motivation for Studying Wavelets

Many mathematical functions can be represented by sum of basis functions. Such representations are known as expansion or series. A well known example being the Fourier expansion given by:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kx}; \quad x \in \mathbf{R} \tag{1.2}$$

which is valid for any well behaved function f with period 1. Here, the basis functions are complex potentials $e^{i2\pi kx}$ each representing a particular frequency indexed by k. The Fourier expansion can be interpreted as follows:
If f is a periodic signal, such as **musical-tone** then (1.2) gives a decomposition of f as a superposition of harmonic modes with frequencies k. This is a good model for vibrations of a guitar string. The coefficient c_k are given by the integral

$$c_k = \int_0^1 f(x) e^{i2\pi kx} dx. \tag{1.3}$$

The series on the righthand side of (1.2) is called the **reconstruction of f**. In theory, reconstruction of f is exact, but in practice this is rarely so. Except in the occasional event where (1.2) can be evaluated analytically, it must be truncated in order to compute numerically. Furthermore, one wants to save computational resources by discarding many of the smallest coefficients c_k which introduce an **approximation error**. Consider the sawtooth function given as follows and shown in Figure-1.1:

$$f(x) = \begin{cases} x, & 0 \le x < 1/2 \\ 1 - x, & 1/2 \le x < 1 \end{cases} \tag{1.4}$$

The Fourier coefficients c_k of the truncated expansion

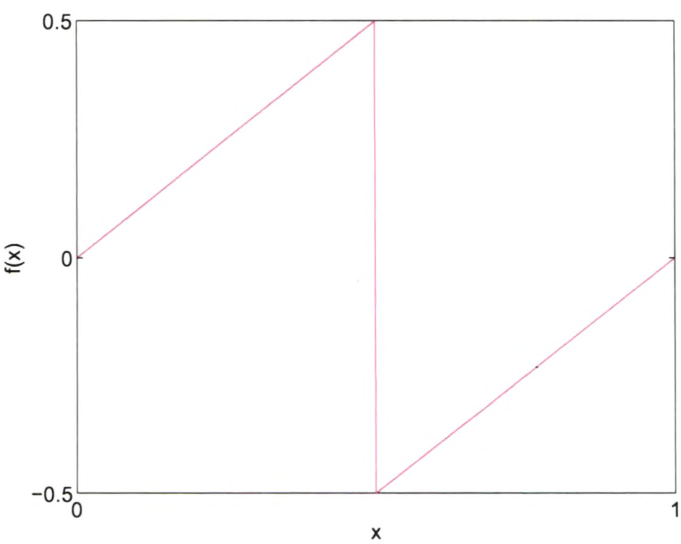

Figure 1.1: **Sawtooth Function**

$$\sum_{k=-(N/2)+1}^{N/2} c_k e^{i2\pi kx}$$

are shown in Figure-1.2 for $N = 1024$.

If, for example, we retain only the 17 largest coefficients, the truncated Fourier expansion with only 17 terms is shown in Figure-1.3 This approximation reflects some of the behavior of f,

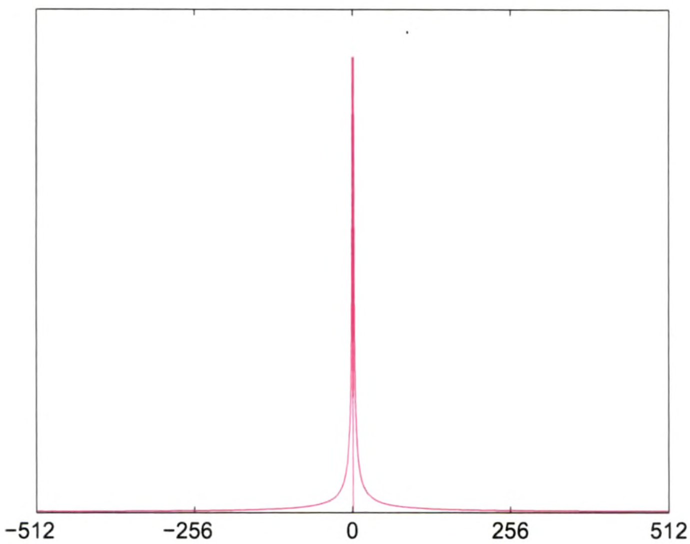

Figure 1.2: **Fourier Coefficients of Sawtooth Function**

and it does not do a good job for the discontinuity at $x = 0.5$. The approximation error is not restricted to the discontinuity but spills into much of the surrounding area. This is known as Gibb's Phenomena. The reason for the poor approximation of the discontinuous function lies in the nature of complex exponentials, as they all cover the entire interval and differ only with respect to frequency. These approximation are not suitable for a discontinuous function. It means Fourier expansion reflects only the locality in frequency and not in time. The problem mentioned above is one way of motivating the use of *wavelets*.

To resolve the discontinuity, we need representation of the function which is both local in space (time) and frequency. Like complex exponentials, wavelets can be used as basis function for the expansion of a function f. They are able to capture the positional information about f as well as about scales (frequency). The wavelet expansion for a function f has the form

$$f(x) = \sum_{k=0}^{2^{J_0}-1} c_{J_0,k}\phi_{J_0,k}(x) + \sum_{j=J_0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x); \quad x \in \mathbf{R} \tag{1.5}$$

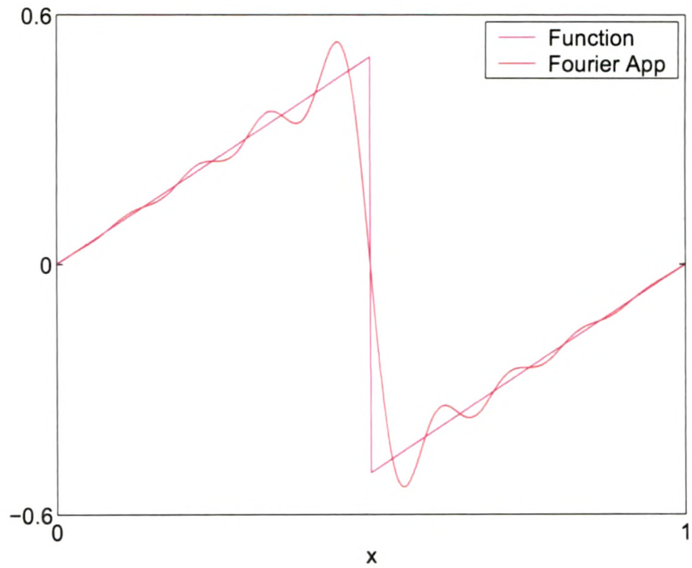

Figure 1.3: **Sawtooth Function and its truncated Fourier expansion with 17 terms**

where J_0 is a non-negative integer. This is similar to the Fourier expansion (1.2). It is a linear combination of a set of basis functions, and the wavelet coefficients are given by

$$c_{J_0,k} = \int_{-\infty}^{\infty} f(x)\phi_{J_0,k}(x)dx;$$

$$d_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx.$$

One immediate difference with respect to the Fourier expansion is the fact that, now we have two types of basis functions and they both are indexed by two different integers. The $\phi_{J_0,k}$ are called scaling functions and $\psi_{J,k}$ are called wavelets. Also both have compact support. We call j the *scale parameter* because it scales the width of support and k the *shift parameter* because it translates the support interval. The scaling function coefficient $c_{J_0,k}$ can be interpreted as local weighted average of f in the region where $\phi_{J_0,k}$ is non-zero. On the other hand, the wavelet coefficients $d_{j,k}$ represent the opposite property, namely the details of f that are lost in weighted average. In practice, the wavelet expansion (like the Fourier expansion) must be truncated at some finest scale which we denote as $J-1$. The truncated wavelet expansion is

$$f(x) = \sum_{k=0}^{2^{J_0}-1} c_{J_0,k}\phi_{J_0,k}(x) + \sum_{j=J_0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x),$$

and the wavelet coefficients are ordered as

$$\left\{ \{c_{J_0,k}\}_{k=0}^{2^{J_0}-1}, \left\{ \{d_{j,k}\}_{k=0}^{2^j-1} \right\}_{j=J_0}^{J-1} \right\},$$

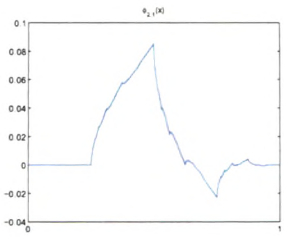

Figure 1.4: **Scaling function in $\tilde{V}_2$**

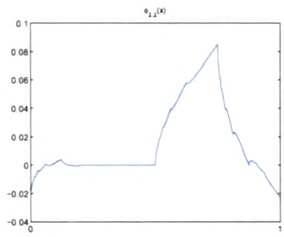

Figure 1.5: **Scaling function in $\tilde{V}_2$**

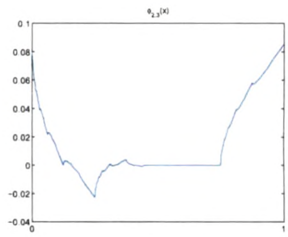

Figure 1.6: **Scaling function in $\tilde{V}_2$**

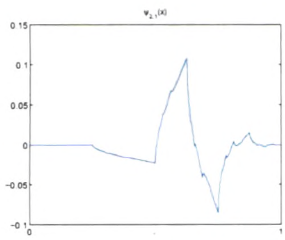

Figure 1.7: **Wavelets in $\tilde{W}_2$**

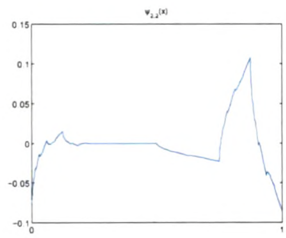

Figure 1.8: **Wavelets in $\tilde{W}_2$**

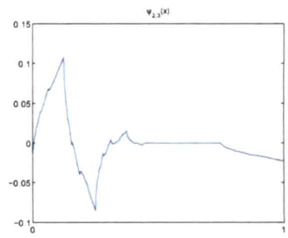

Figure 1.9: **Wavelets in $\tilde{W}_2$**

which are shown in Figure-1.16. The wavelet expansion (1.5) can be understood as follows: The first sum is a coarse approximation of f, where f has been replaced by a linear combination of 2^{J_0} translation of the scaling function $\phi_{J_0,k}$. This corresponds to a Fourier expansion where only low frequency are retained. The remaining terms are the refinements. For each j, a layer represented by 2^j translation of the wavelet $\psi_{j,0}$ is added to obtain a successively more detailed approximation of f. It is convenient to define the approximation spaces

$$V_J = span\{\phi_{j,k}\}_{k=0}^{2j-1},$$

$$W_J = span\{\psi_{j,k}\}_{k=0}^{2j-1}.$$

These spaces are related such that

$$V_J = V_{J_0} \bigoplus W_{J_0} \bigoplus \cdots \bigoplus W_{J-1}.$$

The coarse approximation of f belong to space V_{J_0} and the successive refinements are in the space W_j for $j = J_0, J_0+1, \cdots, J-1$. Together, all of these contribution to constitute a refined approximation of f. Figure-1.4 to Figure-1.15 show the scaling functions and wavelets corresponding to $\tilde{V}_2, \tilde{W}_2$ and $\tilde{W}_3$.

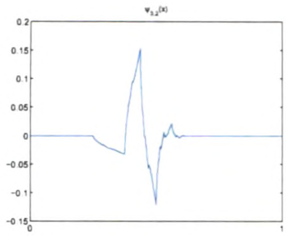

Figure 1.10: **Wavelets in $\tilde{W}_3$**

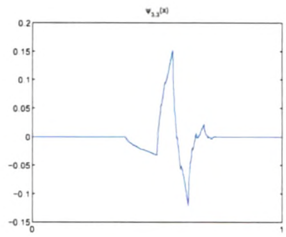

Figure 1.11: **Wavelets in $\tilde{W}_3$**

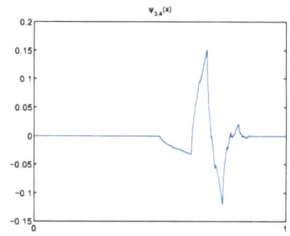

Figure 1.12: **Wavelets in $\tilde{W}_3$**

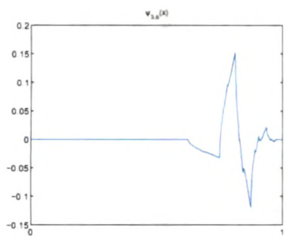

Figure 1.13: **Wavelets in $\tilde{W}_3$**

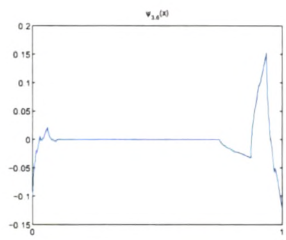

Figure 1.14: **Wavelets in $\tilde{W}_3$**

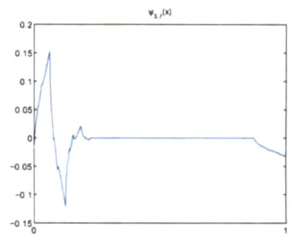

Figure 1.15: **Wavelets in $\tilde{W}_3$**

Figure-1.17 shows the wavelet decomposition of f organized according to scale. Each graph is a projection of f onto one of the approximation spaces mentioned above. The bottom graph is the coarse approximation of f in $\tilde{V}_0$. Those labelled $\tilde{W}_8$ to $\tilde{W}_9$ are successive refinements. Adding these projection yields the graph labelled $\tilde{V}_{10}$.

Figure-1.16 and 1.17 suggest that many of the wavelet coefficients are zero. However, at all scales, there are some non-zero coefficients; and they reveal the position where f is discontinuous. If, as in the Fourier case, we retain only the 17 largest wavelet coefficients, we obtain the approximation shown in Figure-1.18. Because of the way wavelets works, the approximation error is much smaller than that of the truncated Fourier expansion and, very significantly, is highly localized at the point of discontinuity. There are three important facts to note about the wavelet approximations:

1. The good resolution of the discontinuity is a consequence of the large wavelet coefficients appearing at the fine scales. The local high frequency content at the discontinuity is captured much better than that with the Fourier expansion

2. The fact that the error is restricted to a small neighborhood of the discontinuity is a result of the **locality of wavelets**. The behavior of f at one location affects only the coefficients

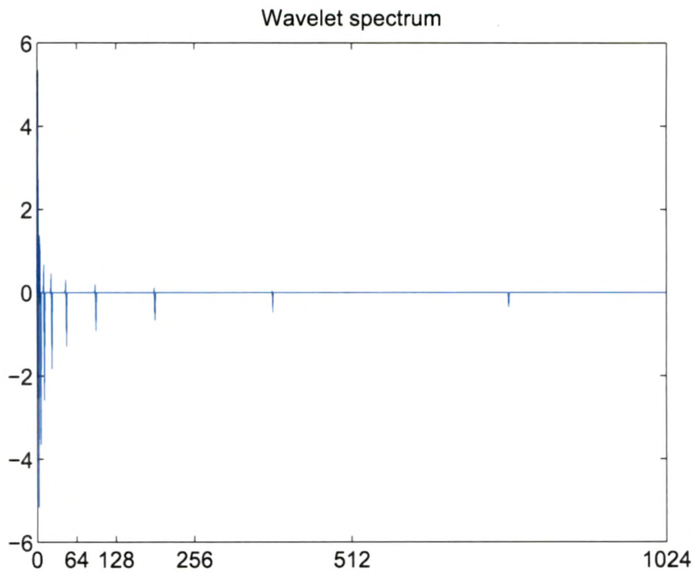

Figure 1.16: **Wavelet Coefficients of $f(x)$**

of wavelets to that location.

3. Most of the linear part of f is represented exactly. In Figure-1.17, one can see that the linear part of f is approximated exactly even in the coarsest approximation space $\tilde{V}_8$ where only a few scaling functions are used. Therefore, no wavelets are needed to add further details to these parts of f.

1.3 Preliminary Requirement

In this section, we present some definitions, and we state some basic theorems.

1.3.1 Some Important Spaces

- $C^n(\mathbf{R})$: The space of all n-times continuously differentiable functions.
- $L^1(\mathbf{R})$: The space of all 1-time integrable function.
- $L^2(\mathbf{R})$: The space of all square-integrable functions.
- In general, $L^p(\mathbf{R})$: The space of all p-times integrable functions where $1 \leq p < \infty$.

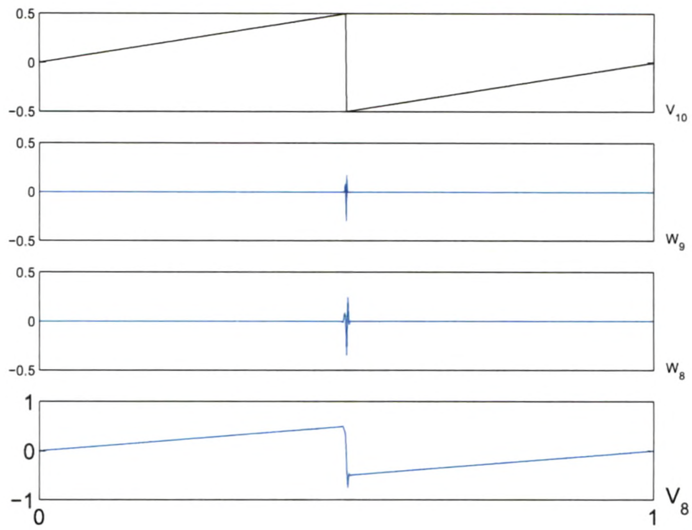

Figure 1.17: **Projection onto a Coarse Space $\tilde{V_6}$ and a sequence of finer spaces $\tilde{W_8}$ - $\tilde{W_9}$.**

- $L^\infty(\mathbf{R})$: The space of all essentially bounded measurable functions.

1.3.2 Inner Product Space

A complex vector space H is said to be an inner product space if for any two elements $x, y \in H$ there exists a complex number $\langle x, y\rangle$ (called the inner product of x and y) that satisfies

- $< x, y >= \overline{< y, x >}$
- $< x + y, z >=< x, z > + < y, z >$, for all $x, y, z \in H$
- $< \alpha x, y >= \alpha < x, y >$, for $x, y \in H$ and $\alpha \in C$
- $< x, x >\geq 0$, for all $x \in H$.
- $< x, x >= 0$, if and only if $x = 0$.

The norm $\|x\|$ of an element $x \in H$ is defined via inner product,

$$\|x\| = \sqrt{< x, x >}.$$

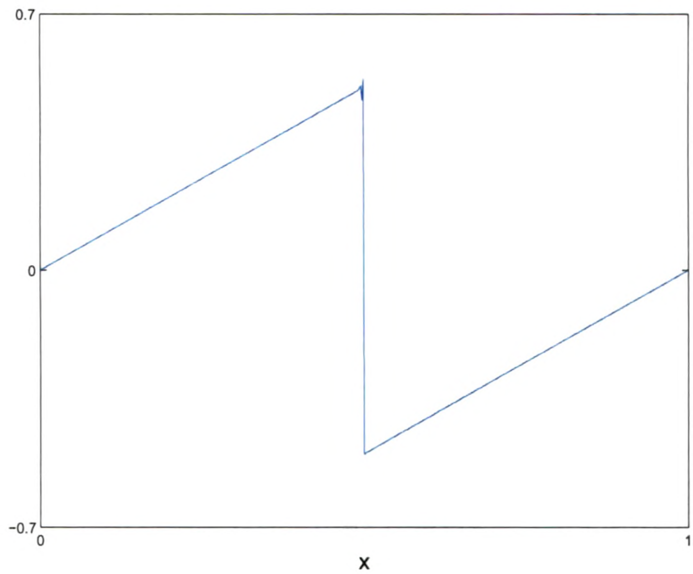

Figure 1.18: **Sawtooth Function and its truncated wavelet expansion**

1.3.3 Compact Support of a Function

The support of a function f denoted by $supp(f)$, is the closer in $\mathbf{R}$ of the set

$$\{x \in \mathbf{R} : f(x) \neq 0\}.$$

The functions whose support are compact are called *function with compact support.*

1.3.4 Properties of Hilbert Space

A linear subspace V of a Hilbert space H is said to be closed subspace of H if V contains all the limiting points, i.e. if $x_n \in V$ and $\|x_n - x\| \to 0$ as $n \to \infty$. The orthogonal complement of a subspace V of H is defined to be the set $V^{\perp}$ of all the elements of H that are orthogonal to every element of V, i.e. $x \in V^{\perp}$ if and only if $< x, y >= 0$, for all $y \in V$. Note that if V is any subset of the Hilbert space H, then $V^{\perp}$ is a closed subspace of H.

1.3.5 Projection

A liner map P from a liner space V to itself is called a projection if $P^2 = P$.

Projection Theorem: Let V be a closed subspace of the Hilbert space H and $x \in H$, then

1. There is a unique element $\hat{x} \in V$ such that

$$\|x - \hat{x}\| = \inf_{y \in V} \|x - y\|,$$

2. $\hat{x} \in V$ and

$$\|x - \hat{x}\| = \inf_{y \in V} \|x - y\|,$$

   if and only if $\hat{x} \in V$, and $(x - \hat{x}) \in V^{\perp}$. Note that the element $\hat{x}$ is called the orthogonal projection of x onto V and is denoted by $P_V x$.

Definition 1.3.1 ***Orthonormal Sets and its Properties:*** *A set $\{e_\lambda, \lambda \in \Lambda\}$ of elements from a given space say V is orthonormal if*

$$< e_s, e_t >= \delta_{s,t}, \quad s, t \in \Lambda.$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal subset of the space V and let M $= span\{e_1, e_2, ..., e_n\}$ then,

1. For any $x \in V$,

$$P_M x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i,$$

2. For any $(a_1, a_2, ..., a_n)$ and $x \in V$

$$\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\| \leq \left\| x - \sum_{i=1}^{n} a_i e_i \right\|;$$

   with equality only for $a_i =< x, e_i >$.

3. Bessel's inequality:

$$\sum_{i=1}^{n} | < x, e_i > |^2 \leq \|x\|^2.$$

The space V is separable if

$$V = \overline{span}\{e_\lambda, \lambda \in \Lambda\}$$

and the set

$$\{e_\lambda, \lambda \in \Lambda\}$$

is finite or countable. Such a set is called a basis. Let V be a separable space with a basis $\{e_n, n \in N\}$. Then,

1. For any $x \in V$ and $\epsilon > 0$, one can find a positive integer N large enough and constants $a_1, a_2, ...a_N$ such that

$$\left\| x - \sum_{n=1}^{N} a_n e_n \right\| < \epsilon.$$

2. $x = \sum_{n=1}^{\infty} < x, e_n > e_n$.

3. Parseval's identity:

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2.$$

4. For any $x, y \in V$,

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \langle e_n, y \rangle.$$

5. If $x = 0$, then for all $n : \langle x, e_n \rangle = 0$.

1.3.6 Fourier Transform and its Properties

The Fourier transform of a function $f \in L^1(R)$ is defined by

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-i\zeta x} dx.$$

If $\hat{f} \in L^1(R)$ is the Fourier transform of $f \in L^1(R)$, then the inverse Fourier transform of f is defined by

$$f(x) = \frac{1}{2\pi} \int \hat{f}(\zeta) e^{i\zeta x} d\zeta$$

at every continuity point of f. The properties are mentioned below:

1. **Boundedness:** $\hat{f} \in L^\infty(R)$, $\|\hat{f}\|_\infty \leq \|f\|_1$.

2. **Uniform Continuity:** $\hat{f}(\omega)$ is uniformly continuous on $-\infty < \omega < \infty$.

3. **Decay:** For $f \in L^1$, $\hat{f}(\omega) \to 0$, when $|\omega| \to \infty$, (Riemann-Lebesgue lemma)

4. **Linearity:** $F[\alpha f(x) + \beta g(x)] = \alpha F[f(x)] + \beta F[g(x)]$.

5. **Derivative:** $F[f^n(x)] = (i\omega)^n \hat{f}(\omega)$.

6. **Plancherel's Identity:** $\langle f, g \rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g} \rangle$. If $g = f$, we obtain Plancherel's Identity as
$$\|f\|^2 = \frac{1}{2\pi}\|\hat{f}\|^2.$$

7. **Shifting:** $F[f(x - x_0)] = e^{-i\omega x_0}\hat{f}(\omega)$.

8. **Symmetry:** $F[F[f(x)]] = 2\pi f(-x)$.

9. **Convolution:** The convolution of f and g is defined as
$$f * g = \int f(x-t)g(t)dt.$$

10. The most important properties of Fourier transform is
$$F[f * g(x)] = \hat{f}(\omega)\hat{g}(\omega).$$

11. **Moment Theorem:**
$$\int_R x^n f(x)dx = (i)^n \left(\frac{d^n \hat{f}(\omega)}{d\omega^n} \right)\bigg|_{\omega=0}$$

1.4 Multiresolution Analysis

There are different types of wavelets available in the literature, for example: Haar wavelets, Shannon wavelets having compact support in the Fourier domain but has a slow decay in the spatial domain, Meyer (see [Mey92]) built wavelets which are infinitely differential and rapidly decreasing functions (Schwartz functions), Battle and Lamarie (see [Dau92]) built polynomial spline wavelets with compact support. All of the above wavelets have unified general framework called the *multiresolution analysis*.

In this section, we discuss the multiresolution analysis which is a general framework for characterizing wavelets, basic scaling functions and basic wavelets with properties and examples. Families of functions

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right); \quad a, b \in \mathbf{R}, a \neq 0,$$

which is derived from a single function ψ by dilation and translation and forms a basis for $L^2(\mathbf{R})$; are known as *wavelets*. Recently, attention has been focused on these families which result in construction of various wavelets with variety of properties. Meyer (see [Mey93]) constructed orthonormal wavelets with $\psi \in C^\infty(\mathbf{R})$, while Daubechies (see [Dau88]) constructed compactly supported wavelets for which $\psi \in C^k(\mathbf{R})$ for arbitrary k. However, it was soon observed by many researchers that a general theory called multiresolution analysis is in the heart of the construction of wavelets. This indeed gives a unified approach that characterizes wavelets in a general way. A natural framework for the wavelet theory is multiresolution analysis (MRA) which is a mathematical construction that characterizes wavelets in a general way. MRA yields fundamental insights into wavelet theory and leads to important algorithms as well. **The goal of MRA is to express an arbitrary function $f \in L^2(\mathbf{R})$ at various levels of detail.**

Definition 1.4.1 ***Multi Resolution Analysis:*** *The MRA is characterized by the following axioms:*

$$0 \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbf{R}) \tag{1.6}$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbf{R}) \tag{1.7}$$

$$\{\phi(x-k)\}_{k\in\mathbf{Z}} \text{ is an orthonormal basis for } V_0 \tag{1.8}$$

$$f \in V_j \Leftrightarrow f(2.) \in V_{j+1} \tag{1.9}$$

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\} \tag{1.10}$$

This describes a sequence of nested approximation spaces V_j in $L^2(\mathbf{R})$ such that the closure of their union equals $L^2(\mathbf{R})$. Projections of a function $f \in L^2(\mathbf{R})$ onto V_j are approximations to f which converges to f as $j \to \infty$. Furthermore, the space V_0 has an orthonormal basis consisting of integral translations of a certain function ϕ, see (1.8). Finally, the spaces are related by the requirement that a function f moves from V_j to V_{j+1} when rescaled by 2. From (1.8), we have the normalization (in the L^2-norm)

$$\|\phi\|_2 \equiv \left(\int_{-\infty}^{\infty} |\phi(x)|^2 dx\right)^{\frac{1}{2}} = 1 \tag{1.11}$$

and for convenience, it is also required that ϕ has unit area (see [Jam94], [Dau92]) i.e.,

$$\int_{-\infty}^{\infty} \phi(x)dx = 1. \tag{1.12}$$

1.4.1 Spaces W_j

Given the nested subspaces in (1.6), we define W_j to be the orthogonal complement of V_j in V_{j+1} i.e., $V_j \perp W_j$ and

$$V_{j+1} = V_j \bigoplus W_j \tag{1.13}$$

Consider now two spaces V_{J_0} and V_J with $J > J_0$. Applying (1.13), recursively, we find that

$$V_j = V_{j_0} \bigoplus \left(\bigoplus_{j=J_0}^{J-1} W_j \right). \tag{1.14}$$

Thus, any function in V_j can be expressed as a linear combination of functions in V_{J_0} and W_j, $j = J_0,\ \ J_0 + 1, \cdots,\ \ J - 1$; hence it can be analyzed separately at different scales. Multiresolution analysis has received its name from this separation of scales.

Continuing the decomposition in (1.14) for $J_0 \to -\infty$ and $J \to \infty$, we obtain in the limits

$$\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbf{R}). \tag{1.15}$$

Note that all W_j are mutually orthogonal.

1.4.2 Basic Scaling Function and Basic Wavelet

The set $\{\phi(x-k)\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0 by axiom (1.8). Now, it follows by repeated application of axiom (1.9) that $\{\phi(2^j x - k)\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_j. Note that the function $\phi(2^j x)$ is translated by $\frac{k}{2^j}$, i.e. it becomes narrower and translations get smaller as j increases. The L^2-norm of one of these basis functions is as follows:

$$\int_{-\infty}^{\infty} |\phi(2^j x - k)|^2 dx.$$

Setting $y = 2^j x - k$, we write $\frac{dy}{2^j} = dx$ and hence, using (1.11), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} |\phi(2^j x - k)|^2 dx &= 2^{-j} \int_{-\infty}^{\infty} |\phi(y)|^2 dy \\ &= 2^{-j} \|\phi\|_2^2 \\ &= 2^{-j} \end{aligned}$$

Therefore, the set $\{2^{\frac{j}{2}}\phi(2^j x - k)\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_j. We call ϕ as the **basic scaling function**, since we generate a whole bunch of basis functions by using dilation and translation of ϕ. Similarly, it is shown in [Dau92] that there exist a function $\psi(x)$ such that $\{2^{\frac{j}{2}}\psi(2^j x - k)\}_{k\in\mathbf{z}}$ is an orthonormal basis for W_j. We call ψ as the **basic wavelet** or **mother wavelet**. Note that, it may not be possible to express either of them (ϕ or ψ) explicitly, but there are efficient methods for calculating the values of ϕ and ψ at any dyadic rational points that we discuss in Chapter 2. For convenience, we now introduce the following notations

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k) \tag{1.16}$$

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) \tag{1.17}$$

and

$$\phi_k(x) = \phi_{0,k}(x) \tag{1.18}$$

$$\psi_k(x) = \psi_{0,k}(x) \tag{1.19}$$

Since $\psi_{j,k} \in W_j$, it follows that $\psi_{j,k}$ is orthogonal to $\phi_{j,k}$ as $\phi_{j,k} \in V_j$ and $V_j \perp W_j$. Note that, all W_j are mutually orthogonal, and hence, the wavelets are orthogonal across scales. Altogether, we have the following orthogonality relations:

$$\int_{-\infty}^{\infty} \phi_{j,k}(x)\phi_{j,l}(x)dx = \delta_{k,l}, \tag{1.20}$$

$$\int_{-\infty}^{\infty} \psi_{i,k}(x)\psi_{j,l}(x)dx = \delta_{i,j}\delta_{k,l}, \tag{1.21}$$

$$\int_{-\infty}^{\infty} \phi_{i,k}(x)\psi_{j,l}(x)dx = 0, j \geq i, \tag{1.22}$$

where $i, j, k, l \in Z$ and $\delta_{k,l}$ is the **Kronecker delta** defined as:

$$\delta_{k,l} = \begin{cases} 0, & k \neq l \\ 1, & k = l. \end{cases} \tag{1.23}$$

1.4.3 Expansion of a Function in V_j

A function $f \in V_j$ can be expanded in various ways. For example, there is the pure scaling function

$$f(x) = \sum_{l=-\infty}^{\infty} c_{J,l}\phi_{J,l}(x), \quad x \in \mathbf{R}, \tag{1.24}$$

where

$$c_{J,l} = \int_{-\infty}^{\infty} f(x)\phi_{J,l}(x)dx. \tag{1.25}$$

We find that

$$\begin{aligned}
\|f_N\|_2^2 &= \langle f_N, f_N \rangle \\
&= \left\langle \sum_{k=0}^{N} c_{J,k}\phi_{J,k}, \sum_{l=0}^{N} c_{J,l}\phi_{J,l} \right\rangle \\
&= \sum_{k=0}^{N}\sum_{l=0}^{N} c_{J,k}c_{J,l}\langle \phi_{J,k}, \phi_{J,l} \rangle \\
&= \sum_{k=0}^{N}\sum_{l=0}^{N} c_{J,k}c_{J,l}\delta_{k,l} \\
&= \sum_{k=0}^{N} |c_{J,k}|^2.
\end{aligned}$$

Applying dominated convergence theorem, we obtain

$$\lim_{N\to\infty} \|f_N\|^2 = \lim_{N\to\infty} \sum_{k=0}^{N} |C_{j,k}|^2.$$

From this, it follows that

$$\|f\|^2 = \sum_{k=-\infty}^{\infty} |C_{j,k}|^2.$$

For any $J_0 \leq J$, there is also the wavelet expansion:

$$f(x) = \sum_{l=-\infty}^{\infty} c_{J_0,l}\phi_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{l=-\infty}^{\infty} d_{j,l}\psi_{j,l}(x), x \in \mathbf{R}, \tag{1.26}$$

where

$$c_{J_0,l} = \int_{-\infty}^{\infty} f(x)\phi_{J_0,l}(x)dx, \tag{1.27}$$

$$d_{j,l} = \int_{-\infty}^{\infty} f(x)\psi_{j,l}(x)dx. \tag{1.28}$$

Further, from the orthogonality property of the wavelets, we obtain

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |c_{J_0,k}|^2 + \sum_{j=J_0}^{J-1}\sum_{k=-\infty}^{\infty} |d_{j,k}|^2.$$

This is Parseval's equation for the wavelet.

Definition 1.4.2 ***Projection Spaces:*** *Let P_{V_j} and P_{W_j} denote the operators that project any $f \in L^2(\mathbf{R})$ orthogonally onto V_j and W_j, respectively. Then*

$$(P_{V_j}f)(x) = \sum_{l=-\infty}^{\infty} c_{j,l}\phi_{j,l}(x),$$

and

$$(P_{W_j}f)(x) = \sum_{l=-\infty}^{\infty} d_{j,l}\psi_{j,l}(x),$$

where

$$c_{j,l} = \int_{-\infty}^{\infty} f(x)\phi_{j,l}(x)dx,$$

and

$$d_{j,l} = \int_{-\infty}^{\infty} f(x)\psi_{j,l}(x)dx.$$

Using decomposition of V_j in (1.14), we can easily obtain

$$P_{V_j}f = P_{V_{J_0}}f + \sum_{j=J_0}^{j-1} P_{W_j}f.$$

1.4.4 Dilation Equation and Wavelets Equation

We observe that, any function in V_0 can be expanded in terms of basis function of V_1, as $V_0 \subset V_1$. Setting $\phi(x) = \phi_{0,0}(x) \in V_0$, we obtain

$$\phi(x) = \sum_{k=-\infty}^{\infty} a_k\phi_{1,k}(x) = \sqrt{2}\sum_{k=-\infty}^{\infty} a_k\phi(2x-k), \tag{1.29}$$

where

$$a_k = \int_{-\infty}^{\infty} \phi(x)\phi_{1,k}(x)dx. \tag{1.30}$$

The scaling function ϕ has compact support if and only if finitely many coefficients $a_k's$ are nonzero (see [Dau92]). For compactly supported ϕ, we have

$$\phi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k\phi(2x-k). \tag{1.31}$$

The equation (1.31) is known as the **dilation equation** where, D is an even positive integer called the **wavelet genus** and the numbers $a_0, a_1, \cdots, a_{D-1}$ are called **filter coefficients**. We now write a relation for the basic wavelets ψ having compact support. Since $\psi \in W_0$ and $W_0 \subset V_1$, we can expand ψ as

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k \phi_(2x-k), \tag{1.32}$$

where the filter coefficient are

$$b_k = \int_{-\infty}^{\infty} \psi(x)\phi_{1,k}(x)dx \tag{1.33}$$

(1.32) is called **wavelet equation**. It turns out that b_k can be expressed in terms of a_k and this relation is given below.

Proposition 1.4.1

$$b_k = (-1)^k a_{D-1-k}, k = 0, 1, \cdots, D-1$$

The proof of the above proposition is based on the orthogonality property of the translation of scaling function (for proof, see [Dau92], or [SN96], or [Nie98]). Also note that $supp(\phi) = supp(\psi) = [0, D-1]$, (see [Dau92], or [SN96]). Thus,

$$supp(\phi_{j,l}) = supp(\psi_{j,l}) = I_{j,l}, \tag{1.34}$$

where

$$I_{j,l} = \left[\frac{l}{2^j}, \frac{l+D-1}{2^j}\right]. \tag{1.35}$$

The formulation of the dilation equation have the following three versions:

$$\phi(x) = \sum_k a_k\phi(2x-k),$$

$$\phi(x) = \sqrt{2}\sum_k a_k\phi(2x-k),$$

$$\phi(x) = 2\sum_k a_k\phi(2x-k).$$

1.4.5 Filter Coefficients

In this section, we shall use properties of ϕ and ψ to derive a number of relations.

1. **Orthonormality Property**
For the filter coefficients, we have

$$\sum_{k=k_1(n)}^{k_2(n)} a_k a_{k-2n} = \delta_{0,n}, n \in \mathbf{Z} \tag{1.36}$$

where

$$k_1(n) = \max\{0, 2n\}$$

and

$$k_2(n) = \min\{D-1, D-1+2n\}.$$

Although, this holds for all $n \in \mathbf{Z}$, it will only yield $D/2$ distinct equations corresponding to $n = 0, 1, 2, \cdots, \frac{D}{2} - 1$. Similarly, it follows from Proposition 1.4.1 that

$$\sum_{k=k_1(n)}^{k_2(n)} b_k b_{k-2n} = \delta_{0,n}, n = 0, 1, 2, \cdots, D/2 - 1. \tag{1.37}$$

2. **Conservation of Area**
Recall from equation (1.12) that $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Integrating both sides of (1.31), we obtain

$$\int_{-\infty}^{\infty} \phi(x)dx = \sqrt{2}\sum_{k=0}^{D-1} a_k \int_{-\infty}^{\infty} \phi(2x-k)dx = \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} a_k \int_{\infty}^{\infty} \phi(y)dy,$$

or

$$\sum_{k=0}^{D-1} a_k = \sqrt{2}. \tag{1.38}$$

3. **Property of Vanishing Moments**
One important property of the scaling function is that it is possible to represent polynomials exactly up to some degree $P-1$. Thus, we require that

$$x^p = \sum_{k=-\infty}^{\infty} M_k^p \phi(x-k), x \in \mathbf{R}, p = 0, 1, \cdots, P-1 \tag{1.39}$$

where

$$M_k^p = \int_{-\infty}^{\infty} x^p \phi(x-k)dx, k \in \mathbf{Z}, p = 0, 1, \cdots, P-1 \tag{1.40}$$

We denote M_k^p, the p^{th} moment of $\phi(x-k)$ and it can be computed by a procedure which is described in Appendix A. Equation (1.39) can be translated into a condition involving the wavelet by taking the inner product with $\psi(x)$. This yields

$$\int_{-\infty}^{\infty} x^p \psi(x)dx = \sum_{k=-\infty}^{\infty} M_k^p \int_{-\infty}^{\infty} \phi(x-k)\psi(x)dx = 0,$$

since ψ and ϕ are orthonormal. Hence, we have the property of P vanishing moments:

$$\int_{-\infty}^{\infty} x^p \psi(x) dx = 0, \quad x \in R; \quad p = 0, 1, \cdots, P-1. \tag{1.41}$$

The property of vanishing moments can be express in terms of the filter coefficients and are as follows:

$$\sum_{l=0}^{D-1} (-1)^l a_l l^p = 0, p = 0, 1, \cdots, P-1. \tag{1.42}$$

Note that the condition (1.36), (1.38) and (1.42) comprise a system of $\left(\frac{D}{2} + 1 + P\right)$ equation for the D filter coefficients $a_k, k = 0, 1, \cdots, D-1$. It turns out that one of the conditions is redundant. For example, (1.42) with $p = 0$ can be obtained from the others (see [New93]). Thus, there are a total of $\left(\frac{D}{2} + P\right)$ equations for the D filter coefficients. The highest number of vanishing moments is

$$P = \frac{D}{2}$$

yielding a total of D equations that must be fulfilled (see [Dau92]). This system can be used to determine filter coefficient for compactly supported wavelets or used to validate coefficients obtained otherwise. Finally, we note two other properties of the filter coefficients. One is

$$\sum_{k=0}^{\frac{D}{2}-1} a_{2k} = \sum_{k=0}^{\frac{D}{2}-1} a_{2k+1} = \frac{1}{\sqrt{2}},$$

and the second is

$$\sum_{l=0}^{\frac{D}{2}-1} \sum_{n=0}^{D-2l-2} a_n a_{n+2l+1} = \frac{1}{2}.$$

1.4.6 Decay of the Wavelet Coefficients

The P vanishing moments have an important role to play for the wavelet coefficients $d_{j,k}$ given in (1.28). The wavelet coefficients decrease rapidly for a smooth function. Further, if a function has a discontinuity in one of its derivatives then the wavelet coefficients decrease slowly only in a neighborhood of that discontinuity and maintain fast decay where the function is smooth. This property makes the wavelets particularly suitable for representing piecewise smooth functions. Below, we discuss an important result on decay of wavelet coefficients (see [PS]).

Theorem 1.4.2 *Let $P = \frac{D}{2}$ be the number of vanishing moments for a wavelet $\psi_{j,k}$ and let $f \in C^P(\mathbf{R})$. Then the wavelet coefficient given in (1.28) decay as follows:*

$$|d_{j,k}| \leq C_P 2^{-j(P+1/2)} \max_{\xi \in I_{j,k}} |f^P(\xi)|$$

where C_P is a constant independent of j, k and f; and $I_{j,k} = supp(\psi_{j,k}) = [k/2^j, (k+D-1)/2^j]$.

Proof: For $x \in I_{j,k}$, we write the Taylor expansion for f around $x = k/2^j$.

$$f(x) = \left(\sum_{p=0}^{P-1} f^{(p)}(k/2^j)\frac{\left(x-\frac{k}{2^j}\right)^p}{p!}\right) + f^P(\xi)\frac{\left(x-\frac{k}{2^j}\right)^P}{P!} \tag{1.43}$$

where $\xi \in [k/2^j, x]$. Note that ξ depends on x. Substituting (1.43) into (1.28) and restricting the integral to the support of $\psi_{j,k}$ we find that

$$\begin{aligned} d_{j,k} &= \int_{I_{j,k}} f(x)\psi_{j,k}(x)dx \\ &= \left(\sum_{p=0}^{P-1} f^{(p)}(k/2^j)\frac{1}{p!}\int_{I_{j,k}}\left(x-\frac{k}{2^j}\right)^p \psi_{j,k}(x)dx\right) \\ &\quad +\frac{1}{P!}\int_{I_{j,k}} f^{(P)}(\xi)\left(x-\frac{k}{2^j}\right)^P \psi_{j,k}(x)dx. \end{aligned}$$

Note that ξ depends on x, so $f^{(P)}(\xi)$ is not constant and must remain under the last integral sign. Consider the integrals where $p = 0, 1, ...P-1$. Using

$$I_{j,k} = \left[\frac{k}{2^j}, \frac{k+D-1}{2^j}\right]$$

and set $y = 2^jx - k$ then $dx = \frac{dy}{2}$. For $x = k/2^j$, we have $y = 0$ and for $x = k + D - 1/2^j$, we have $y = D - 1$. Thus, we obtain

$$\begin{aligned} \int_{k/2^j}^{(k+D-1)/2^j}\left(x-\frac{k}{2^j}\right)^p 2^{j/2}\psi(2^jx-k)dx &= 2^{j/2}\int_0^{(D-1)}\left(\frac{y}{2^j}\right)^p \psi(y)2^{-j}dy \\ &= 2^{-j(p+1/2)}\int_0^{(D-1)} y^p\psi(y)dy \\ &= 0; \quad p = 0, 1, ..., P-1. \end{aligned}$$

Here in the last step we have used the P vanishing moments property (1.41). Therefore, the wavelet coefficient is determined from the remainder term alone. Hence,

$$\begin{aligned} |d_{j,k}| &= \frac{1}{P!}\left|\int_{I_{j,k}} f^{(P)}(\xi)\left(x-\frac{k}{2^j}\right)^P 2^{j/2}\psi(2^jx-k)dx\right| \\ &\leq \frac{1}{P!}\max_{\eta\in I_{j,k}}\left|f^{(P)}(\eta)\right|\int_{I_{j,k}}\left|\left(x-\frac{k}{2^j}\right)^P 2^{j/2}\psi(2^jx-k)\right|dx \\ &= 2^{-j(P+1/2)}\frac{1}{P!}\max_{\eta\in I_{j,k}}\left|f^{(P)}(\eta)\right|\int_0^{D-1}|y^P\psi(y)|dy. \end{aligned}$$

With

$$C_P = \frac{1}{P!}\int_0^{D-1} |y^P \psi(y)|dy, \tag{1.44}$$

we find that

$$|d_{j,k}| \leq C_P 2^{-j(P+1/2)} \max_{\xi\in I_{j,k}} |f^{(P)}(\xi)| \tag{1.45}$$

and this completes the rest of the proof.

1.5 Wavelets and the Fourier Transform

Very often, it is useful to consider the behavior of the Fourier transform of a function rather than the function itself. Therefore, in this section, we derive some relations between the basic scaling function and the basic wavelet. For this purpose, we define the (continuous) Fourier transform as

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x)e^{-i\xi x}dx, \quad \xi \in \mathbf{R}.$$

Since, we need ϕ to have unit area, hence (1.12) translates into

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{1.46}$$

We would like to express $\hat{\phi}$ at other values of ξ in the dilation equation (1.31). Taking the Fourier transform on both sides, we find that

$$\hat{\phi}(\xi) = \sqrt{2}\sum_{k=0}^{D-1} a_k \int_{-\infty}^{\infty} \phi(2x-k)e^{-i\xi x}dx.$$

With change of variables, i.e., $y = 2x - k$, we rewrite

$$\begin{aligned}\hat{\phi}(\xi) &= \sqrt{2}\sum_{k=0}^{D-1} a_k \int_{-\infty}^{\infty} \phi(y)e^{-i\xi(y+k)/2}dy/2 \\ &= \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} a_k e^{-ik\xi/2} \int_{-\infty}^{\infty} \phi(y)e^{-i(\xi/2)y}dy \\ &= A\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right),\end{aligned} \tag{1.47}$$

where

$$A(\xi) = \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} a_k e^{-ik\xi}, \quad \xi \in \mathbf{R}. \tag{1.48}$$

Some Properties of $A(\xi)$:

1. $A(\xi)$ is a 2π-periodic function.
2. If ψ has P vanishing moments, then
$$A(0) = 1$$
$$\left.\frac{d^P A(\xi)}{d\xi^P}\right|_{\xi=\pi} = 0, p = 0, 1, \ldots.., P-1.$$
3.
$$A(n\pi) = \begin{cases} 0; & \text{when } n \text{ is even,} \\ 1; & \text{when } n \text{ is odd.} \end{cases}$$

Equation (1.47) can be repeated for $A(\xi)$ and then we obtain

$$\hat{\phi}(\xi) = A\left(\frac{\xi}{2}\right) A\left(\frac{\xi}{4}\right) \hat{\phi}\left(\frac{\xi}{4}\right).$$

After N such steps, we find that

$$\hat{\phi}(\xi) = \prod_{j=1}^{N} A\left(\frac{\xi}{2^j}\right) \hat{\phi}\left(\frac{\xi}{2^N}\right). \tag{1.49}$$

It follows from (1.48) and (1.38) that $|A(\xi)| \leq 1$, so that the product converges for $N \to \infty$ and hence, we obtain

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} A\left(\frac{\xi}{2^j}\right) \hat{\phi}(0).$$

Using (1.46), we arrive at the following product formula

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} A\left(\frac{\xi}{2^j}\right), \quad \xi \in \mathbf{R}. \tag{1.50}$$

Lemma 1.5.1

$$\hat{\phi}(2\pi n) = \delta_{0,n}, \quad n \in \mathbf{Z}.$$

A consequence of Lemma-1.5.1 is the following basic property of ϕ:

Theorem 1.5.2

$$\sum_{n=-\infty}^{\infty} \phi_{j,0}(x+n) = 2^{-j/2}, j \leq 0, \quad x \in \mathbf{R}.$$

Proof: See [Nie98].

The above theorem states that if π is a zero of the function $A(\xi)$ then the constant function can be represented by a linear combination of the translates of $\phi_{j,0}(x)$, which is equivalent to the zeroth vanishing moment condition (1.41). If the number of vanishing moment P is greater than one, then a similar argument can be used to show the following more general statement (see [SN96]).

Theorem 1.5.3 *If π is a zero of $A(\xi)$ of multiplicity P, i.e., if*

$$\left.\frac{d^p(A(\xi))}{d(\xi)^p}\right|_{\xi=\pi} = 0; p = 0, 1, \cdots, P-1,$$

then

1. *The integral translates of $\phi(x)$ can be reproduce polynomials of degree less than P.*
2. *The wavelet translates of $\psi(x)$ has P vanishing moments.*
3. *$(\hat{\phi})^p(2\pi n) = 0$ for $n \in \mathbf{Z}$, $n \neq 0$ and $p < P$.*
4. *$(\hat{\phi})^p(0)$ for $p < P$.*

1.5.1 The Wavelet Equation

In the beginning of this Subsection, we obtain the relation (1.47) for scaling functions in the frequency domain. Using (1.32), it is straight forward to obtain a similar relation for the wavelets in the frequency domain, i.e., for $\hat{\psi}$ as:

$$\hat{\psi}(\xi) = \sqrt{2}\sum_{k=0}^{D-1} b_k \int_{-\infty}^{\infty} \phi(2x-k)e^{-i\xi x}dx.$$

Let

$$y = 2x - k \Rightarrow \frac{dy}{dx} = 2 \Rightarrow \frac{dy}{2} = dx$$

This gives

$$x = \frac{y+k}{2}.$$

Hence, we can write

$$\begin{aligned}\hat{\psi}(\xi) &= \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} b_k e^{-i\xi k/2}\int_{-\infty}^{\infty}\phi(y)e^{-i(\xi/2)y}dy \\ &= B\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right),\end{aligned}$$

where

$$B(\xi) = \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} b_k e^{-ik\xi}. \tag{1.51}$$

Using Proposition-1.4.1, we express $B(\xi)$ in terms of $A(\xi)$

$$\begin{aligned}B(\xi) &= \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1}(-1)^k a_{D-1-k}e^{-ik\xi} \\ &= \frac{1}{\sqrt{2}}\sum_{k=0}^{D-1} a_{D-1-k}e^{-ik(\xi+\pi)}.\end{aligned}$$

Let $l = D-1-k$, then

$$\begin{aligned}B(\xi) &= \frac{1}{\sqrt{2}}\sum_{l=0}^{D-1} a_l e^{-i(D-1-l)(\xi+\pi)} \\ &= e^{-i(D-1)(\xi+\pi)}\frac{1}{\sqrt{2}}\sum_{l=0}^{D-1} a_l e^{il(\xi+\pi)} \\ &= e^{-i(D-1)(\xi+\pi)}\overline{A(\xi+\pi)}.\end{aligned}$$

Thus, this leads to the wavelet equation in the frequency domain,

$$\hat{\psi} = e^{-i(D-1)(\xi/2+\pi)}\overline{A(\xi/2+\pi)}\hat{\psi}(\xi/2). \tag{1.52}$$

This leads to the following property of $\hat{\Psi}$

$$\hat{\Psi}(4\pi n) = 0, n \in \mathbf{Z}.$$

1.5.2 Orthonormality in the Frequency Domain

By Plancherel's identity, the inner product of ϕ with its integral translates in the physical domain equals the inner product in the frequency domain (except for a factor of 2π). Hence,

$$\begin{aligned} f_k &= \int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\xi)\overline{\hat{\phi}(\xi)e^{-(i\xi k)}}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{\phi}(\xi)\right|^2 e^{i\xi k}d\xi \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{2n\pi}^{2(n+1)\pi} \left|\hat{\phi}(\xi)\right|^2 e^{i\xi k}d\xi. \end{aligned}$$

Let $\xi = \xi - 2\pi n$, then we rewrite

$$f_k = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty}\left|\hat{\phi}(\xi+2\pi n)\right|^2 e^{i\xi k}d\xi. \tag{1.53}$$

Define

$$F(\xi) = \sum_{n=-\infty}^{\infty} \|\hat{\phi}(\xi+2\pi n)\|^2, \xi \in \mathbf{R}. \tag{1.54}$$

Then, we see from (1.53) that f_k is the k^{th} Fourier coefficient of $F(\xi)$. Thus,

$$F(\xi) = \sum_{k=-\infty}^{\infty} f_k e^{-ik\xi} \tag{1.55}$$

Since ϕ is orthogonal to its integral translations, it follows from (1.20) that

$$f_k = \begin{cases} 0; & k \neq 0, \\ 1; & k = 0. \end{cases}$$

Hence, (1.55) leads to

$$F(\xi) = 1, \xi \in \mathbf{R}.$$

Thus, we have proved the following Lemma 1.5.4.

Lemma 1.5.4 *The translates $\phi(x-k), k \in \mathbf{Z}$ are orthonormal if and only if*

$$F(\xi) \equiv 1.$$

We can now translate the condition on F to a condition on $A(\xi)$. From (1.47) and (1.54), we find that splitting the sum into two sums according to whether n is even or odd and using the periodicity of $A(\xi)$, we obtain

$$\begin{aligned} F(2\xi) &= \sum_{n=-\infty}^{\infty} \left|\hat{\phi}(\xi + 2\pi n)\right|^2 |A(\xi + 2\pi n)|^2 + \sum_{n=-\infty}^{\infty} \left|\hat{\phi}(\xi + \pi + 2\pi n)\right|^2 |A(\xi + \pi + 2\pi n)|^2 \\ &= |A(\xi)|^2 \sum_{n=-\infty}^{\infty} \left|\hat{\phi}(\xi + 2\pi n)\right|^2 + |A(\xi + \pi)|^2 \sum_{n=-\infty}^{\infty} \left|\hat{\phi}(\xi + \pi + 2\pi n)\right|^2 \\ &= |A(\xi)|^2 F(\xi) + |A(\xi + \pi)|^2 F(\xi + \pi). \end{aligned}$$

If $F(\xi) = 1$ then $|A(\xi)|^2 + |A(\xi + \pi)|^2 \equiv 1$ and the converse is also true. Thus, we have the following lemma:

Lemma 1.5.5

$$F(\xi) = 1 \Longleftrightarrow |A(\xi)|^2 + |A(\xi + \pi)|^2 \equiv 1.$$

Finally, the Lemma 1.5.4 and Lemma 1.5.5 yields the following theorem:

Theorem 1.5.6 *The translates $\phi(x - k), k \in \mathbf{Z}$ are orthonormal if and only if*

$$|A(\xi)|^2 + |A(\xi + \pi)|^2 \equiv 1.$$

1.5.3 Periodized Wavelets

Up till now, we have discussed results pertaining to the functions $f \in L^2(R)$ having applications in audio processing signals with unknown length. In practical applications such as image processing, data fitting or problems involving differential equations, the space domain is a finite interval which are dealt by using periodized scaling functions and wavelets which we define as follows:

Definition 1.5.1 ***Periodic Scaling Function and Wavelet:*** *Let $\phi \in L^2(\mathbf{R})$ and $\psi \in L^2(\mathbf{R})$ be the basic function and the basic wavelet, respectively, from a MRA as defined for $f \in L^2(\mathbf{R})$. For any $j, l \in \mathbf{Z}$, we define the 1-periodic scaling function*

$$\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x + n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \phi(2^j(x + n) - l), \quad x \in \mathbf{R}, \tag{1.56}$$

and the 1-periodic wavelet

$$\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x + n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \psi(2^j(x + n) - l), \quad x \in \mathbf{R}. \tag{1.57}$$

The 1-periodicity can be verified as follows:

$$\begin{aligned}\tilde{\phi}_{j,l}(x+1) &= \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n+1) \\ &= \sum_{m=-\infty}^{\infty} \phi_{j,l}(x+m) \\ &= \tilde{\phi}_{j,l}(x) \end{aligned} \tag{1.58}$$

and similarly

$$\tilde{\psi}_{j,l}(x+1) = \tilde{\psi}_{j,l}(x).$$

Some important cases of Periodized wavelets are given in the following theorem:

Theorem 1.5.7 *Let the basic scaling function ϕ and the basic wavelet ψ have compact support $[0, D-1]$, and let $\tilde{\phi}_{j,l}$ and $\tilde{\psi}_{j,l}$ be defined as in Definition-1.5.1. Then*

- $j \leq 0, \quad l \in \mathbf{Z}, \quad x \in \mathbf{R}:$

$$\begin{aligned}\tilde{\phi}_{j,l}(x) &= 2^{j/2} \\ \tilde{\psi}_{j,l}(x) &= 0, \quad j \leq -1\end{aligned}$$

- $j > 0, \quad x \in \mathbf{R}:$

$$\begin{aligned}\tilde{\phi}_{j,l+2^j p}(x) &= \tilde{\phi}_{j,l}(x) \\ \tilde{\psi}_{j,l+2^j p}(x) &= \tilde{\psi}_{j,l}(x)\end{aligned}$$

- $j > J_0 \geq [\log_2(D-1)], \; x \in [0,1]:$

$$\tilde{\phi}_{j,l}(x) = \begin{cases} \phi_{j,l}(x) & x \in I_{j,l} \\ \phi_{j,l}(x+1) & x \notin I_{j,l} \end{cases}$$

$$\tilde{\psi}_{j,l}(x) = \begin{cases} \psi_{j,l}(x) & x \in I_{j,l} \\ \psi_{j,l}(x+1) & x \notin I_{j,l} \end{cases}$$

1.5.4 Periodized MRA in $L^2(0,1)$

Many of the properties of the non-periodic scaling functions and wavelets carry over to the periodized version restricted to the interval $[0,1]$. Wavelet orthonormality, for example, is preserved for the scales; $i, j > 0$:

$$\begin{aligned}
\int_0^1 \tilde{\psi}_{i,k}(x)\tilde{\psi}_{j,l}(x)dx &= \int_0^1 \sum_{m=-\infty}^{\infty} \psi_{i,k}(x+m)\tilde{\psi}_{j,l}(x)dx \\
&= \sum_{m=-\infty}^{\infty} \int_m^{m+1} \psi_{i,k}(y)\tilde{\psi}_{j,l}(y-m)dy \\
&= \sum_{m=-\infty}^{\infty} \int_m^{m+1} \psi_{i,k}(y)\tilde{\psi}_{j,l}(y)dy \\
&= \int_{-\infty}^{\infty} \psi_{i,k}(y)\tilde{\psi}_{j,l}(y)dy
\end{aligned}$$

Using (1.57) for the second function and invoking the orthogonality relation for non-periodic wavelets given in equation (1.21) gives

$$\begin{aligned}
\int_0^1 \tilde{\psi}_{i,k}(x)\tilde{\psi}_{j,l}(x)dx &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{i,k}(x)\psi_{j,l-2^j n}(x)dx \\
&= \delta_{i,j} \sum_{n=-\infty}^{\infty} \delta_{k,l-2^j n}.
\end{aligned}$$

If $i = j$, then $\delta_{i,j} = 1$ and $\delta_{k,l-2^j n}$ contributes only when $n = 0$ and $k = l$ because $k, l \in [0, 2^j - 1]$. Hence,

$$\int_0^1 \tilde{\psi}_{i,k}(x)\tilde{\psi}_{j,l}(x)dx = \delta_{i,j}\delta_{k,l}. \tag{1.59}$$

as desired. By a similar analysis, one can establish the relations

$$\int_0^1 \tilde{\phi}_{j,k}(x)\tilde{\phi}_{j,l}(x)dx = \delta_{k,l}, j \geq 0.$$

$$\int_0^1 \tilde{\phi}_{j,k}(x)\tilde{\phi}_{j,l}(x)dx = 0, j \geq i \geq 0.$$

The periodized wavelets and scaling functions restricted to $[0, 1]$ generate a multiresolution analysis of $L^2([0, 1])$ analogous to that of $L^2(R)$. The relevant subspaces are given by the following definition:

Definition 1.5.2 *We can define the Periodic subspaces in the similar way as that in the non-periodic case as follows:*

$$\tilde{V}_j = span\left\{\tilde{\phi}_{j,l}, x \in [0,1]\right\}_{l=0}^{2^j-1}$$

$$\tilde{W}_j = span\left\{\tilde{\psi}_{j,l}, x \in [0,1]\right\}_{l=0}^{2^j-1}$$

It turns out that the $\tilde{V}_j$ are nested as in the non-periodic MRA (see [Dau92]),

$$\tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots \subset L^2([0,1]),$$

and that the

$$\overline{\bigcup_{j=0}^{\infty} \tilde{V}_j} = L^2([0,1]).$$

In addition, the orthogonality relations imply

$$\tilde{V}_j \bigoplus \tilde{W}_j = \tilde{V}_{j+1}. \tag{1.60}$$

So, we have the decomposition

$$L^2([0,1]) = \tilde{V}_0 \bigoplus \left(\bigoplus_{j=0}^{\infty} \tilde{W}_j \right). \tag{1.61}$$

From Theorem-1.5.7 and (1.61), we have seen that the system

$$\left\{ 1, \left\{ \left\{ \tilde{\psi}_{j,k} \right\}_{k=0}^{2^j-1} \right\}_{j=0}^{\infty} \right\} \tag{1.62}$$

is an orthonormal basis for $L^2([0,1])$. This basis is canonical in the sense that the space $L^2([0,1])$ is fully decomposed as in (1.61); i.e. the orthonormal decomposition process cannot be continued further because, as stated in Theorem 1.5.7, $\tilde{W}_j = \{0\}$ for $j \leq -1$. Note that the scaling functions no longer appear explicitly in the expansion since they have been replaced by constant 1 according to Theorem 1.5.7. Sometimes one wants to use the basis associated with the decomposition

$$L^2([0,1]) = \tilde{V}_{J_0} \bigoplus \left(\bigoplus_{j=J_0}^{\infty} \tilde{W}_j \right)$$

for some $J_0 > 0$. We recall that if $J_0 \geq \log_2(D-1)$ then the non-periodic basis functions do not overlap. This property is exploited in the parallel algorithm described in Chapter 3.

1.5.5 Expansions of Periodic Functions

Let $f \in \tilde{V}_J$ and let J_0 satisfy $0 \leq J_0 \leq J$. The decomposition

$$\tilde{V}_J = \tilde{V}_{J_0} \bigoplus \left(\bigoplus_{j=J_0}^{J-1} \tilde{W}_j \right),$$

which is obtained from (1.60), leads to two expansions of f, namely the pure periodic scaling function expansion

$$f(x) = \sum_{l=0}^{2^J-1} c_{J,l}\tilde{\phi}_{J,l}(x), \quad x \in [0,1], \tag{1.63}$$

and the periodic wavelet expansion

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x), \quad x \in [0,1]. \tag{1.64}$$

If $J_0 = 0$, then (1.64) becomes

$$f(x) = c_{0,0} + \sum_{j=0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x). \tag{1.65}$$

The MRA for periodic wavelets leads to the following expansion of f:

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x), x \in [0,1]. \tag{1.66}$$

If $J_0 = 0$, then (1.66) becomes

$$f(x) = c_{0,0} + \sum_{j=0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x). \tag{1.67}$$

Let $\tilde{f}$ be the periodic extension of f, i.e.,

$$\tilde{f} = f(x - \lfloor x \rfloor), \quad x \in \mathbf{R}. \tag{1.68}$$

Then $\tilde{f}$ is 1-periodic, since $\lfloor x \rfloor$ is an integer, we have

$$\tilde{\phi}(x - \lfloor x \rfloor) = \tilde{\phi}(x)$$

and

$$\tilde{\psi}(x - \lfloor x \rfloor) = \tilde{\psi}(x); \quad x \in \mathbf{R}.$$

Thus, we obtain

$$\tilde{f}(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x), x \in \mathbf{R} \tag{1.69}$$

The coefficients are defined by

$$c_{j,l} = \int_0^1 f(x)\tilde{\phi}_{j,l}(x)dx,$$

$$d_{j,l} = \int_0^1 f(x)\tilde{\psi}_{j,l}(x)dx.$$

But we observe that these are in fact the same as those of the non-periodic expansion. to see this, we use the fact that

$$f(x) = \tilde{f}(x), \quad x \in [0,1],$$

and we write

$$c_{j,l} = \int_0^1 \tilde{f}(x)\tilde{\phi}_{j,l}(x)dx = \int_{-\infty}^{\infty} f(y)\phi_{j,l}(y)dy \tag{1.70}$$

$$d_{j,l} = \int_0^1 \tilde{f}(x)\tilde{\psi}_{j,l}(x)dx = \int_{-\infty}^{\infty} f(y)\psi_{j,l}(y)dy. \tag{1.71}$$

Also, periodicity in $\tilde{f}$ induces periodicity in the wavelet coefficients:

$$c_{j,l+(2^j)p} = c_{j,l}, \tag{1.72}$$

$$d_{j,l+(2^j)p} = d_{j,l}. \tag{1.73}$$

Definition 1.5.3 ***Projection Spaces in*** $L^2([0,1])$***:*** *Let* $P_{\tilde{V}_j}$ *and* $P_{\tilde{W}_j}$ *denote the operators that project any* $f \in L^2([0,1])$ *orthogonally onto* $\tilde{V}_j$ *and* $\tilde{W}_j$*, respectively. Then*

$$(P_{\tilde{V}_j}f)(x) = \sum_{l=-\infty}^{\infty} c_{j,l}\tilde{\phi}_{j,l}(x),$$

and

$$(P_{\tilde{W}_j}f)(x) = \sum_{l=-\infty}^{\infty} d_{j,l}\tilde{\psi}_{j,l}(x),$$

where

$$c_{j,l} = \int_0^1 f(x)\tilde{\phi}_{j,l}(x)dx,$$

and

$$d_{j,l} = \int_0^1 f(x)\tilde{\psi}_{j,l}(x)dx.$$

Hence, we can write

$$P_{\tilde{V}_j}f = P_{\tilde{V}_{J_0}}f + \sum_{j=J_0}^{j-1} P_{\tilde{W}_j}f.$$